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Original Study

A new approach to the Benjamin-Bona-Mahony equation via ultraspherical wavelets collocation method

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Abstract

In this paper, we develop a precise and efficient ultraspherical wavelet method for a famous Benjamin-Bona-Mahony (BBM) mathematical model. The suggested technique uses the collocation method and ultraspherical wavelets. The proposed scheme is applied to linear and nonlinear BBM equations to inspect the efficiency and accuracy of the proposed technique. The effectiveness of this practical approach is verified. Moreover, the method based on the ultraspherical wavelets is simple, accurate, fast, flexible, and convenient. The results are analyzed using tables and graphs and compared with other methods in literature. As we know, many partial differential equations (PDEs) don't have exact solutions, and some semi-analytical methods work based on controlling parameters, but this is a controlling parameter-free technique. Also, it is pretty simple to implement and consumes less time to execute the programs. The recommended wavelet-based numerical approach is interesting, productive, and efficient. The proposed technique's convergence analysis is also presented through the theorem.

Keywords: Benjamin-Bona-Mahony equation, ultraspherical wavelet, collocation method.
AMS 2020 codes: 35G16; 35G31; 49M15; 65M70; 65T60.

1 Introduction

Linear and nonlinear phenomena are essential in physics, applied mathematics, and engineering problems. Optical fibers, fluid mechanics, plasma physics, electricity, fluid dynamics, wave circulation phenomena, elastic media, movement of heat, and other fields are related to wave phenomena. PDEs, such as Advection, Burgers, Boussinesq, Fisher equations, and many others, are of endless interest in nonlinear optics and quantum mechanics. Occasionally, in addition to helping authors fully understand the process they are describing, solving nonlinear equations enlightens them on facts that are not readily apparent through everyday observation. Additionally, finding precise answers to these problems is somewhat unexplored. However, numerical analysis [1] has become increasingly popular for linear and nonlinear partial differential equations, including BBM equations, in recent years. We look into two physical scenarios with an initial-boundary value problem of the BBM equations type. In 1972, Benjamin, Bona, and Mahony [2] enhanced the equation has the following form:

$$y_t(x,t)+y_x(x,t)+y(x,t)y_x(x,t)-y_{xxt}(x,t)=0, \tag{1}$$

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with different constraints,

$$y(x,0) = f(x), \quad 0 \le x \le 1,$$

$$y(0,t) = g(t),\ y(1,t) = h(t), \quad t \ge 0.$$

The BBM equation, an alternate smoothness model for the KdV equation [3], is given in equation (1).

A lot of mathematicians have also recently introduced several new techniques for solving differential equations, such as the Hermite wavelet technique [4], the Bäcklund transformation method [5], the Fibonacci wavelet scheme for solving hyperbolic PDE, dispersive PDE, the Rosenau-Hyman equation [6–8], the first integral method for MBBM equation [9], Hirota's bilinear method [10], Bernoulli wavelet scheme for nonlinear Murray equation [11], BBMB equations by Exp-Function method [12], (2+1) dimensional Sobolev equation via wavelet technique [13], the Haar wavelet method for the BBM equations [14], ultraspherical wavelet scheme for spectral solutions of Riccati equations [15], ultraspherical wavelet technique for solving 2nth-order boundary value problems [16], ultraspherical operational matrices of derivatives [17], clique polynomial and Adomian decomposition method for solving differential equations [18], and Laguerre wavelets scheme for solving delay differential equations [19], Bernoulli wavelet technique for solving biological models [20], explicit solution of atmosphere-soil-land plant carbon cycle system [21], study on Kudryashov-Sinelshchikov dynamical equation [22], study on Caudrey-Dodd-Gibbon-Sawada-Kotera partial differential equation [23], structure of the analytic solutions for Schrödinger equation [24], solutions for Konopelchenko-Dubrovsky equation [25], solutions of Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equation [26], solutions of the Korteweg-de Vries-Zakharov-Kuznetsov equation [27].

Wavelets are utilized in numerous locations in numerical analysis as ideal instruments to provide a proper mathematical approach to scientific phenomena, often modelled by linear or nonlinear differential equations. The ultraspherical wavelets approach is one of such methods due to its unique properties called orthogonality, which compactly supports and holds multiresolution analysis. Also, these wavelets are continuous in respective domains; hence they play a vital role in applied mathematics. Our literature survey found only a few work-related to ultraspherical wavelets led by Youssri et al. [15–17]. In literature, no one has solved BBM equations by ultraspherical wavelets; this impels us to propose a new technique called the ultraspherical wavelets method (UWM).

The main goal of this paper is to describe and explain a novel numerical method UWM for obtaining a numerical solution to the nonlinear PDEs. The results are compared with wavelet and non-wavelet techniques [14]. According to the current literature survey, UWM has not been used to solve BBM equations.

The structure of this article is organized as follows. Section 2 is dedicated to the properties of ultraspherical wavelets. Section 3 gives the projected technique's convergence analysis. Section 4 is dedicated to the ultraspherical wavelets method. The numerical experiment, outcomes, and error analysis of the examples are covered in Section 5. In Section 6, the suggested work's conclusion is covered.

2 Preliminaries of ultraspherical wavelets

The family of functions known as wavelets is descended from a single function called the mother wavelet through dilatation and translation. When the translation parameter b and the dilation parameter a are continuously variable, we have a family of continuous wavelets as

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}} \psi\left(\frac{x-b}{a}\right),\ a,b \in \mathbb{R},\ a \neq 0.$$

If the parameters a and b are restricted to discrete values as

$$a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0,$$

the discrete wavelets family that we have is as follows

$$\psi_{k,n}(x) = |a_0|^{\frac{1}{2}} \psi(a_0^k x - nb_0).$$

Where $L^2(R)$'s wavelet basis is formed by $\psi_{k,n}$. The orthonormal basis $\psi_{k,n}(x)$ is particularly formed when $a_0 = 2$ and $b_0 = 1$. The ultraspherical wavelets $\psi^{\alpha}_{n,m}(x) = \psi(k,n,m,\alpha,x)$ involve five arguments where k is supposed to be any positive integer, $n = 1,2,3,\cdots,2^{k-1}$, m be the degree of the ultraspherical polynomials, x be the normalized time, and α is the known ultraspherical parameter. The ultraspherical wavelets be defined on the interval $[0,1]$ as in [16],

$$\psi^{\alpha}_{n,m}(x) = \begin{cases} 2^{\frac{k+1}{2}} \mu_{m,\alpha}\, c^{\alpha}_m(2^{k+1}x - 2n - 1), & \frac{n}{2^k} \le x < \frac{n+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where, $\mu_{m,\alpha} = 2^{\alpha}\Gamma(\alpha)\sqrt{\frac{m!(m+\alpha)}{2\pi\Gamma(m+2\alpha)}}$ and $m = 0,1,2,\cdots,M-1$. Here c^{α}_m is the ultraspherical polynomial having degree m and satisfies the recursive formula as follows, $c^{\alpha}_0 = 1$ and $c^{\alpha}_1 = x$, [17],

$$c^{\alpha}_{m+1}(x) = \frac{2(m+\alpha)xc^{\alpha}_m(x) - mc^{\alpha}_{m-1}(x)}{m+2\alpha}, \qquad m = 1,2,3,\cdots.$$

3 Convergence of ultraspherical wavelets

Theorem 1. *The bounded and continuous function $y(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ defined on $[0,1]\times[0,1]$, then the ultraspherical wavelets approximation of $y(x,t)$ converges uniformly to it.*

Proof. Since $y(x,t)$ is bounded, $\exists\, \mu \in \mathbb{R} \ni |y(x,t)| \le \lambda$. The approximation is as follows,

$$y(x,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m}\psi^{\alpha}_{n,m}(x)\psi^{\alpha}_{n,m}(t).$$

The ultraspherical wavelet coefficients of $y(x,t)$ can be defined as

$$a_{n,m} = \int_0^1\int_0^1 y(x,t)\psi^{\alpha}_{n,m}(x)\psi^{\alpha}_{n,m}(t)dxdt,$$

$$a_{n,m} = 2^{\frac{k+1}{2}}\mu_{m,\alpha}\int_0^1\int_I y(x,t)c^{\alpha}_m(2^{k+1}x-2n-1)dx\psi^{\alpha}_{n,m}(t)dt,$$

where, $\mu_{m,\alpha} = 2^{\alpha}\Gamma(\alpha)\sqrt{\frac{m!(m+\alpha)}{2\pi\Gamma(m+2\alpha)}}$, and $I = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$, put $2^{k+1}x - 2n - 1 = e$, we get

$$a_{n,m} = 2^{\frac{k+1}{2}}\mu_{m,\alpha}\int_0^1\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} y\left(\frac{e+2n+1}{2^{k+1}},t\right)c^{\alpha}_m(e)\frac{de}{2^{k+1}}\psi^{\alpha}_{n,m}(t)dt,$$

$$a_{n,m} = 2^{-\frac{(k+1)}{2}}\mu_{m,\alpha}\int_0^1\left[\int_{-1}^1 y\left(\frac{e+2n+1}{2^{k+1}},t\right)c^{\alpha}_m(e)de\right]\psi^{\alpha}_{n,m}(t)dt.$$

By using the generalized mean value theorem for integrals, we get

$$a_{n,m} = 2^{-\frac{(k+1)}{2}}\mu_{m,\alpha}\int_0^1 y\left(\frac{\xi+2n+1}{2^{k+1}},t\right)\psi^{\alpha}_{n,m}(t)dt\left[\int_{-1}^1 c^{\alpha}_m(e)de\right].$$

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Here, $\xi \in (-1,1)$ and c_m^α is an integrable function on $(-1,1)$. Therefore $\int_{-1}^{1} c_m^\alpha(e)de = \Lambda$.

$$a_{n,m} = \Lambda \mu_{m,\alpha}^2 \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} y\left(\frac{\xi+2n+1}{2^{k+1}}, t\right) c_m^\alpha(2^{k+1}t - 2n - 1).$$

Put $2^{k+1}t - 2n - 1 = f$, we get

$$a_{n,m} = \Lambda \mu_{m,\alpha}^2 \int_{-1}^{1} y\left(\frac{\xi+2n+1}{2^{k+1}}, \frac{f+2n+1}{2^{k+1}}\right) c_m^\alpha(f)\frac{df}{2^{k+1}},$$

$$a_{n,m} = \Lambda \mu_{m,\alpha}^2 2^{-(k+1)} \int_{-1}^{1} y\left(\frac{\xi+2n+1}{2^{k+1}}, \frac{f+2n+1}{2^{k+1}}\right) c_m^\alpha(f)df.$$

Since c_m^α is integrable on $(-1,1)$, therefore, $\int_{-1}^{1} c_m^\alpha(f)df = \Lambda_1$, and by generalized mean value theorem for integrals, we get

$$a_{n,m} = \Lambda\Lambda_1 \mu_{m,\alpha}^2 2^{-(k+1)} y\left(\frac{\xi+2n+1}{2^{k+1}}, \frac{\eta+2n+1}{2^{k+1}}\right), \text{where}, \eta \in (-1,1).$$

Since y is bounded by λ,

$$|a_{n,m}| = \left|\frac{\Lambda\Lambda_1\lambda}{2^{k+1}}\right| \left|2^{2\alpha}\Gamma(\alpha)\Gamma(\alpha)\frac{m!(m+\alpha)}{2\pi\Gamma(m+2\alpha)}\right|.$$

Therefore, $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m}$ is convergent. Thus, the ultraspherical wavelet expansion of $y(x,t)$ uniformly converges.

4 Description of ultraspherical wavelets scheme

The generalized BBM equation of the form

$$\alpha y_t(x,t) + \beta y_x(x,t) + \chi y(x,t)y_x(x,t) - \delta y_{xxt}(x,t) = \mu(x,t), \tag{3}$$

with initial-boundary conditions,

$$y(x,0) = f(x), \qquad 0 \le x \le 1,$$

and

$$y(0,t) = g_0(t), \qquad y(1,t) = g_1(t), \qquad t \ge 0,$$

where $\alpha, \beta, \delta, \chi$ are constants, and $f(x), g_0(t), g_1(t), \mu(x,t)$ are continuous real functions.
Consider,

$$y_{xxt}(x,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{n,m}\psi_{n,m}(x), \tag{4}$$

truncating the (4), we get

$$y_{xxt}(x,t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{n,m}\psi_{n,m}(x), \tag{5}$$

where ultraspherical coefficients $a_{n,m}$ have to be determined. Integrate (5) with respective t from 0 to t.

$$y_{xx}(x,t) = y_{xx}(x,0) + t\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{n,m}\psi_{n,m}(x). \tag{6}$$

Now integrate (6) concerning x from 0 to x.

$$y_x(x,t) = y_x(x,0) + y_x(x,0) - y_x(0,0) + \int_0^x t \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \psi_{n,m}(x). \tag{7}$$

Now integrate (7) concerning x from 0 to x.

$$y(x,t) = y(0,t) + x(y_x(0,t) - y_x(0,0)) + y(x,0) - y(0,0) + t \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \int_0^x \psi_{n,m}(x). \tag{8}$$

Put $x = 1$ in (8), and by given conditions, we get

$$y_x(0,t) - y_x(0,0) = g_1(t) - g_0(t) + f(0) - f(1) - t \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \int_0^x \psi_{n,m}(x) dx dx \right) \Big|_{x=1}. \tag{9}$$

Substitute (9) in (7) and (8), we get

$$y_x(x,t) = y_x(x,0) + \left[g_1(t) - g_0(t) + f(0) - f(1) - t \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \int_0^x \psi_{n,m}(x) dx dx \right) \Big|_{x=1} \right] + t \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \psi_{n,m}(x) dx, \tag{10}$$

and

$$y(x,t) = y(0,t) + x \left[g_1(t) - g_0(t) + f(0) - f(1) - t \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \int_0^x \psi_{n,m}(x) dx dx \right) \Big|_{x=1} \right] + y(x,0) - y(0,0) + t \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \int_0^x \psi_{n,m}(x) dx dx. \tag{11}$$

Now differentiate (11) concerning t, and we get

$$y_t(x,t) = y_t(0,t) + x \left[g_1(t) - g_0(t) - \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \int_0^x \psi_{n,m}(x) dx dx \right) \Big|_{x=1} \right] + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{n,m} \int_0^x \int_0^x \psi_{n,m}(x) dx dx. \tag{12}$$

Substituting (5), (10), (11), and (12) in (3) by assuming $t = \frac{1}{M}$ and using the collocation points $x_i = \frac{2i-1}{2M}$ where $i = 1, 2, \cdots, M$ the obtained equation (3) can be converted into a system of algebraic equations. The system is solved by Newton's iterative method. We get the ultraspherical coefficients $a_{n,m}$, then subsitute these $a'_{n,m}s$ in (11) will contribute to the ultraspherical wavelets-based approximate solution of a chosen equation. $AE = |y_e(x,t) - y_a(x,t)|$, where $y_a(x,t)$ and $y_e(x,t)$ approximate and exact solutions will determine the absolute error (AE).

5 Numerical experiments

We adapt the UWM mentioned in section 4 to various BBM equations in this part.

Problem 1. Consider the following BBM equation [14]

$$y_t(x,t) - 2y_{xxt}(x,t) + y_x(x,t) = 0, \tag{13}$$

with initial-boundary conditions

$$y(x,0) = e^{-x}, \quad 0 \le x \le 1, \tag{14}$$

and

$$y(0,t) = e^{-t},\ y(1,t) = e^{-1-t}, \quad t \ge 0. \tag{15}$$

The exact solution is $y(x,t) = e^{-x-t}$. Method of implementation at $k = 1, M = 6$, and $\alpha = \frac{1}{2}$. Consider

$$y_{xxt}(x,t) \approx \sum_{m=0}^{5} a_{1,m}\psi_{1,m}(x). \tag{16}$$

Using (14), (15), and the procedure explained in section 4, we get

$$y_x(x,t) = \left[e^{-1-t} - e^{-t} - e^{-1} + 1 - t\left(\sum_{m=0}^{5}\int_0^x\int_0^x a_{1,m}\psi_{1,m}(x)dxdx\right)\Big|_{x=1}\right] - e^{-x} + t\sum_{m=0}^{5}\int_0^x a_{1,m}\psi_{1,m}(x)dx, \tag{17}$$

$$y(x,t) = e^{-t} + e^{-x} - 1 + x\left[e^{-1-t} + e^{-t} - e^{-1} + 1 - t\left(\sum_{m=0}^{5}\int_0^x\int_0^x a_{1,m}\psi_{1,m}(x)dxdx\right)\Big|_{x=1}\right] + t\sum_{m=0}^{5}\int_0^x\int_0^x a_{1,m}\psi_{1,m}(x)dxdx, \tag{18}$$

$$y_t(x,t) = -e^{-t} + x\left[-e^{-1-t} + e^{-t} - \left(\sum_{m=0}^{5}\int_0^x\int_0^x a_{1,m}\psi_{1,m}(x)dxdx\right)\Big|_{x=1}\right] + \sum_{m=0}^{5}\int_0^x\int_0^x a_{1,m}\psi_{1,m}(x)dxdx. \tag{19}$$

Substituting (17), (18), and (19) into (13) by taking $t = \frac{1}{6}$. Then collocate the obtained equation using the discrete points $x_i = \frac{2i-1}{12}$ where $i = 1,2,\cdots,6$, we obtain a system of six equations. Solving this system, we get the ultraspherical wavelets coefficients $a_{1,m}$'s. That is $[-0.2850, 0.0717, -0.0060, 2.9699 \times 10^{-4}, -1.1741 \times 10^{-5}, -3.4727 \times 10^{-8}]$. Substituting these $a_{1,m}$'s in (18) will contribute to the ultraspherical wavelets-based estimated solution of a given equation (13) and compared with the exact solution for different values of M and t is represented in Table 1 ($\alpha = \frac{1}{2}$). The present method solution is compared with the finite difference method (FDM) [14], Haar wavelet method (HWM) [14], and exact solutions are represented in Table 2. Fig. 1 shows a graphical comparison of UWM, FDM, and HWM with exact solutions. Fig. 2 shows a two-dimensional graphical judgment of FDM, UWM, and HWM with exact solutions.

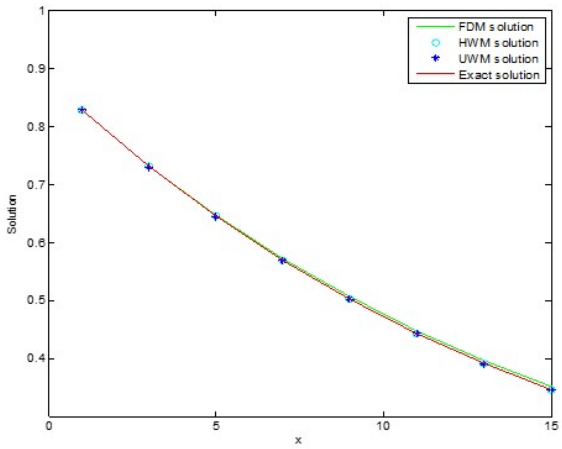


Fig. 1 Comparison of the projected scheme solution with other solutions for Problem 1 at $t = \frac{1}{9}$.

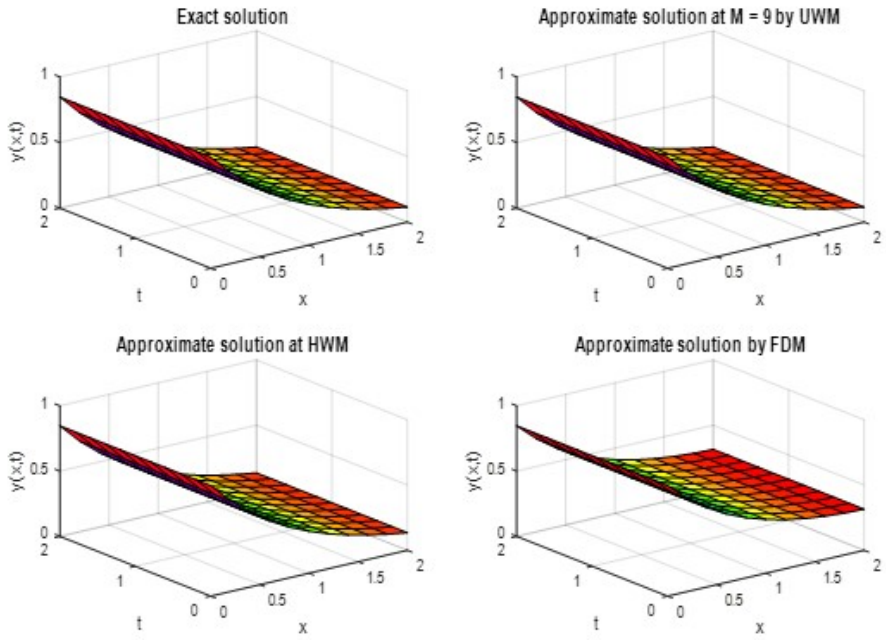


Fig. 2 Graphical demonstration of the solution of present and other methods for Problem 1.

Problem 2. Consider the following non-homogeneous BBM equation [14]

$$y_t(x,t) - 2y_{xxt}(x,t) + e^{x+t} = 0, \tag{20}$$

with initial-boundary conditions

$$y(x,0) = e^x, \quad 0 \le x \le 1, \tag{21}$$

and

$$y(0,t) = e^t,\ y(1,t) = e^{1+t}, \quad t \ge 0. \tag{22}$$

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Table 1 AE of the present scheme for Problem 1.

x	AE at $M=6$	AE at $M=7$	AE at $M=8$	AE at $M=9$
0.1	3.9376×10^{-4}	2.9425×10^{-4}	2.2818×10^{-4}	1.8209×10^{-4}
0.2	6.7769×10^{-4}	5.0624×10^{-4}	3.9246×10^{-4}	3.1312×10^{-4}
0.3	8.6162×10^{-4}	6.4339×10^{-4}	4.9864×10^{-4}	3.9775×10^{-4}
0.4	9.5446×10^{-4}	7.1244×10^{-4}	5.5200×10^{-4}	4.4021×10^{-4}
0.5	9.6427×10^{-4}	7.1949×10^{-4}	5.5729×10^{-4}	4.4433×10^{-4}
0.6	8.9835×10^{-4}	6.7003×10^{-4}	5.1883×10^{-4}	4.1357×10^{-4}
0.7	7.6329×10^{-4}	5.6907×10^{-4}	4.4052×10^{-4}	3.5106×10^{-4}
0.8	5.6505×10^{-4}	4.2109×10^{-4}	3.2587×10^{-4}	2.5963×10^{-4}
0.9	3.0900×10^{-4}	2.3018×10^{-4}	1.7807×10^{-4}	1.4184×10^{-4}

Table 2 Comparison of the projected scheme with other techniques for $N=M=8$ of Problem 1.

$x=(/16)$	FDM [14]	HWM [14]	UWM	Exact
1	0.82942254	0.82957675	0.82887868	0.82902911
3	0.73277957	0.73244494	0.73124041	0.73161562
5	0.64756878	0.64621771	0.64514042	0.64564852
7	0.57245382	0.56995069	0.56922344	0.56978282
9	0.50625604	0.50267060	0.50229344	0.50283157
11	0.44793598	0.44342497	0.44329494	0.44374731
13	0.39657711	0.39130681	0.39129648	0.39160562
15	0.35137142	0.34546659	0.34547605	0.34559075

The exact solution is $y(x,t)=e^{x+t}$. Solving this problem using the procedure discussed in section 4, the obtained solution is compared with HWM, FDM, and exact solutions, represented in Table 4. Table 3 illustrates an AE obtained by the estimated solution of the projected method with the exact solution for different M and t. Fig. 3 shows a graphical comparison of FDM, UWM, and HWM with exact solutions. Fig. 4 exhibits a geometrical interpretation of HWM, UWM, and FDM with exact solutions.

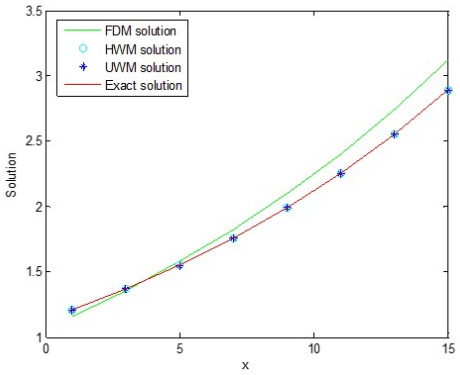


Fig. 3 Comparison of the projected scheme solution with other solutions for Problem 2 at $t=\frac{1}{6}$.

Table 3 AE of the present scheme for Problem 2.

x	AE at $M = 6$	AE at $M = 7$	AE at $M = 8$	AE at $M = 9$
0.1	1.0354×10^{-3}	7.4856×10^{-4}	5.6626×10^{-4}	4.4325×10^{-4}
0.2	1.8990×10^{-3}	1.3729×10^{-3}	1.0386×10^{-3}	8.1295×10^{-4}
0.3	2.5727×10^{-3}	1.8600×10^{-3}	1.4070×10^{-3}	1.1014×10^{-3}
0.4	3.0365×10^{-3}	2.1953×10^{-3}	1.6607×10^{-3}	1.2999×10^{-3}
0.5	3.2684×10^{-3}	2.3630×10^{-3}	1.7875×10^{-3}	1.3992×10^{-3}
0.6	3.2441×10^{-3}	2.3454×10^{-3}	1.7742×10^{-3}	1.3888×10^{-3}
0.7	2.9364×10^{-3}	2.1230×10^{-3}	1.6060×10^{-3}	1.2571×10^{-3}
0.8	2.3157×10^{-3}	1.6742×10^{-3}	1.2665×10^{-3}	9.9137×10^{-4}
0.9	1.3490×10^{-3}	9.7533×10^{-4}	7.3780×10^{-4}	5.7753×10^{-4}

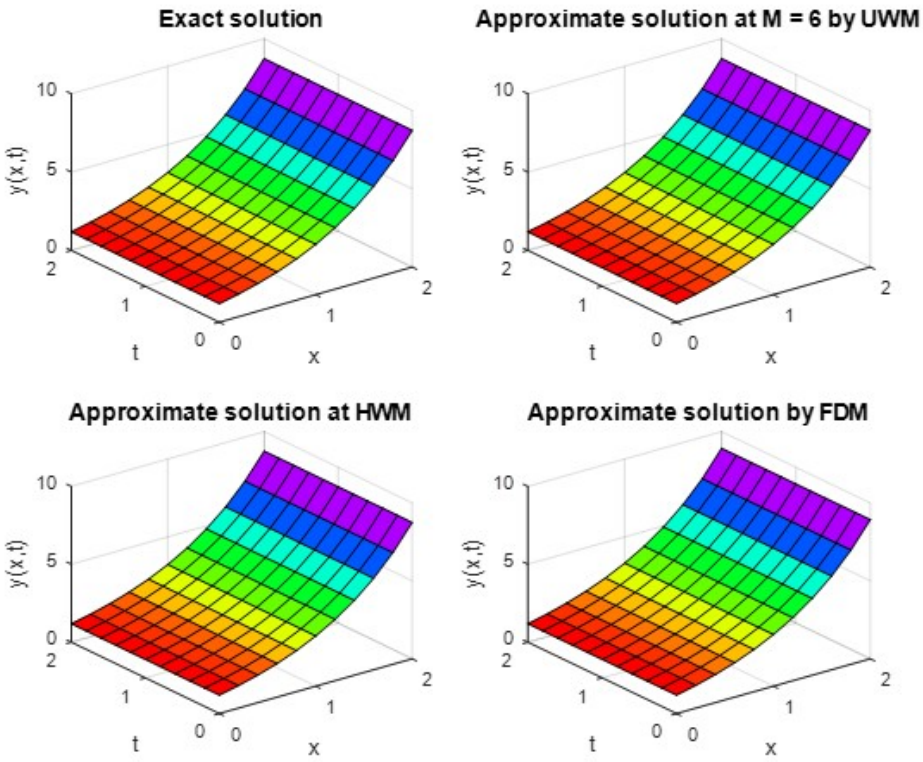


Fig. 4 Graphical demonstration of the solution of present and other methods for Problem 2.

Table 4 Comparison of the projected scheme with other techniques for $N = M = 8$ of Problem 2.

$x = (/16)$	FDM [14]	HWM [14]	UWM	Exact
1	1.15741935	1.20750077	1.20586582	1.20623024
3	1.35750043	1.36926498	1.3658529	1.36683794
5	1.57881432	1.55109039	1.5473850	1.54883029
7	1.82434828	1.75639170	1.7533308	1.75505465
9	2.09744632	1.98892088	1.9869409	1.98873746
11	2.40185758	2.25280954	2.2518987	2.25353478
13	2.74179096	2.55257492	2.5523783	2.55358945
15	3.12197659	2.89308458	2.8931091	2.89359594

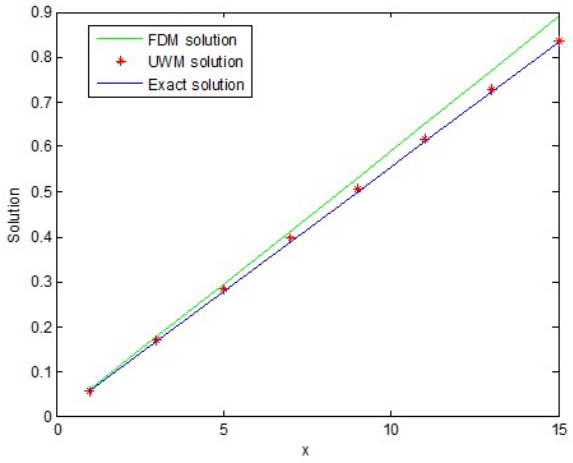


Fig. 5 Comparison of the projected scheme solution with other solutions for Problem 3 at $t = \frac{1}{6}$.

Problem 3. Consider the following nonlinear BBM equation [14]

$$y_t(x,t) - y_{xxt}(x,t) + y(x,t)y_x(x,t) = 0, \tag{23}$$

with initial-boundary conditions

$$y(x,0) = x, \quad 0 \le x \le 1, \tag{24}$$

and

$$y(0,t) = 0,\ y(1,t) = \frac{1}{1+t}, \quad t \ge 0. \tag{25}$$

The exact solution is $y(x,t) = \frac{x}{1+t}$. Solving this problem using the procedure discussed in section 4, the obtained solution is compared with FDM and exact solutions, represented in Table 6. Table 5 represents an AE obtained by the approximate solution of the projected method with the exact solution for different M and t. Fig. 5 displays a graphical comparison of UWM and FDM with exact solutions. Fig. 6 exhibits a geometrical presentation of HWM, UWM, and FDM with exact solutions.

Table 5 AE of the present scheme for Problem 3.

x	AE at $M = 6$	AE at $M = 7$	AE at $M = 8$	AE at $M = 9$
0.1	2.1089×10^{-3}	1.8448×10^{-3}	1.6396×10^{-3}	1.5825×10^{-3}
0.2	4.0983×10^{-3}	3.5849×10^{-3}	3.1859×10^{-3}	2.9895×10^{-3}
0.3	5.8473×10^{-3}	5.1141×10^{-3}	4.5446×10^{-3}	3.1664×10^{-3}
0.4	7.2314×10^{-3}	6.3236×10^{-3}	5.6186×10^{-3}	4.8969×10^{-3}
0.5	8.1208×10^{-3}	7.0998×10^{-3}	6.3072×10^{-3}	5.9892×10^{-3}
0.6	8.3788×10^{-3}	7.3234×10^{-3}	6.5045×10^{-3}	6.3058×10^{-3}
0.7	7.8595×10^{-3}	6.8674×10^{-3}	6.0980×10^{-3}	5.2771×10^{-3}
0.8	6.4062×10^{-3}	5.5955×10^{-3}	4.9672×10^{-3}	3.7137×10^{-3}
0.9	3.8484×10^{-3}	3.3601×10^{-3}	2.9818×10^{-3}	1.0753×10^{-3}

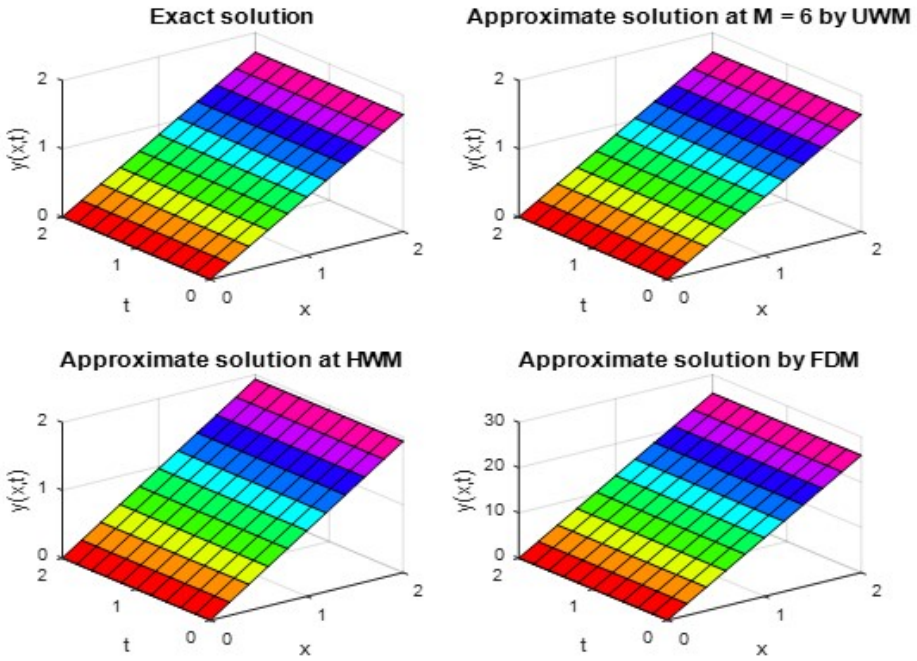


Fig. 6 Graphical demonstration of the solution of present and other methods for Problem 3.

Table 6 Comparison of the projected scheme with other techniques for $N = M = 8$ of Problem 3.

$x = (/16)$	FDM [14]	UWM	Exact
1	0.05887280	0.05658619	0.05555555
3	0.17667011	0.16966762	0.16666666
5	0.29462366	0.28247422	0.27777777
7	0.41284152	0.39481645	0.38888888
9	0.53143772	0.50649507	0.50000000
11	0.65053500	0.61729700	0.61111111
13	0.77026763	0.72699079	0.72222222
15	0.89078469	0.83532175	0.83333333

6 Conclusion

The present study used the ultraspherical wavelet method to find the numerical solution of BBM equations. The proposed approach is easily computer implementable and supported with illustrative examples. We also observed that this method is computationally efficient and gives better results. The present approach was compared with Haar wavelet and finite difference methods through tables and graphs. We have drawn 1-D and 2-D graphs of UWM, HWM, and FDM. From the figures and tables, we observed that the present technique has more accuracy than HWM and FDM, which are available in literature [14]. By raising the M, the precision of the solution is improved. As a result, the suggested approach is highly efficient and straightforward for solving linear and nonlinear BBM equations. Additionally, this approach can be extended to higher-order equations by making minor adjustments to the suggested procedure.

7 Declarations

7.1 Conflict of interest:

The authors state that they do not have any conflict of interests.

7.2 Author's contributions:

M.M.-Conceptualization, Methodology, Formal analysis, Writing-Review and Editing, Supervision. K.S.-Resources, Writing-Original Draft, Methodology, Validation. All authors read and approved the final submitted version of this manuscript.

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7.5 Data availability statement:

All data that support the findings of this study are included within the article.

7.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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