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Original Study

Symmetry analysis and invariant solutions of generalized coupled Zakharov-Kuznetsov equations using optimal system of Lie subalgebra

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Abstract

This research focuses on the examination of nonlinear evolution equations, with a specific emphasis on the generalized coupled Zakharov-Kuznetsov (CZK) equations serving as a primary application. Given the wide application of classical Lie symmetry methods in this field, this study employs a Lie symmetry analysis to investigate the CZK equations, as detailed in this research. Our methodology involves the construction of a nine-dimensional optimal system by leveraging the fundamental elements of the Lie algebra. Subsequently, we apply similarity reductions to the equations using each subalgebra. The resulting invariant solutions find diverse applications within the realm of physics and can also be adapted to solve a broad range of related nonlinear evolution equations. We meticulously validate all these solutions through a straightforward verification process. To enhance our comprehension of the physical implications of these solutions, we employ Mathematica simulations to visually represent various solution scenarios. Additionally, to preserve conservation laws, we incorporate Ibragimov's novel conservation law theorem as a crucial component of our analysis.

Keywords: Classical Lie symmetry method, CZK equations, sub-algebra, optimal system, Mathematica simulations.
AMS 2020 codes: 22E05; 35C07; 35G50; 350R03; 35C79; 37K06.

1 Introduction

Physical sciences rely heavily on nonlinear partial differential equations (PDEs), which are engaged in complex nonlinear physical mechanisms. The generalized CZK equations are significant models for a wide range of physical phenomena, including shallow and multilayered internal waves, ion-acoustic waves, plasma physics, hydrodynamics, and waves in nonlinear LC circuits by way of mutual inductance among surrounding inductors and many others [1–4]. Several disciplines of applied sciences and engineering make use of generalized CZK equations. As many natural phenomena, including vibration and self-reinforcing solitary waves, they are characterized by exact solutions of nonlinear coupled PDEs. These models can play a vital role in the understanding of these physical phenomena. As a result, the work of determining the exact solutions has become essential and

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significant in nonlinear science. Both mathematicians and physicists have put a lot of effort into this subject in recent years and have demonstrated a variety of helpful techniques, such as Hirota's bilinear technique [5], the mapping method [6], the reciprocal Bäcklund transformation method [7], the homogeneous balance mechanism [8], the Painleve expansion [9], the Exp-function approach [10], the Jacobi elliptic function expansion method [11, 12], the rational expansion approach [13] and numerous other methods [14, 15] are just a few of them.

Initially, the Zakharov Kuznetsov (ZK) equation [16] was developed to describe lossless plasma with strongly magnetized, weakly nonlinear ion-acoustic waves in two dimensions. In order to properly consider the nonlinear water wave model in a coastal design, seaport, and other areas, civil engineers, coastal, need to have a solid understanding of the solutions to such PDEs. Hence, a key interest in fluid dynamics is finding various kinds of exact solutions to these equations. Many articles have been written to locate the exact and numerical solutions to these equations. The ZK equation, the modified ZK equation, and the extended forms of these equations have all been studied for exact solutions using various methodologies [16–20]. The fractional form of the ZK equation was studied by [21]. Here, we take a look at a CZK equation [22] that is described as

$$\begin{aligned} \mu_t + \mu_{xxx} + \mu_{yyx} - 6\mu\mu_x - \nu_x = 0, \\ \nu_t + \alpha_1\nu_{xxx} + \alpha_2\nu_{yyx} + \alpha_3\nu_x - 6\alpha_4\nu\nu_x - \alpha_5\mu_x = 0, \end{aligned} \tag{1}$$

where α_1, α_2, α_3, α_4 and α_5 are arbitrary parameters, μ, ν are dependent variables and x, y, t are independent variables. In these equations, the initial term represents the system's evolution, while the combination of the second and third terms accounts for dispersion effects. The second-to-last term in the first equation and the fifth term in the second equation are related to convection, while the final terms, along with the fourth term in the second equation are wedge terms, introduce coupling between different aspects of the system. Solitons emerge due to a delicate equilibrium between dispersion and nonlinearity, where these waves maintain their shape and amplitude while propagating through the medium. The use of symmetry approaches to deal with the precise solutions of the nonlinear CZK equations is interesting as it provides a systematic algorithm.

Symmetry analysis [23–28] is a mathematical technique used to study and understand the symmetries inherent in equations or physical systems. Symmetries represent invariance properties under certain transformations. In the context of differential equations, symmetry analysis involves identifying transformations (often generated by Lie groups) that leave the equation unchanged or transform it into an equivalent form. These symmetries can reveal valuable information about the equations and may lead to simplified solutions. An optimal system of Lie subalgebra [29–31] refers to a specific set of subalgebras that are particularly useful for analyzing the symmetries of a given mathematical equation or system. These subalgebras are chosen in a way that simplifies the symmetry analysis and allows for the most efficient reduction of the equation. This systematic approach aids in identifying the key symmetries that can help solve the equation or uncover important properties. Invariant solutions are solutions to a differential equation or mathematical problem that remain unchanged (invariant) under certain symmetrical transformations. These solutions are particularly valuable because they can simplify complex equations and make them more tractable. Invariant solutions are often derived by exploiting the symmetries of the equation, which reduces the problem to a more manageable form.

The importance of Lie theory in nonlinear and engineering disciplines is enormous, and it has numerous uses. At the present day, there is a wealth of literature on Lie theory [32–38]. Many well-known nonlinear PDEs can only be solved with symmetry methods. We use symmetry approaches to accomplish our objectives since we are interested in exact solutions to the nonlinear CZK equations (1). To our knowledge, there is no prior documentation regarding the solutions we found here. Our obtained solutions exhibit a unique form of wave behavior, where waves travel at a constant speed while maintaining their characteristics, and can be used in the theory of waves. We shall plot the results acquired here to emphasize their physical significance.

The following sections serve as an introduction to the technique and problem discussed in this article. In Section 2, the Lie symmetry approach is explained along with some findings for optimal systems that make use of the Lie algebra that was acquired as the basis. Potential similarity reductions for the CZK equations are covered in Section 3. In Section 4, we examine the step response for the structural dynamics of the presented solutions. In Section 5, we investigate the solution profiles for the CZK equations (1). In Section 6, we introduce our analysis.

2 Symmetries and classification of the invariant solutions

In this section, we will explore the Lie symmetries and optimal system of equation (1). Consider the one-parameter Lie group of transformation

$$\begin{aligned}
\tilde{x} &\to x+\varsigma\varphi_1(x,y,t,\mu,\nu)+O(\varsigma^2),\\
\tilde{y} &\to x+\varsigma\varphi_2(x,y,t,\mu,\nu)+O(\varsigma^2),\\
\tilde{t} &\to t+\varsigma\varphi_3(x,y,t,\mu,\nu)+O(\varsigma^2),\\
\tilde{\mu} &\to \mu+\varsigma\rho_1(x,y,t,\mu,\nu)+O(\varsigma^2),\\
\tilde{\nu} &\to x+\varsigma\rho_2(x,y,t,\mu,\nu)+O(\varsigma^2),
\end{aligned} \tag{2}$$

where ς is the group parameter. The infinitesimal generator associated with the above transformations is

$$Q=\varphi_1(x,y,t,\mu,\nu)\frac{\partial}{\partial x}+\varphi_2(x,y,t,\mu,\nu)\frac{\partial}{\partial y}+\varphi_3(x,y,t,\mu,\nu)\frac{\partial}{\partial t}+\rho_1(x,y,t,\mu,\nu)\frac{\partial}{\partial \mu}+\rho_2(x,y,t,\mu,\nu)\frac{\partial}{\partial \nu}. \tag{3}$$

The coefficient functions $\varphi_1,\varphi_2,\varphi_3,\rho_1$ and ρ_2 are to be found, and the operator Q fulfills the Lie symmetry condition

$$\begin{aligned}
Q^{[3]}(\mu_t+\mu_{xxx}+\mu_{yyx}-6\mu\mu_x-\nu_x)|_{(1)}&=0,\\
Q^{[3]}(\nu_t+\alpha_1\nu_{xxx}+\alpha_2\nu_{yyx}+\alpha_3\nu_x-6\alpha_4\nu\nu_x-\alpha_5\mu_x)|_{(1)}&=0,
\end{aligned} \tag{4}$$

where $Q^{[3]}$ is the third extension of Q.

By solving equation (4), we obtain the expressions for infinitesimals $\varphi_1,\varphi_2,\varphi_3,\rho_1$ and ρ_2 which leads to four symmetry generators given by,

$$Q_1=\frac{\partial}{\partial t},\quad Q_2=\frac{\partial}{\partial x},\quad Q_3=\frac{\partial}{\partial y},\quad Q_4=t\frac{\partial}{\partial x}-\frac{1}{6}\frac{\partial}{\partial \mu}-\frac{1}{6\alpha_4}\frac{\partial}{\partial \nu}. \tag{5}$$

Under the bracket operator, the commutator Table 1 is defined as

$$[Q_i,Q_i]=Q_iQ_j-Q_jQ_i.$$

Table 1 Commutator table.

$[Q_i,Q_j]$	Q_1	Q_2	Q_3	Q_4
Q_1	0	0	0	Q_2
Q_2	0	0	0	0
Q_3	0	0	0	0
Q_4	$-Q_2$	0	0	0

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The adjoint representation Table 2 is given by

$$Ad(\exp(\varepsilon Q_i).Q_j) = Q_j - \varepsilon[Q_i, Q_j] + \frac{\varepsilon^2}{2!}[Q_i,[Q_i, Q_j]] - \cdots . \tag{6}$$

Table 2 Adjoint table.

$[Q_i, Q_j]$	Q_1	Q_2	Q_3	Q_4
Q_1	Q_1	Q_2	Q_3	$Q_4 - \varepsilon Q_2$
Q_2	Q_1	Q_2	Q_3	Q_4
Q_3	Q_1	Q_2	Q_3	Q_4
Q_4	$Q_1 + \varepsilon Q_2$	Q_2	Q_3	Q_4

Theorem 1. *Let $\mathscr{L}^4$ be the Lie algebra of equation (1) with basis (5). The optimal system of one-dimensional subalgebras is then generated by the following generators:*

$$\begin{aligned}
&\mathscr{M}^1 = \langle Q_2 \rangle,\\
&\mathscr{M}^2 = \langle Q_3 \rangle,\\
&\mathscr{M}^3 = \langle Q_2 + \beta Q_3 \rangle, \beta \neq 0,\\
&\mathscr{M}^4 = \langle Q_1 \rangle,\\
&\mathscr{M}^5 = \langle Q_1 + \beta Q_3 \rangle, \beta \neq 0,\\
&\mathscr{M}^6 = \langle Q_4 \rangle,\\
&\mathscr{M}^7 = \langle Q_3 + \beta Q_4 \rangle, \beta \neq 0,\\
&\mathscr{M}^8 = \langle Q_1 + \beta Q_4 \rangle, \beta \neq 0,\\
&\mathscr{M}^9 = \langle Q_1 + \beta Q_3 + \gamma Q_4 \rangle,\ \beta, \gamma \neq 0.
\end{aligned} \tag{7}$$

Proof: Consider a general element $Q \in \mathscr{L}^4$. We have,

$$Q = k_1 Q_1 + k_2 Q_2 + k_3 Q_3 + k_4 Q_4. \tag{8}$$

Case 1: $k_4 = 0$, $k_1 = 0, k_3 = 0$. Then, we have

$$Q = k_2 Q_2. \tag{9}$$

We get

$$\mathscr{M}^1 = Q_2. \tag{10}$$

Case 2: $k_4 = 0$, $k_1 = 0, k_3 \neq 0, k_2 = 0$. Then, we have

$$Q = k_3 Q_3. \tag{11}$$

We get

$$\mathscr{M}^2 = Q_3. \tag{12}$$

Case 3: $k_4 = 0$, $k_1 = 0, k_3 \neq 0, k_2 \neq 0$. Then, we have

$$Q = k_2 Q_2 + k_3 Q_3. \tag{13}$$

Take $k_2 = 1$, then, we have

$$\mathscr{M}^3 = Q_2 + \beta Q_3, \quad \beta \neq 0. \tag{14}$$

Case 4: $k_4 = 0$, $k_1 \neq 0, k_3 = 0$. Then, we have

$$Q = k_1 Q_1 + k_2 Q_2, \tag{15}$$

$$Q' = Ad(e^{\varepsilon Q_4})Q = k_1 Q_1, \tag{16}$$

$$\mathscr{M}^4 = Q_1. \tag{17}$$

Case 5: $k_4 = 0$, $k_1 \neq 0, k_3 \neq 0, k_1 = 1$. Then, we have

$$Q = Q_1 + k_2 Q_2 + k_3 Q_3, \tag{18}$$

$$Q' = Ad(e^{\varepsilon Q_4})Q = Q_1 + k_3 Q_3, \tag{19}$$

$$\mathscr{M}^5 = Q_1 + \beta Q_3, \ \ \beta \neq 0. \tag{20}$$

Case 6: $k_4 \neq 0$, $k_1 = 0, k_3 = 0$. Then, we have

$$Q = k_2 Q_2 + k_4 Q_4, \tag{21}$$

$$Q' = Ad(e^{\varepsilon Q_1})Q = k_4 Q_4, \tag{22}$$

$$\mathscr{M}^6 = Q_4. \tag{23}$$

Case 7: $k_4 \neq 0$, $k_1 = 0, k_3 \neq 0$. Then, we have

$$Q = k_2 Q_2 + k_3 Q_3 + k_4 Q_4, \tag{24}$$

$$Q' = Ad(e^{\varepsilon Q_1})Q = k_3 Q_3 + k_4 Q_4. \tag{25}$$

Take $k_3 = 1$, Then, we have

$$\mathscr{M}^7 = Q_3 + \beta Q_4, \ \ \beta \neq 0. \tag{26}$$

Case 8: $k_4 \neq 0$, $k_1 \neq 0, k_3 = 0, k_1 = 1$. Then, we have

$$Q = Q_1 + k_2 Q_2 + k_4 Q_4, \tag{27}$$

$$Q' = Ad(e^{\varepsilon Q_4})Q = Q_1 + k_4 Q_4, \tag{28}$$

$$\mathscr{M}^8 = Q_1 + \beta Q_4, \ \ \beta \neq 0. \tag{29}$$

Case 9: $k_4 \neq 0$, $k_1 \neq 0, k_3 \neq 0, k_1 = 1$. Then, we have

$$Q = Q_1 + k_2 Q_2 + k_3 Q_3 + k_4 Q_4, \tag{30}$$

$$Q' = Ad(e^{\varepsilon Q_4})Q = Q_1 + k_3 Q_3 + k_4 Q_4, \tag{31}$$

$$\mathscr{M}^9 = Q_1 + \beta Q_3 + \gamma Q_4, \ \ \beta, \gamma \neq 0. \tag{32}$$

3 Invariant solutions by similarity transformations

In this section, we find the invariant group solutions under the reduction of different symmetries.

3.1 Reduction by $\langle \mathscr{M}^1, \mathscr{M}^2 \rangle = \langle Q_2, Q_3 \rangle$

First, consider the vector field $Q_2 = \frac{\partial}{\partial x}$. We get the similarity transformations $\mu = f(r,s), v = g(r,s)$, where $r = t, s = y$. Using these transformations, we obtain the reduced system given by,

$$\begin{aligned} f_r &= 0, \\ g_r &= 0. \end{aligned} \tag{33}$$

Now, take $Q_3 = \frac{\partial}{\partial y}$. Q_3 in new variables can be written as $\tilde{Q}_3 = \frac{\partial}{\partial s}$. The similarity variables are written as, $f = \theta(z), g = \vartheta(z)$ where $z = r$. The reduced ODE system obtained by using above transformations is given by,

$$\begin{aligned} \theta' &= 0, \\ \vartheta' &= 0, \end{aligned} \tag{34}$$

this implies,

$$\begin{aligned} \theta &= c_1, \\ \vartheta &= c_2. \end{aligned} \tag{35}$$

So, we get

$$\begin{aligned} f &= c_1, \\ g &= c_2. \end{aligned} \tag{36}$$

Hence, the invariant solution of Eq. (1) in original variables becomes

$$\begin{aligned} \mu_1(x,y,t) &= c_1, \\ v_1(x,y,t) &= c_2. \end{aligned} \tag{37}$$

3.2 Reduction by $\langle \mathscr{M}^4, \mathscr{M}^3 \rangle = \langle Q_1, Q_2 + \beta Q_3 \rangle$

First, consider the vector field $Q_1 = \frac{\partial}{\partial t}$. We get the similarity transformations $\mu = f(r,s), v = g(r,s)$, where $r = x, s = y$. Using these transformations, we obtain the reduced system given by,

$$\begin{aligned} \alpha_1 g_{rrr} + \alpha_2 g_{rss} + (\alpha_3 - 6\alpha_4 g) g_r - \alpha_5 f_r = 0, \\ -6 f f_r + f_{rrr} + f_{rss} - g_r = 0. \end{aligned} \tag{38}$$

Now, take $Q_2 + \beta Q_3 = \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$. $Q_2 + \beta Q_3$ in new variables can be written as $\tilde{Q}_2 + \beta \tilde{Q}_3 = \frac{\partial}{\partial r} + \beta \frac{\partial}{\partial s}$. The similarity variables are written as, $f = \theta(z), g = \vartheta(z)$ where $z = -\beta r + s$. The reduced ODE system obtained by using above transformations is given by,

$$\begin{aligned} -\beta((\alpha_1 \beta^2 + \alpha_2)\vartheta''' + (\alpha_3 - 6\alpha_4 \vartheta)\vartheta' + \alpha_5 \theta') = 0, \\ -\beta(\beta^2 \theta''' - 6\theta\theta' - \vartheta' + \theta''') = 0. \end{aligned} \tag{39}$$

Since $\beta \neq 0$. It follows that,

$$\begin{aligned} (\alpha_1 \beta^2 + \alpha_2)\vartheta''' + (\alpha_3 - 6\alpha_4 \vartheta)\vartheta' + \alpha_5 \theta' = 0, \\ \beta^2 \theta''' - 6\theta\theta' - \vartheta' + \theta''' = 0. \end{aligned} \tag{40}$$

We can furnish the solution of (40) by the tanh ansatz [39]

$$\theta(z) = a_0 + a_1 T^1 + a_2 T^2, \;\; \vartheta(z) = b_0 + b_1 T^1 + b_2 T^2, \tag{41}$$

where $T = \tanh(z)$. By substituting the values (41) into equation (40), and by the back substitution, we get the solution

$$\begin{aligned}
\mu_2(x,y,t) &= \frac{-1}{6\alpha_1 C_{13}(C_{13}^2+C_{14}^2)}(8C_{13}^5\alpha_1 + 16C_{13}^3C_{14}^2\alpha_1 + 8C_{13}C_{14}^4\alpha_1 + C_{13}^3\alpha_3 - C_{13}^3C_{15}\alpha_1 \\
&\quad + C_{13}C_{14}^2\alpha_5 - C_{14}^2C_{15}\alpha_1) + (2C_{13}^2+2C_{14}^2)\tanh^2(C_{12}+C_{13}x+C_{14}y+C_{15}t), \\
\nu_2(x,y,t) &= \frac{-1}{6\alpha_1 C_{13}(C_{13}^2\alpha_3+C_{14}^2\alpha_5)}(8C_{13}^5\alpha_3^2 + 16C_{13}^3C_{14}^2\alpha_3\alpha_5 + 8C_{13}C_{14}^4\alpha_5^2 \\
&\quad - C_{13}^3\alpha_3\alpha_4 + C_{13}^3\alpha_1\alpha_2 - C_{13}C_{14}^2\alpha_5\alpha_4 + C_{14}^2C_{13}\alpha_1\alpha_2 - C_{13}^2C_{15}\alpha_3 - C_{14}^2C_{15}\alpha_5) \\
&\quad + \frac{(2C_{13}^2\alpha_3+2C_{14}^2\alpha_5)}{\alpha_1}\tanh^2(C_{12}+C_{13}x+C_{14}y+C_{15}t),
\end{aligned} \tag{42}$$

where $C_i (i = 1,2,\cdots,15)$ are constants.

We can furnish the solution of (40) by the coth ansatz

$$\theta(z) = a_0 + a_1T^1 + a_2T^2, \;\; \vartheta(z) = b_0 + b_1T^1 + b_2T^2, \tag{43}$$

where $T = \coth(z)$. By substituting the values (43) into equation (40), and by the back substitution, we get the solution as

$$\begin{aligned}
\mu_3(x,y,t) &= \frac{-1}{6\alpha_1 C_5^2(C_5^2+C_6^2)}(8C_5^5\alpha_1 + 16C_5^3C_6^2\alpha_1 + 8C_5C_6^4\alpha_1 + C_5^3\alpha_3 - C_5^2C_8\alpha_1 \\
&\quad + C_5C_6^2\alpha_5 - C_6^2C_8\alpha_1) + (2C_5^2+2C_6^2)\coth^2(C_1+C_5x+C_6y+C_8t), \\
\nu_3(x,y,t) &= \frac{-1}{6\alpha_1 C_5(C_5^2\alpha_3+C_6^2\alpha_5)}(8C_5^5\alpha_3^2 + 16C_5^3C_6^2\alpha_3\alpha_5 + 8C_5C_6^4\alpha_5^2 - C_5^3\alpha_3\alpha_4 \\
&\quad + C_5^3\alpha_1\alpha_2 - C_5C_6^2\alpha_5\alpha_4 + C_6^2C_5\alpha_1\alpha_2 - C_5^2C_8\alpha_3 - C_6^2C_8\alpha_5) \\
&\quad + \frac{(2C_5^2\alpha_3+2C_6^2\alpha_5)}{\alpha_1}\coth^2(C_1+C_5x+C_6y+C_8t),
\end{aligned} \tag{44}$$

where $C_i (i = 1,2,\cdots,8)$ are constants.

3.3 Reduction by $\langle \mathscr{M}^4, \mathscr{M}^2 \rangle = \langle Q_1, Q_3 \rangle$

First, consider the vector field $Q_1 = \frac{\partial}{\partial t}$. We get the similarity transformations $\mu = f(r,s), v = g(r,s)$, where $r = x, s = y$. Using these transformations, we obtain the reduced system given by,

$$\begin{aligned}
\alpha_1 g_{rrr} + \alpha_2 g_{rss} + (\alpha_3 - 6\alpha_4 g)g_r - \alpha_5 f_r = 0, \\
-6ff_r + f_{rrr} + f_{rss} - g_r = 0.
\end{aligned} \tag{45}$$

Now, take $Q_3 = \frac{\partial}{\partial y}$. $Q_2 + \beta Q_3$ in new variables can be written as $\tilde{Q}_3 = \frac{\partial}{\partial s}$. The similarity variables are written as, $f = \theta(z), g = \vartheta(z)$ where $z = r$. The reduced ODE system obtained by using the above transformations is given by,

$$\begin{aligned}
\alpha_1\vartheta''' + (\alpha_3 - 6\alpha_4\vartheta)\vartheta' - \alpha_5\theta' = 0, \\
\theta''' - 6\theta\theta' - \vartheta' = 0.
\end{aligned} \tag{46}$$

We can also find the solution of (46) by the tanh − coth ansatz [40]

$$\theta(z) = \sum_{\varsigma=0}^{2} a_\varsigma \mathrm{T}^\varsigma + \sum_{\varsigma=1}^{2} a_\varsigma \mathrm{T}^{-\varsigma}, \tag{47}$$

$$\vartheta(z) = \sum_{\varsigma=0}^{2} b_{\varsigma} \mathrm{T}^{\varsigma} + \sum_{\varsigma=1}^{2} b_{\varsigma} \mathrm{T}^{-\varsigma}, \tag{48}$$

where $\mathrm{T} = \tanh(z)$. By substituting the values into equation (46), and by the back substitution, we get two sets of solutions for (1)

$$\begin{aligned}
\mu_4(x,y,t) &= \frac{-1}{6\alpha_1 C_2 (C_2^2 + C_3^2)} (8C_2^5\alpha_1 + 16C_2^3 C_3^2 \alpha_1 + 8C_2 C_3^4 \alpha_1 + C_2^3\alpha_3 - C_2^2 C_5 \alpha_1 \\
&\quad + C_2 C_3^2 \alpha_5 - C_3^2 C_5 \alpha_1) + (2C_2^2 + 2C_3^2) \tanh^2(C_1 + C_2 x + C_3 y + C_5 t), \\
\nu_4(x,y,t) &= \frac{-1}{6\alpha_1 C_2 (C_2^2\alpha_3 + C_3^2\alpha_5)} (8C_2^5\alpha_3^2 + 16C_2^3 C_3^2 \alpha_3\alpha_5 + 8C_2 C_3^4 \alpha_5^2 - C_2^3\alpha_3\alpha_4 \\
&\quad + C_2^3\alpha_1\alpha_2 - C_2 C_3^2 \alpha_5\alpha_4 + C_3^2 C_2 \alpha_1\alpha_2 - C_2^2 C_5 \alpha_3 - C_3^2 C_5 \alpha_5) \\
&\quad + \frac{(2C_2^2\alpha_3 + 2C_3^2\alpha_5)}{\alpha_1} \tanh^2(C_1 + C_2 x + C_3 y + C_5 t),
\end{aligned} \tag{49}$$

$$\begin{aligned}
\mu_5(x,y,t) &= \frac{-1}{6\alpha_1 C_2 (C_2^2 + C_3^2)} (8C_2^5\alpha_1 + 16C_2^3 C_3^2 \alpha_1 + 8C_2 C_3^4 \alpha_1 + C_2^3\alpha_3 - C_2^2 C_5 \alpha_1 \\
&\quad + C_2 C_3^2 \alpha_5 - C_3^2 C_5 \alpha_1) + 2(C_2^2 + C_3^2) \coth^2(C_1 + C_2 x + C_3 y + C_5 t), \\
\nu_5(x,y,t) &= \frac{-1}{6\alpha_1 C_2 (C_2^2\alpha_3 + C_3^2\alpha_5)} (8C_2^5\alpha_3^2 + 16C_2^3 C_3^2 \alpha_3\alpha_5 + 8C_2 C_3^4 \alpha_5^2 - C_2^3\alpha_3\alpha_4 \\
&\quad + C_2^3\alpha_1\alpha_2 - C_2 C_3^2 \alpha_5\alpha_4 + C_3^2 C_2 \alpha_1\alpha_2 - C_2^2 C_5 \alpha_3 - C_3^2 C_5 \alpha_5) \\
&\quad + \frac{(2C_2^2\alpha_3 + 2C_3^2\alpha_5)}{\alpha_1} \coth^2(C_1 + C_2 x + C_3 y + C_5 t).
\end{aligned} \tag{50}$$

We can also find the solution of (46) by the JacobiNS ansatz

$$\theta(z) = \vartheta(z) = \mathrm{JacobiNS}. \tag{51}$$

By substituting the values (51) in the equation (46) and then by back substituting, we have the solution for (1)

$$\begin{aligned}
\mu_6(x,y,t) &= \frac{-1}{6\alpha_1 C_6 (C_6^2 + C_8^2)} (4C_1^2 C_6^5 \alpha_1 + 8C_1^2 C_6^3 C_8^2 \alpha_1 + 4C_1^2 C_6 C_8^4 \alpha_1 \\
&\quad + 4C_6^5\alpha_1 + 8C_6^3 C_8^2 \alpha_1 + 4C_6 C_8^4 \alpha_1 + C_6^3\alpha_3 - C_6^3 C_9^4 \alpha_1 + C_6 C_8^2 \alpha_5 - C_9 C_8^2 \alpha_1) \\
&\quad + 2(C_6^2 + C_8^2)\mathrm{JacobiNS}^2(C_1, C_5 + C_6 x + C_8 y + C_9 t), \\
\nu_6(x,y,t) &= \frac{-1}{6\alpha_1 C_6 (C_6^2\alpha_3 + C_8^2\alpha_5)} (4C_1^2 C_6^5 \alpha_3^2 + 8C_1^2 C_6^3 C_8^2 \alpha_3\alpha_5 + 4C_1^2 C_6 C_8^4 \alpha_5^2 \\
&\quad + 4C_6^5\alpha_3^2 + 8C_6^3 C_8^2 \alpha_3\alpha_5 + 4C_6 C_8^4 \alpha_5^2 - C_6^3\alpha_3\alpha_4 - C_6^3\alpha_1\alpha_2 - C_6 C_8^2 \alpha_5\alpha_4 \\
&\quad + C_6 C_8^2 \alpha_2\alpha_1 - C_9 C_6^2 \alpha_3 - C_9 C_8^2 \alpha_5) \\
&\quad + 2\frac{(C_6^2\alpha_3 + C_8^2\alpha_5)}{\alpha_1} \mathrm{JacobiNS}^2(C_1, C_5 + C_6 x + C_8 y + C_9 t).
\end{aligned} \tag{52}$$

We can also find the solution of (46) by the JacobiSN ansatz

$$\theta(z) = \vartheta(z) = \mathrm{JacobiSN}. \tag{53}$$

By substituting the values (53) in the equation (46) and then by back substituting, we have the solution for (1)

$$\begin{aligned}\mu_7(x,y,t) &= \frac{-1}{6\alpha_1 C_6(C_6^2+C_8^2)}(4C_1^2C_6^5\alpha_1+8C_1^2C_6^3C_8^2\alpha_1+4C_1^2C_6C_8^4\alpha_1+4C_6^5\alpha_1+8C_6^3C_8^2\alpha_1\\ &+4C_6C_8^4\alpha_1+C_6^3\alpha_3-C_6^3C_9^4\alpha_1+C_6C_8^2\alpha_5-C_9C_8^2\alpha_1)+2(C_1^2C_6^2\\ &+C_1^2C_8^2)\text{JacobiSN}^2(C_1,C_5+C_6x+C_8y+C_9t),\\ \nu_7(x,y,t) &= \frac{-1}{6\alpha_1 C_6(C_6^2\alpha_3+C_8^2\alpha_5)}(4C_1^2C_6^5\alpha_3^2+8C_1^2C_6^3C_8^2\alpha_3\alpha_5+4C_1^2C_6C_8^4\alpha_5^2\\ &+4C_6^5\alpha_3^2+8C_6^3C_8^2\alpha_3\alpha_5+4C_6C_8^4\alpha_5^2-C_6^3\alpha_3\alpha_4+C_6^3\alpha_2\alpha_1-C_6C_8^2\alpha_5\alpha_4\\ &+C_6C_8^2\alpha_2\alpha_1-C_9C_6^2\alpha_3-C_9C_8^2\alpha_5)+2\frac{(C_6^2\alpha_3+C_8^2\alpha_5)}{\alpha_1}\text{JacobiSN}^2(C_1\\ &,C_5+C_6x+C_8y+C_9t).\end{aligned} \tag{54}$$

Next we furnish the solution of (46) by the WeierstrassP ansatz

$$\theta(z)=\vartheta(z)=WeierstrassP. \tag{55}$$

By substituting the values (55) in the equation (46) and then by back substituting, we have the solution for (1)

$$\begin{aligned}\mu_8(x,y,t) &= -\frac{-C_{10}C_8^2\alpha_1-C_{10}C_9^2\alpha_1+C_8^3\alpha_3+C_8C_9^2\alpha_5}{6\alpha_1C_8(C_8^2+C_9^2)}+2(C_8^2\\ &+C_9^2)\text{WeierstrassP}(C_1,C_6+C_8x+C_9y+C_{10}t),\\ \nu_8(x,y,t) &= \frac{C_8^3(\alpha_3\alpha_4-\alpha_1\alpha_2)+C_8C_9^2(\alpha_5\alpha_4-\alpha_1\alpha_2)+C_{10}C_8^2\alpha_3+C_9^2C_{10}\alpha_5}{6\alpha_1C_8(C_8^2\alpha_3+C_9^2\alpha_5)}\\ &+\frac{2(C_8^2\alpha_3+C_9^2\alpha_5)}{\alpha_1}\text{WeierstrassP}(C_1,C_6+C_8x+C_9y+C_{10}t),\end{aligned} \tag{56}$$

where $C_i(i=1,2,\cdots,15)$ are constants.

3.4 Reduction by $\langle \mathscr{M}^1,\mathscr{M}^5\rangle=\langle Q_2,Q_1+\beta Q_3\rangle$

First, consider the vector field $Q_2=\frac{\partial}{\partial x}$. We get the similarity transformations $\mu=f(r,s),v=g(r,s)$, where $r=t,s=y$. Using these transformations, we obtain the reduced system given by,

$$\begin{aligned}f_r&=0,\\ g_r&=0.\end{aligned} \tag{57}$$

Now, take $Q_1+\beta Q_3=\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial y}$. $Q_1+\beta Q_3$ in new variables can be written as $\tilde{Q}_1+\beta\tilde{Q}_3=\frac{\partial}{\partial r}+\beta\frac{\partial}{\partial s}$. The similarity variables are written as, $f=\theta(z),g=\vartheta(z)$ where $z=\beta r-s$. The reduced ODE system obtained by using the above transformations is given by,

$$\begin{aligned}\theta'&=0,\\ \vartheta'&=0,\end{aligned} \tag{58}$$

this implies,

$$\begin{aligned}\theta&=c_1,\\ \vartheta&=c_2.\end{aligned} \tag{59}$$

So, we get

$$\begin{aligned} f &= c_1, \\ g &= c_2. \end{aligned} \tag{60}$$

Hence, the invariant solution of equation (1) in orginal variables becomes

$$\begin{aligned} \mu_9(x,y,t) &= c_1, \\ \nu_9(x,y,t) &= c_2. \end{aligned} \tag{61}$$

3.5 Reduction by $\langle \mathscr{M}^2, \mathscr{M}^6 \rangle = \langle Q_3, Q_4 \rangle$

First, consider the vector field $Q_4 = t\frac{\partial}{\partial x} - \frac{1}{6}\frac{\partial}{\partial \mu} - \frac{1}{6\alpha_4}\frac{\partial}{\partial v}$. We get the similarity transformations $\mu = -\frac{x}{6t} + f(r,s), v = -\frac{x}{6\alpha_4 t} + g(r,s)$, where $r = t, s = y$. Using these transformations, we obtain the reduced system given by,

$$\begin{aligned} 6\alpha_4 r f_r + 6\alpha_4 f + 1 &= 0, \\ 6\alpha_4 r g_r + 6\alpha_4 g + \alpha_4\alpha_5 - \alpha_3 &= 0. \end{aligned} \tag{62}$$

Now, take $Q_3 = \frac{\partial}{\partial y}$. Q_3 in new variables can be written as $\tilde{Q}_3 = \frac{\partial}{\partial s}$. The similarity variables are written as, $f = \theta(z), g = \vartheta(z)$ where $z = r$. The reduced ODE system obtained by using above transformations is given by,

$$\begin{aligned} 6\alpha_4 z\theta' + 6\alpha_4\theta + 1 &= 0, \\ 6\alpha_4 z\vartheta' + 6\alpha_4\vartheta + \alpha_4\alpha_5 - \alpha_3 &= 0, \end{aligned} \tag{63}$$

this implies,

$$\begin{aligned} \theta &= -\frac{1}{6\alpha_4} + \frac{c_1}{z}, \\ \vartheta &= -\frac{\alpha_5}{6} + \frac{\alpha_3}{6\alpha_4} + \frac{c_2}{z}. \end{aligned} \tag{64}$$

So, we get

$$\begin{aligned} f &= -\frac{1}{6\alpha_4} + \frac{c_1}{r}, \\ g &= -\frac{\alpha_5}{6} + \frac{\alpha_3}{6\alpha_4} + \frac{c_2}{r}. \end{aligned} \tag{65}$$

Hence, the invariant solution of equation (1) in original variables becomes:

$$\begin{aligned} \mu_{10}(x,y,t) &= \frac{(-x+6c_1)\alpha_4 - t}{6\alpha_4 t}, \\ \nu_{10}(x,y,t) &= \frac{-x+\alpha_4 t + \alpha_4(6c_2 - \alpha_5 t)}{6\alpha_4 t}, \end{aligned} \tag{66}$$

where $c_i (i = 1, 2, \cdots, 4)$ are constants.

4 Conservation laws for the CZK (1)

Regarding [41], Ibragimov introduced a groundbreaking theorem that deals with conserved vectors in the context of differential equations. This theorem is particularly relevant in systems of differential equations where

the number of equations matches the number of dependent variables. What sets this theorem apart is its remarkable independence from the presence of a classical Lagrangian. Ibragimov's approach establishes a vital link between each infinitesimal generator and a conserved vector. This concept is expressed through a specially crafted adjoint equation designed for nonlinear differential equations. Looking ahead, we will provide a comprehensive overview of this theorem.

Let us examine a differential system comprising k differential equations.

$$\mathscr{P}_\gamma(x,\mu,\mu_{(1)},\cdots,\mu_{(k)})=0,\quad \gamma=1,2,\cdots,k. \tag{67}$$

In this context, we have a differential system with n independent variables denoted as $x=(x^1,x^2,\cdots,x^n)$ and a system consisting of k dependent variables denoted as $\mu=(\mu^{(1)},\cdots,\mu^{(k)})$. The variational derivative is defined as follows

$$\frac{\delta}{\delta\mu}=\frac{\partial}{\partial\mu}+\sum_{i=1}^{\infty}(-1)^s\mathfrak{D}_{i_1}\cdots\mathfrak{D}_{i_s}\frac{\partial}{\partial\mu_{i_1\cdots i_s}}. \tag{68}$$

This operator, known as the Euler-Lagrange operator, is represented by the term $\mathfrak{D}_i$, which takes the following form:

$$\mathfrak{D}_i=\frac{\partial}{\partial x_i}+\mu_i\frac{\partial}{\partial\mu}+\mu_{ij}\frac{\partial}{\partial\mu_j}+\cdots. \tag{69}$$

Theorem 2. *Any form of symmetry, (whether it is a Lie point symmetry, Lie-Bäcklund, or nonlocal symmetry), is denoted as*

$$Q=\varphi^i(x,\mu,\mu_{(1)},\cdots)\frac{\partial}{\partial x^i}+\rho^\gamma(x,\mu,\mu_{(1)},\cdots)\frac{\partial}{\partial\mu^\gamma}, \tag{70}$$

is inherited by the adjoint system. Particularly, the operator

$$Q=\varphi^i\frac{\partial}{\partial x^i}+\rho^\gamma\frac{\partial}{\partial\mu}+\rho^\gamma_*\frac{\partial}{\partial\nu}, \tag{71}$$

accompanied by a appropriately chosen coefficient $\rho^\gamma_*=\rho^\gamma_*(x,\mu,\nu,\cdots)$, is incorporated into the system, which includes equation (67) and its corresponding adjoint equation. This expression can be stated as follows

$$\mathscr{P}^\star_\gamma(x,\mu,\nu,\mu_{(1)},\nu_{(1)},\cdots\mu_{(k)},\nu_{(m)})\equiv\frac{\delta(v^i\mathscr{P}_i)}{\delta\mu^\gamma}=0,\qquad \gamma=1,2,\cdots,k. \tag{72}$$

Furthermore, when examining the system formed by equations (67) and (72), it displays a conservation law represented as $\mathfrak{D}_i\,(\mathscr{W}^i)=0$, where

$$\begin{aligned}\mathscr{W}^i=&\varphi^i\mathscr{L}+\mathscr{H}^\gamma\left[\frac{\partial\mathscr{L}}{\partial\mu_i}-\mathfrak{D}_j\left(\frac{\partial\mathscr{L}}{\partial\mu_{ij}}\right)+\mathfrak{D}_j\mathfrak{D}_k\left(\frac{\partial\mathscr{L}}{\partial\mu_{ijk}}\right)+\cdots\right]\\&+\mathfrak{D}_j(\mathscr{H}^\gamma)\left[\frac{\partial\mathscr{L}}{\partial\mu_{ij}}-\mathfrak{D}_k\left(\frac{\partial\mathscr{L}}{\partial\mu_{ijk}}\right)+\cdots\right]+\mathfrak{D}_j\mathfrak{D}_k(\mathscr{H}^\gamma)\left[\frac{\partial\mathscr{L}}{\partial\mu_{ijk}}+\cdots\right].\end{aligned} \tag{73}$$

In this context, function $\mathscr{H}^\gamma$ is associated with the existence of conserved vectors and is commonly known as the Lie characteristic function

$$\mathscr{H}^\gamma=\varphi^\gamma-\varphi^i\mu^\gamma. \tag{74}$$

Now, by applying the aforementioned theorem, we will derive the nonlocal conservation law for the system (1).

Theorem 3. *The adjoint equations for the system of equations (1) are provided as follows*

$$\begin{aligned}\mathscr{P}^\star_1&=\frac{\delta\mathscr{P}_1}{\delta\mu}=w_t-6w_x\mu-\alpha_5z_x+w_{xxx}+w_{yyx}=0,\\\mathscr{P}^\star_2&=\frac{\delta\mathscr{P}_2}{\delta\nu}=-6\alpha_4\nu z-z_t+w_x-\alpha_3z_x+6\alpha_4z_x\nu+6\alpha_4\nu_xz-\alpha_1z_{xxx}-\alpha_2z_{yyx}=0,\end{aligned} \tag{75}$$

where the formal Lagrangian is given by

$$\mathscr{L} = w(\mu_t + \mu_{xxx} + \mu_{yyx} - 6\mu\mu_x - \nu_x) + z(\nu_t + \alpha_1\nu_{xxx} + \alpha_2\nu_{yyx} + \alpha_3\nu_x - 6\alpha_4\nu\nu_x - \alpha_5\mu_x), \tag{76}$$

and $\mu = \mu(x,y,t),\ \nu = \nu(x,y,t)$ *are functions.*

Following the Ibragimov theorem, each symmetry generator corresponds to a conserved vector. Therefore, we can now proceed with the calculation of the conserved vectors using the formulation provided in Theorem 2.

(I) When we consider $Q_1 = \frac{\partial}{\partial t}$, it becomes evident that both the $\mathscr{H}^1 = -\mu_t$ and $\mathscr{H}^2 = -\mu_t$ are nonzero, indicating the presence of conserved quantities. To derive the associated conserved vector, we can follow as

$$\begin{aligned}
\mathscr{W}_1^t =& w(\mu_{xxx} + \mu_{yyx} - 6\mu\mu_x - \nu_x) + z(\nu_t + \alpha_1\nu_{xxx} + \alpha_2\nu_{yyx} + \alpha_3\nu_x - 6\alpha_4\nu\nu_x - \alpha_5\mu_x) - \mu_t z,\\
\mathscr{W}_1^x =& \mu_t(6w\mu + \alpha_5 z + w - \alpha_3 z + \alpha_4\nu z - w_{yy} - \alpha_2 z z_{yy}) - \mu_t w_{xx} - \alpha_1\mu_t z_{xx} - \mu_{xt} w_x\\
&- \alpha_1\mu_{xt} z - \mu_{xt} w + \alpha_1\mu_{xxt} z - \mu_{yt} w_y - \mu_{yt} z_y - \mu_{yyt} w - \alpha_2\mu_{yyt} z,\\
\mathscr{W}_1^y =& -\mu_t(w_{yy} + \alpha_2 z_{yy}) - \mu_{xt} w_x - \alpha_2\mu_{xt} z_x - \mu_{yyt} w - \alpha_2\mu_{yyt} z.
\end{aligned} \tag{77}$$

(II) When we consider $Q_2 = \frac{\partial}{\partial x}$, it becomes evident that both the $\mathscr{H}^1 = -\mu_x$ and $\mathscr{H}^2 = -\mu_x$ are nonzero, indicating the presence of conserved quantities. To derive the associated conserved vector, we can follow as

$$\begin{aligned}
\mathscr{W}_2^t =& -\mu_x(w + z),\\
\mathscr{W}_2^x =& w(\mu_t + \mu_{xxx} - \nu_x) + z(\nu_t + \alpha_1\nu_{xxx} + \alpha_2\nu_{yyx} + \alpha_3\nu_x - 6\alpha_4\nu\nu_x - \alpha_5\mu_x)\\
&+ \mu_x(\alpha_5 z + w - \alpha_3 z + 6\alpha_4\nu z) - \mu_x w_{xx} - \alpha_1 z_{xx}\mu_x - \mu_{xx} w_x - \alpha_1\mu_{xx} z\\
&- \mu_{xxt} w - \alpha_1\mu_{xxt} z - \mu_x w_{yy} - \alpha_2\mu_x z_{yy} + \mu_{yx} w_y + \mu_{yx} z_y - \mu_{yyx} w - \alpha_2\mu_{yyx} z,\\
\mathscr{W}_2^y =& -\mu_x(w_{yy} + \alpha_2 z_{yy}) + \mu_{xx} w_x + \alpha_2\mu_{xx} z_x - \mu_{yyx} w - \alpha_2\mu_{yyx} z.
\end{aligned} \tag{78}$$

(III) When we consider $Q_3 = \frac{\partial}{\partial y}$, it becomes evident that both the $\mathscr{H}^1 = -\mu_y$ and $\mathscr{H}^2 = -\mu_y$ are nonzero, indicating the presence of conserved quantities. To derive the associated conserved vector, we can follow as

$$\begin{aligned}
\mathscr{W}_3^t =& -\mu_y(w + z),\\
\mathscr{W}_3^x =& \mu_y(6w\mu + \alpha_5 z + w - \alpha_3 z + 6\alpha_4\nu z - w_{xx} - \alpha_1\mu_y z_{xx} - w_{yy} - \alpha_2 z_{yy}) - \alpha_1\mu_{yz} z\\
&- \mu_{yy} w_x + \mu_{yxx} w - \alpha_1\mu_{yxx} z - \mu_{yy} w_y - \mu_{yy} z_y - \mu_{yy} w - \alpha_2\mu_{yy} z,\\
\mathscr{W}_3^y =& w(\mu_t + \mu_{xxx} + \mu_{yyx} - 6\mu\mu_x - \nu_x) + z(\nu_t + \alpha_1\nu_{xxx} + \alpha_2\nu_{yyx} + \alpha_3\nu_x - 6\alpha_4\nu\nu_x\\
&- \alpha_5\mu_x) - \mu_y w_{yy} - \alpha_2\mu_y z_{yy} + \mu_{yx} w_x + \alpha_2\mu_{yx} z_x - \mu_{yyy} w - \alpha_2\mu_{yyy} z.
\end{aligned} \tag{79}$$

(IV) When we consider $Q_4 = t\frac{\partial}{\partial x} - \frac{1}{6}\frac{\partial}{\partial \mu} - \frac{1}{6\alpha_4}\frac{\partial}{\partial v}$, it becomes evident that both the $\mathscr{H}^1 = -\frac{1}{6} - t\mu_x$ and $\mathscr{H}^2 = -\frac{1}{6\alpha_4} - t\mu_x$ are nonzero, indicating the presence of conserved quantities. To derive the associated conserved vector, we can follow as

$$\begin{aligned}
\mathscr{W}_4^t =& -w(\frac{1}{6}+t\mu_x) - z(\frac{1}{6\alpha_4}+t\mu_x),\\
\mathscr{W}_4^x =& t\mathscr{L} + (\frac{1}{6}+t\mu_x)(6w\mu+\alpha_5 z) - (\frac{1}{6\alpha_4}+t\mu_x)(w-\alpha_3 z+6\alpha_4 v z) - (\frac{1}{6}+t\mu_x)w_{xx}\\
&- \alpha_1(\frac{1}{6\alpha_4}+t\mu_x)z_{xx} + (\frac{1}{6}+t\mu_x)w_x + \alpha_1(\frac{1}{6\alpha_4}+t\mu_x)z - (\frac{1}{6}+t\mu_x)w\\
&- \alpha_1(\frac{1}{6\alpha_4}+t\mu_x)z - (\frac{1}{6}+t\mu_x)w_{yy} - \alpha_2(\frac{1}{6\alpha_4}+t\mu_x)z_{yy} + (\frac{1}{6}+t\mu_{yx})w_y + (\frac{1}{6\alpha_4}+t\mu_{yx})z_y\\
&- (\frac{1}{6}+t\mu_{yyx})w - \alpha_2(\frac{1}{6\alpha_4}+t\mu_{yyx})z,\\
\mathscr{W}_4^y =& -(\frac{1}{6}+t\mu_x)w_{yy} - \alpha_2(\frac{1}{6\alpha_2}+t\mu_x)z_{yy} + tw_x\mu_{xx} + \alpha_2 t\mu_{xx}z_x - t\mu_{yyx}w - \alpha_2 tz\mu_{yyx}.
\end{aligned} \tag{80}$$

5 Solution profiles for the CZK equations (1)

Together with dealing with the exact solutions of the nonlinear PDEs, we must deal with the physical significance of the solutions that are obtained. To get around this problem, we here concentrate on the dynamical characteristics of the CZK equations (1) solutions. We have found solutions that are hyperbolic, rational, Jacobi functions, and Weierstrass functions using the symmetry approach. The obtained results include dark, bright, combined dark-bright solitons, and periodic solutions in the form of elliptic and Weierstrass functions. The solutions depend upon five arbitrary parameters and are verified by a simple Maple check. We have plotted the obtained solutions for 2D and 3D surface views. All the solutions are plotted with a particular set of parameters. The obtained solutions are shown graphically in Figures (1-8).

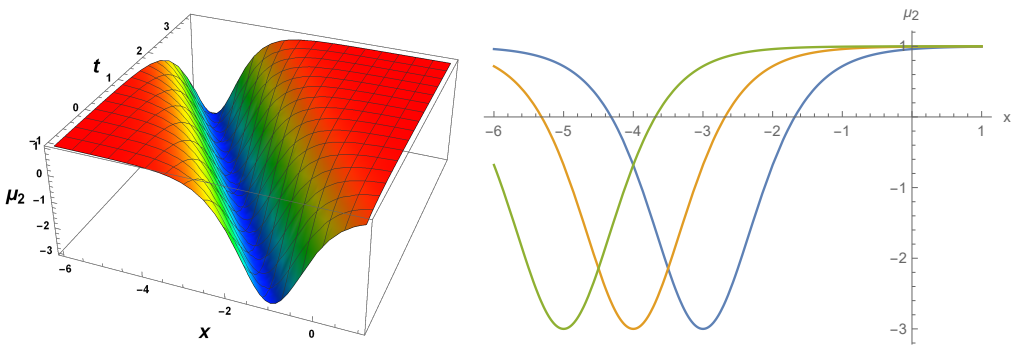


Fig. 1 Structural dynamics of (1) by (42) when all parameters are set to 1 and at $t = 1, 2, 3$.

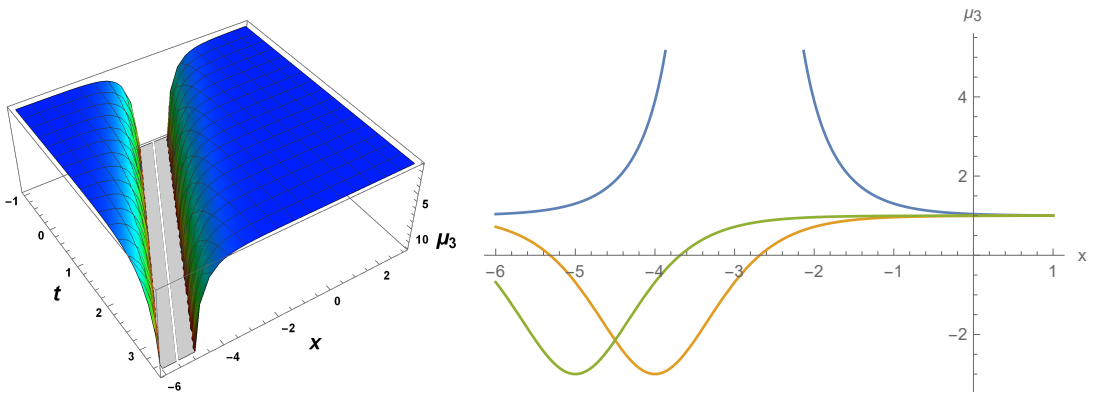


Fig. 2 Structural dynamics of (1) by (44) when all parameters are set to 1 and at $t = 1, 2, 3$.

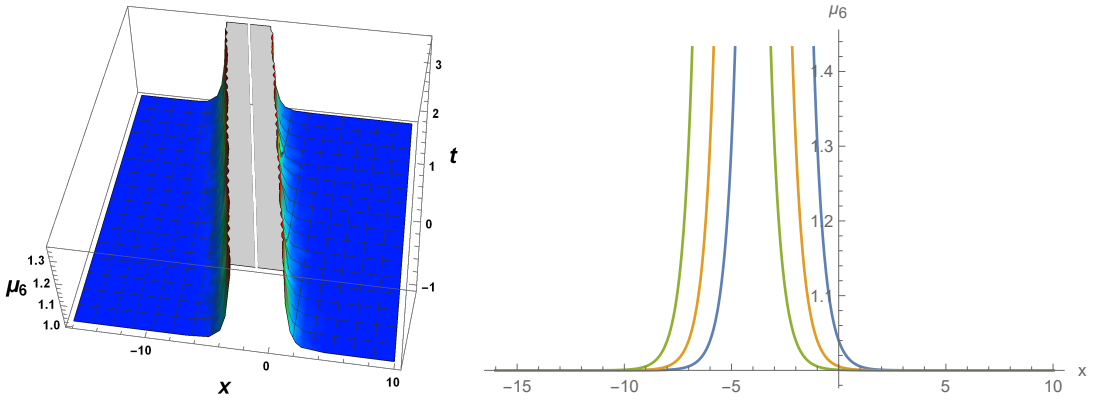


Fig. 3 Structural dynamics of (1) by (52) when all parameters are set to 1 and at $t = 1, 2, 3$.

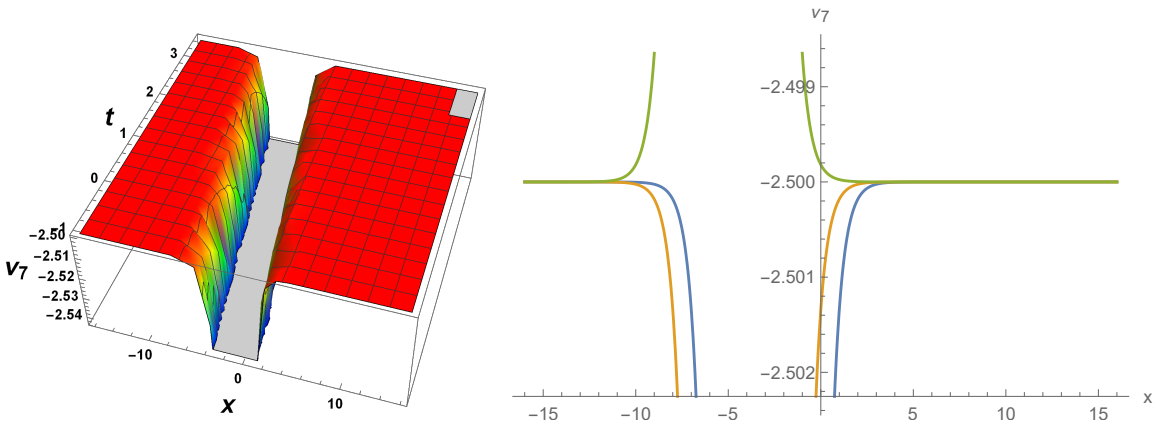


Fig. 4 Structural dynamics of (1) by (54) when all parameters are set to 1 and at $t = 1, 2, 3$.

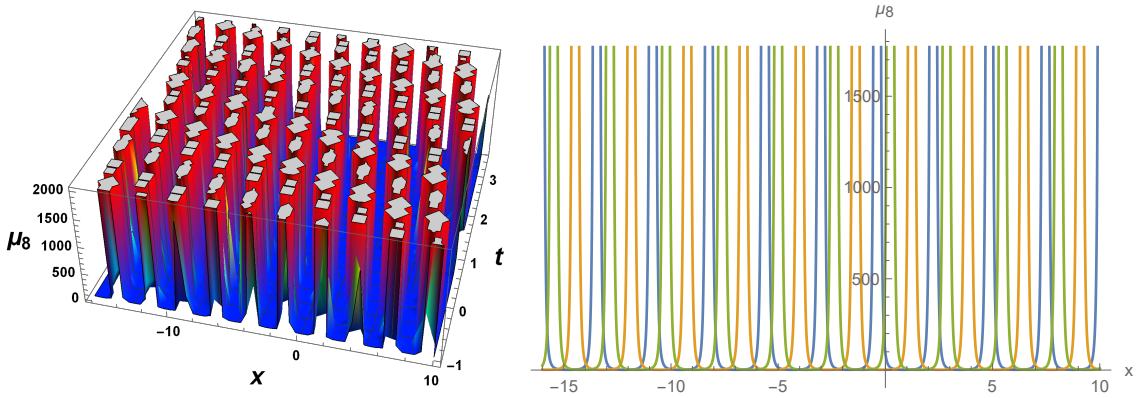


Fig. 5 Periodic dynamics of (1) by (56) when all parameters are set to 1 and at $t = 1, 2, 3$.

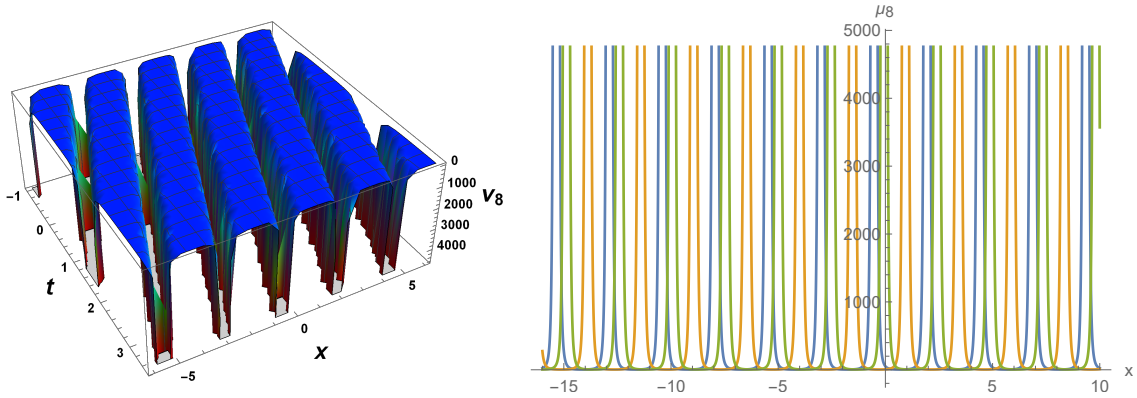


Fig. 6 Periodic dynamics of (1) by (56) when all parameters are set to 1 and at $t = 1, 2, 3$.

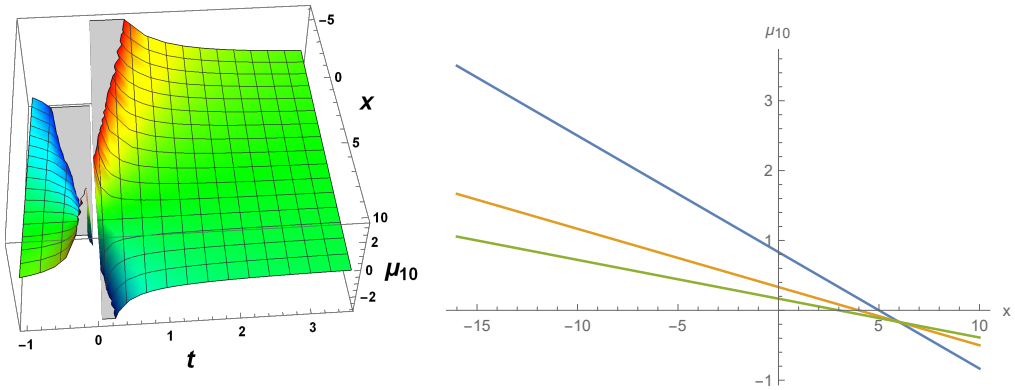


Fig. 7 Structural dynamics of (1) by (66) when all parameters are set to 1 and at $t = 1, 2, 3$.

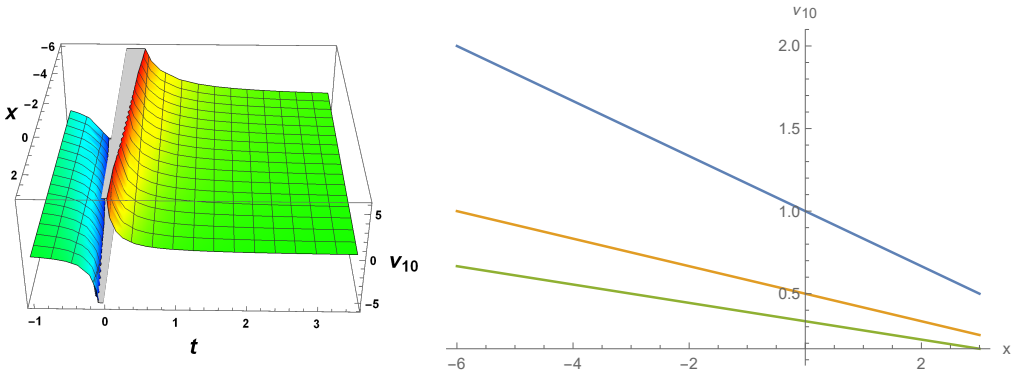


Fig. 8 Structural dynamics of (1) by (66) when all parameters are set to 1 and at $a\alpha_5 = -1$, $t = 1, 2, 3$.

6 Conclusion

The generalized coupled CZK equations were studied in this paper as an application of the nonlinear evolution equations. We were successful in applying the Lie symmetry analysis of the CZK equations, which was addressed in this study, due to the popularity of the classical Lie symmetry methods. The basic elements of the Lie algebra were used to build a nine-dimensional optimal system. Each subalgebra case was then followed by similarity reductions. The obtained results contained various types of solutions such as dark, bright, combined dark-bright solitons, and periodic solutions in the form of elliptic and Weierstrass functions. Current invariant solutions have a variety of applications in physics, and this method can be utilized to solve a sizable class of related nonlinear evolution equations. Mathematica simulations were plotted for a variety of solutions to aid in the physical understanding of the obtained solutions. We are inspired to explore the same category of nonlinear evolution equations using the Lie symmetry method for future models due to its precision and effectiveness in handling such equations.

7 Declarations

7.1 Conflict of interest:

The authors declare that there is no conflict of interest regarding the publication of this paper.

7.2 Author's contributions:

M.U. and A.H.-Methodology, Writing-Review, Software, Plotting, Visualization, Editing and Supervision. F.Z.-Validation, Supervision, Conceptualization, Formal Analysis. N.A.-Resources, Writing-Original Draft, Methodology, Data Curation, Investigation, Plotting, Visualization. The paper has been submitted with the knowledge and consent of all authors.

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7.5 Data availability statement:

All data that support the findings of this study are included within the article.

7.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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