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Original Study

Solving coupled non-linear higher order BVPs using improved shooting method

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Abstract

The purpose of this work is to propose a new efficient initial guess algorithm rather than the conventional Newton method to meet the adjoint terminal conditions, rapidly. Thus, the focus of this paper is to use better algorithms for obtaining refined initial guesses with shooting method to solve the coupled boundary value problems. The boundary value problem is formulated as a system of equations i.e. initial value problems with one unknown initial condition. The efficiency and accuracy of the Shooting method is enhanced by improving the initial guess and then solving the problem iteratively.

Keywords: Shooting method, coupled non-linear BVPs, RK-4 method.
AMS 2020 codes: 35G20; 35G25; 35G30; 34A34; 34B15.

1 Introduction

Boundary value problems (BVPs) are highly significant because of their wide applications in many disciplines, such as applied mathematics, which covers boundary layer theory in fluid mechanics, theoretical physics, engineering, control, and optimization theory. Since analytical solutions to BVPs are not always attainable, efficient and precise numerical techniques must be used to solve such boundary value problem (BVP). The shooting method is one of the most commonly utilised procedures for the aforementioned objective. The shooting technique is based on turning a BVP into a system of initial value problems (IVPs) [1, 2]. As a result, the precision and efficiency of an initial value problem (IVP) solver may be used to solve a BVP. Numerous researchers have extensively explored shooting techniques in order to get trustworthy solutions for non-linear BVPs [3–10].

Jones in [11] explored the shooting technique for obtaining eigenvalues of fourth-order BVPs. Wang et al. investigated second order multi-point integral BVPs [12]. Wang et al. applied the shooting method for solving Stieltjes integral BVPs [13]. Kwong and Wong discussed the shooting technique for solving non-homogeneous multipoint BVPs of second-order ordinary differential equations (ODEs) [14]. Granas et al. proposed the shooting technique for a class of nonlinear BVPs [15]. Russell and Shampine investigated several numerical approaches for solving single BVPs [16, 17].

The shooting method is based on converting a BVP into an equivalent system of IVPs. In order to obtain

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accurate results of the reformulated system of IVPs, an appropriate initial guess should be made to initiate a recursive procedure. Initial guesses are usually acquired by using iterative methods for finding roots of algebraic equations, such as the Newton method, the secant method and interpolation formulae, etc.

This paper proposes new ways to obtain efficient initial guesses of higher order methods which work far better than conventional methods and yield faster convergence. In this study, a family of iterative initial approximation algorithms is suggested and its use to modify initial guesses in order to swiftly reach the adjoint terminal condition is demonstrated. These early approximation methods can be used to obtain rapid convergence. A BVP is recast into an associated system of IVPs, which are then solved by refining an initial guess. The Newton-Raphson formula is commonly employed in many softwares to solve a BVP, although the formula fails to predict results when the first derivative of a function is 0 or undefined. To solve the above mentioned problems, a collection of higher order iterative techniques is employed to efficiently estimate the initial guess for solving non-linear BVPs. The innovation of this work is that it introduces and applies a family of iterative initial approximation algorithms utilising the shooting technique rather than the traditional Newton approach.

The rest of this paper is arranged as follows: Section 2 shows new algorithms for refining initial estimates. Section 3 presents the suggested shooting technique. Section 4 comprises the implementation of existing problems with results from certain non-linear BVPs, and Section 5 summarises the paper's conclusions and future directions.

2 Improved shooting method

Consider a non-linear second-order BVP,

$$y'' = f(x,y,y'), \quad \text{for } a \le x \le b \text{ where } y(a) = \alpha \text{ and } y(b) = \beta. \tag{1}$$

The BVP can be rewritten as

$$y' = z, \quad z' = f(x,y,z), \qquad y(a) = \alpha \text{ and } y(b) = \beta. \tag{2}$$

Then, BVP is transformed to IVP by replacing the boundary condition (BC) at $x = b$ as

$$y' = z, \qquad z' = f(x,y,z), \qquad y(a) = \alpha \text{ and } y'(a) = t. \tag{3}$$

Afterwards, the parameter $t = t_k$ is selected in such a way that it satisfies

$$\lim_{k\to\infty} y(b,t_k) = y(b) = \beta,$$

where $y(x,t_k)$ represents the IVP solution being Eq.(3) considering $t = t_k$ and $y(x)$ represents the BVP solution, given by Eq.(1).

Firstly, start hitting the target initially with a parameter $t = t_0$ if $y(b,t_0)$ is not sufficiently close to β, then another elevation i.e. t_1 is chosen until $y(b,t_0)$ is sufficiently close to hitting the target β, which favourably satisfies the level of tolerance chosen and the discretization scheme used in the time stepping (c.f. Figure 1). Generally, the Newton method, the Bisection or Secant methods are being used to choose initial guesses for solving BVP.

The IVP given by Eq.(3), emphasizes that the solution is depending on both variables such as x and t, as

$$y''(x,t) = f(x,y(x,t),y'(x,t)), \qquad \text{for} \quad a \le x \le b, \tag{4}$$

where $y(a,t) = \alpha$ and $y'(a,t) = t$. Some higher order algorithms (such as, Algorithms 2 and 3) can be used to hit the target, say β, efficiently. For this, one has to calculate $\frac{\partial y}{\partial t}(b,t_k)$.

Differentiating Eq.(4) w.r.t t yields as follows:

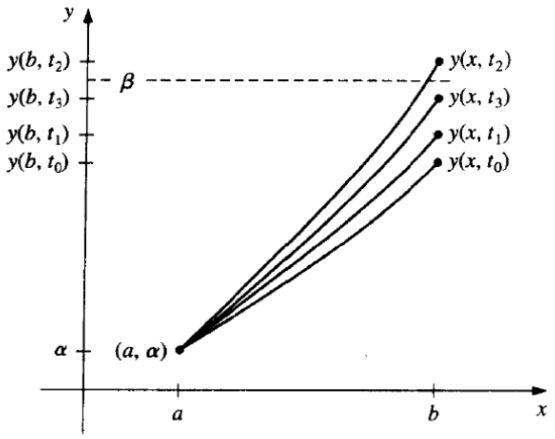

Fig. 1 Shooting Method

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial t}(x,y(x,t),y'(x,t))$$
$$= \frac{\partial f}{\partial x}(x,y(x,t),y'(x,t))\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x,y(x,t),y'(x,t))\frac{\partial y}{\partial t}(x,t) + \frac{\partial f}{\partial y'}(x,y(x,t),y'(x,t))\frac{\partial y'}{\partial t}(x,t). \tag{5}$$

Eq.(5) is then transformed to

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}(x,y(x,t),y'(x,t))\frac{\partial y}{\partial t}(x,t) + \frac{\partial f}{\partial y'}(x,y(x,t),y'(x,t))\frac{\partial y'}{\partial t}(x,t), \tag{6}$$

for $a \leq x \leq b$, with the following initial conditions

$$\frac{\partial y}{\partial t}(a,t) = 0 \text{ and } \frac{\partial y'}{\partial t}(a,t) = 1.$$

Denoting $\frac{\partial y}{\partial t}(x,t) = z(x,t)$. Eq.(6) finally takes the form

$$z''(x,t) = f_y(x,y,y')z(x,t) + f_{y'}(x,y,y')z'(x,t), \text{ for } a \leq x \leq b, \tag{7}$$

where $z(a,t) = 0$ and $z'(a,t) = 1$. Now, two IVPs (c.f. Eqs. (3) and (7)) are required to solve for each iteration [18]. From Eq.(7), the value of $z(b,t_k)$ will be obtained and used in the next subsection to find the sequence of t_k for quickly reaching the right BC i.e. $y(b) = \beta$.

3 Iterative algorithms to refine initial guess

In recent years, several higher order iterative techniques have been presented in literature to solve non-linear equations [19–21]. These developed algorithms are based on several methods such as the Taylor series, the decomposition method, the variational iteration technique, the homotopy perturbation method, and the quadrature formula. Various researchers used these techniques to develop several one and two step iterative methods to achieve higher order convergence [22].

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This study proposes the use of two or three step methods instead of applying Newton and secant methods. The suggested algorithms proved fast convergence as compared to conventional methods. In the next section, the applied algorithms are presented and the Newton method is used to compare the results of BVPs.

3.1 Algorithm 1: Newton method

One of the most used methods for obtaining the initial guess is the Newton method which converges quadratically [23].
For a given t_0, the rest values of t_k can be calculated with the following formulas:

$$t_{k+1} = t_k - \frac{y(b,t_k)-\beta}{z(b,t_k)}. \tag{8}$$

3.2 Algorithm 2: The fifth order convergent algorithm

Algorithm 2 is a fifth order convergent algorithm [24]. For each iteration this algorithm requires two evaluations of the function $y(b,s_k)$ and $y(b,t_k)$ and two evaluations of its first derivative $z(b,t_k)$ and $z(b,t_{k^*})$. In this manner the effectiveness of the proposed algorithm can be improved and is in good comparison with Newton method.
For a given t_0, the rest values of t_k can be calculated with the following formula:

$$t_{k+1} = s_k - \frac{z(b,t_{k^*})[y(b,s_k)-\beta]}{[3z(b,t_{k^*})-2z(b,t_k)]z(b,t_k)}, \tag{9}$$

where $s_k = t_k - \frac{y(b,t_k)-\beta}{z(b,t_{k^*})}$, $r_k = t_k - \frac{y(b,t_k)-\beta}{z(b,t_k)}$ and $t_{k^*} = \frac{1}{2}(t_k + r_k)$.

3.3 Algorithm 3: The sixth-order convergent method

The sixth-order convergent method [25] requires three evaluations of the function $y(b,s_k)$, $y(b,r_k)$ and $y(b,t_k)$ and one evaluation of its first derivative $z(b,t_k)$ is also applied to refine the initial guess. For a given t_0, the rest values of t_k can be calculated with the following formulas:

$$t_{k+1} = s_k - (1+2\mu+\mu^2+\mu^3)\frac{y(b,s_k)-\beta}{z(b,t_k)}, \tag{10}$$

where $s_k = r_k - \frac{[y(b,t_k)-\beta][y(b,r_k)-\beta]}{[[y(b,t_k)-\beta]-2[y(b,r_k)-\beta]]z(b,t_k)}$, $r_k = t_k - \frac{y(b,t_k)-\beta}{z(b,t_k)}$ and $\mu = \frac{y(b,r_k)-\beta}{y(b,t_k)-\beta}$.

The error and convergence analysis of Algorithm 2 and Algorithm 3 are presented in literature [24, 25].

3.4 Algorithm 3: Runge Kutta-4 method

The RK method of order 4 is a well known method to solve IVPs. The formula of RK-4 is given as [18]:

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{6}(k_0 + 2k_1 + 2k_2 + k_3), \\ x_{n+1} &= x_n + h, \end{aligned} \tag{11}$$

where

$$\begin{aligned} k_0 &= f(x_n, y_n), \\ k_1 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_0), \\ k_2 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\ k_3 &= f(x_n + h, y_n + hk_2), \quad \text{for } n = 0,1,2,3,\cdots. \end{aligned}$$

The proposed method is then applied to solve applications such as non-linear and coupled BVPs.

4 Applications of the improved shooting method

4.1 Test problem-I

In order to solve the non-linear BVP, RK-4 method is applied (c.f. Section 3.4) to solve the resulting system of IVPs with proposed algorithms for updating the initial guess.
Here, a problem in which the internal heat generation effects on flow and heat transfer effects in a thin liquid film considering stretching sheet is studied. The equations of the problem were transformed to non-linear BVP by similarity transformation [26].
The BVP is governed by

$$f'''(\eta)+\gamma(f(\eta)f''(\eta)-(f'(\eta))^2-S_1f'(\eta))=0 \tag{12}$$
$$\frac{1}{Pr}\theta''(\eta)+\gamma(f(\eta)\theta'(\eta)-r_1f'(\eta)\theta(\eta)-\frac{1}{2}S_1\eta\theta'(\eta)-r_2S_1\theta(\eta)+\frac{1}{Pr}B^*\theta(\eta))=0,$$

with BCs

$$f(0)=0,\quad f'(0)=1,\quad \theta(0)=1,$$
$$f(1)=\frac{1}{2}S_1,\quad f''(1)=0,\quad \theta'(1)=0,$$

where S_1 is the measure of unsteadiness. The Prandtl number is denoted by Pr. The dimensionless parameters of film thickness are denoted by γ, r_1 and r_2. Moreover, B^* represents the temperature-dependent parameter.
Numerical solutions of Eq.(12) are obtained by using our proposed improved shooting method to meet the asymptotic BCs employing RK-4.

The numerical solution of Eq.(12) is obtained by first converting the BVP to corresponding IVP using the proposed improved shooting method for the satisfaction of asymptotic BCs.

After selecting appropriate values for $f''(0)=t$ and $\theta'(0)=s$ (some initial guess using initial guess algorithms e.g Bisection, Secant, Newton method etc) which satisfies the condition $f(1)=\frac{1}{2}S_1$, $f''(1)=0$ and $\theta'(1)=0$, the resulting IVP is integrated using RK-4. The best possible initial guess leads to fast convergence to the adjoining boundary conditions. For this, we refined our initial guess using proposed higher order Algorithms 2 and 3 in comparison to the existing technique, named as, Newton method (Algorithm 1).

The physical parameters, S_1 the measure of unsteadiness, Pr the Prandtl number, r_1 and r_2 the positive power indices, temperature-dependent parameter B^* and the film thickness γ occurring in BVP given by equation (12) are assigned some numerical values to obtain the numerical solutions of the afore discussed problem,

$$\gamma=2.151994,\quad Pr=1,\quad r_1=2,\quad r_2=3/2,\quad B^*=0,\quad S_1=0.8.$$

Based on convergence criteria, different initial guesses were made for different values of the parameters and the suitable initial guesses are taken as $t_0=-2.680943$ and $s_0=-3.591125$ which are revealed by the numerical solutions in literature [26]. Then, the efficient and accurate fifth and sixth order Algorithms 2 and 3 are implemented for obtaining initial guesses (c.f. Section 3.4) and the results are compared with second order algorithm 1 (c.f. Section 3.4). The initial guesses say t_4 and s_4 after four successive iterations are given below.

Algorithm 1 gives $t_4=-2.6809589589908$ and $s_4=-3.5959727116775$,
Algorithm 2 gives $t_4=-2.6809636175929$ and $s_4=-3.5959908500960$,
Algorithm 3 gives $t_4=-2.6809659852767$ and $s_4=-3.5959915546254$.

Using these aforementioned initial guesses, RK-4 suggested in Section 3.4 is applied to implement on the test problem. The results acquired from the chosen algorithms together with absolute percentage errors and number of iterations taken to reach the right boundary $f(1) = 0.4$, $f''(1) = 0$ and $\theta'(1) = 0$, are presented in Table 1 and Table 2. The absolute percentage error was calculated using the formula given below:

$$\text{Absolute percentage error} = |\text{Approximate solution - Exact solution}| \times 100\%. \tag{13}$$

It can be seen that the proposed Algorithm 2 and Algorithm 3 are approaching the target (right boundary) accurately and rapidly as compared to Algorithm 1 (Newton method). It is clear from the table that sixth order Algorithm 3 converged in 4 iterations for $f(1) = 0.4$, $f''(1) = 0$ and $\theta'(1) = 0$ and reached the right boundary rapidly while Algorithm 1 took 21 iterations and Algorithm 2 hit in 10 iterations. Results reveal that sixth order Algorithm 3 is more accurate and efficient. This is also quantitatively analyzed from absolute percentage errors in Table 2. Figures 2 and 3 present a comparison among Algorithms 1, 2 and 3.

Table 1 Comparison of Algorithms 1, 2 and 3 in hitting right BCs $f(1) = 0.4$, $f''(1) = 0$ and $\theta'(1) = 0$.

Algo	t	s	f	f''	θ'	No. of IT
1	-2.6809589589908	-3.5959727116775	0.400006	0.000041	0.000168	21
2	-2.6809636175929	-3.5959908500960	0.400002	0.000014	0.000013	10
3	-2.6809659852767	-3.5959915546254	0.400000	0.000000	0.000000	4

Table 2 Comparison of absolute percentage error of Algorithms 1, 2 and 3 in hitting right BCs $f(1) = 0.4$, $f''(1) = 0$ and $\theta'(1) = 0$.

Algo	Ab err (%) f	Ab err (%) f''	Ab err (%) θ'
1	$5.999999999950489 \times 10^{-4}$	0.0041000	0.0168000
2	2×10^{-4}	0.0014000	0.0013000
3	0.0000000000	0.0000000	0.0000000

The solution of coupled BVP is presented in Figures 4 and 5 using step size $h = 0.001$. It can be easily seen from figures that the results obtained by using Algorithms 2 and 3 are more accurate as compared to Algorithm 1. As Algorithm 1 and Algorithm 2 did not hit the target in fourth iteration, we repeated the procedure until the right boundary conditions were met. This makes clear that Algorithm 1 and 2 requires more computational time as compared to Algorithm 3. Therefore, Algorithm 3 is proven a more suitable algorithm to acquire initial guess in shooting method for solving BVP. On the basis of these results, one can infer the inclusion of better initial approximation algorithms with the shooting method that produces more accurate results and converges fast.

4.2 Test problem-II

In the presence of a heat source, we examined a fluid dynamics model consisting of MHD boundary layer driven convection flow along a decreasing surface with changing heat flux. The fluid flow is caused by the linear shrinkage of the sheet. The momentum and energy equations are then similarly transformed into non-linear ODEs [27, 28].
The BVP is given as

$$\begin{aligned} &f'''(\eta) + f(\eta)f''(\eta) - (f'(\eta))^2 - M^2 f'(\eta) = 0 \\ &\theta''(\eta) + Prf(\eta)\theta'(\eta) - nPrf'(\eta)\theta(\eta) + PrB\theta(\eta) = 0 \end{aligned} \tag{14}$$

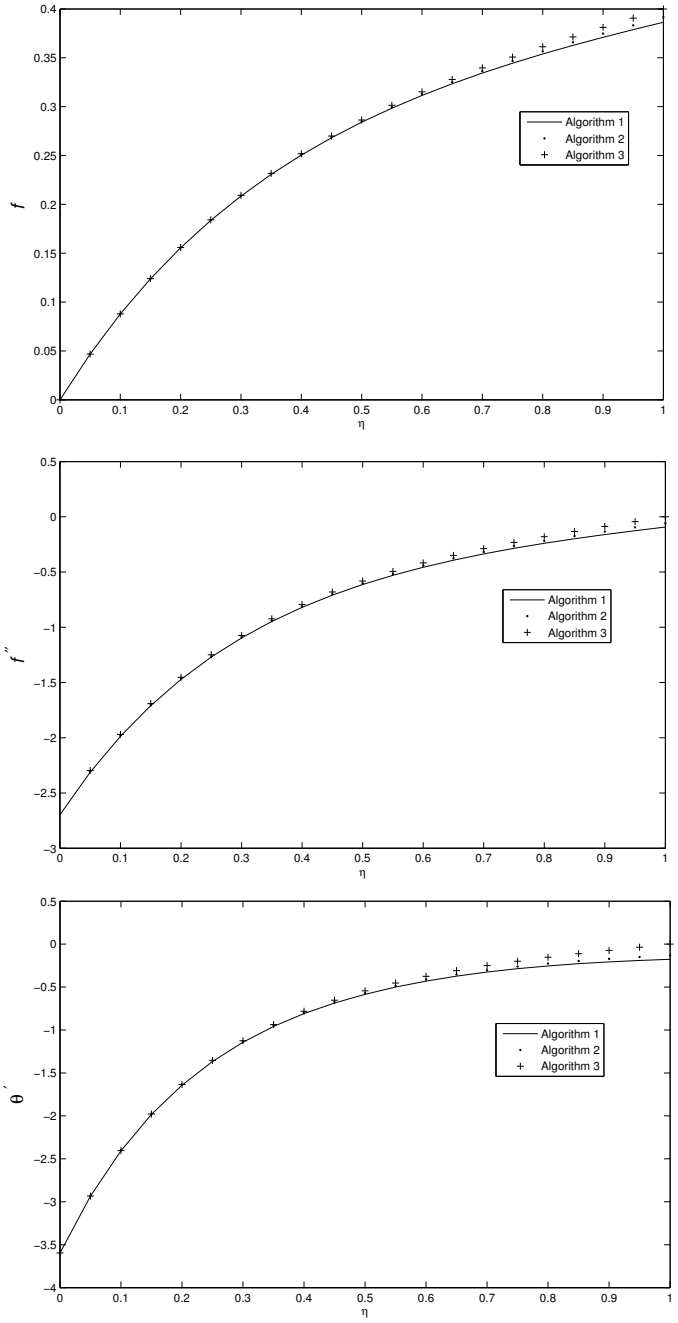

Fig. 2 Problem 1: Comparison of Algorithm 1, 2 and 3 in approaching $f(1) = 0.4$, $f''(1) = 0$ and $\theta'(1) = 0$ after 4 time recursive iterations with $h = 0.001$.

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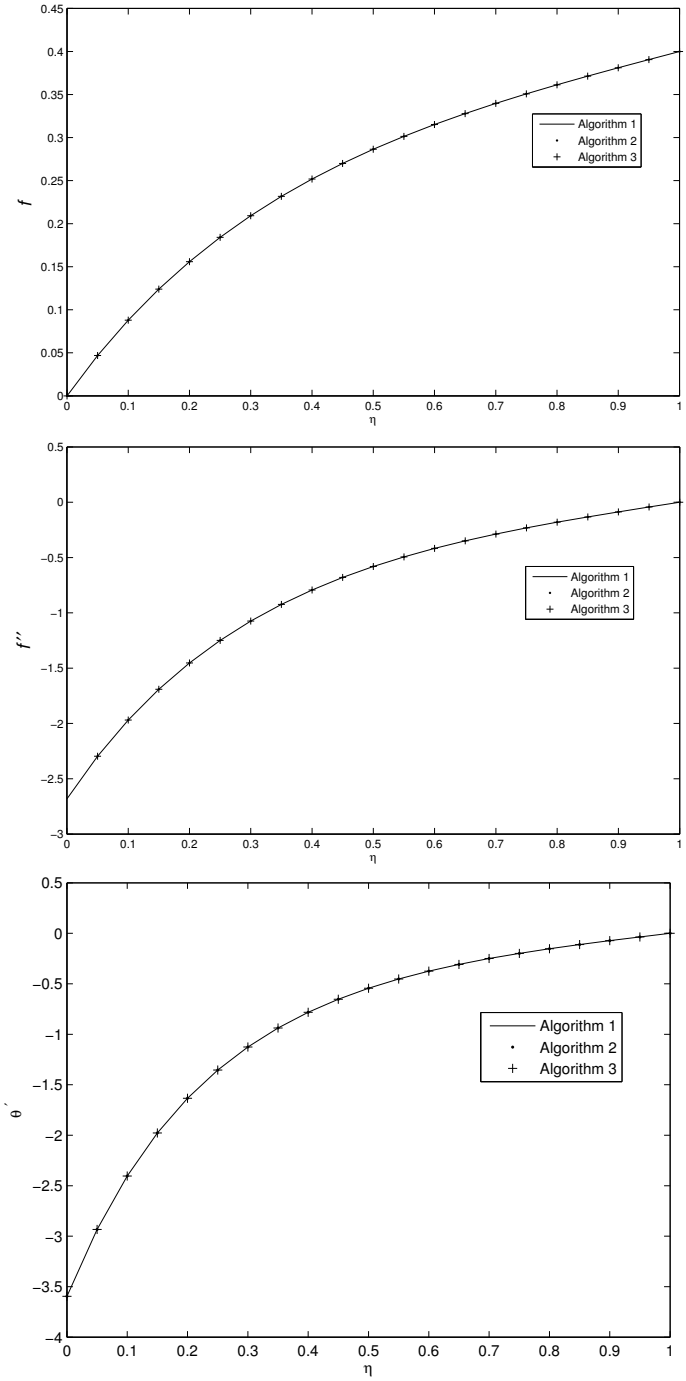

Fig. 3 Problem 1: Comparison of Algorithm 1, 2 and 3 in approaching $f(1) = 0.4$, $f''(1) = 0$ and $\theta'(1) = 0$ after 21 time recursive iterations with $h = 0.001$.

with BCs

$$f(0) = S, \quad f'(0) = \varepsilon \text{ and } \quad f'(\infty) = 0,$$
$$\theta'(0) = -1, \quad \theta(\infty) = 0,$$

where M^2 is magnetic parameter, B is heat source parameter, Pr is prandtl number, n is heat flux parameter, S is suction parameter and ε is stretching/shrinking parameter.
We obtained the numerical solution of Eq.(14) using our proposed improved shooting method to meet the asymptotic BCs employing RK-4.
We need to integrate momentum and energy equation occurring in BVP Eq.(14), for this, we need $f''(0)$ and $\theta(0)$. So, we converted the above BVP to IVP. Taking convergence criterion into account we made various initial estimates for various values of pertinent parameters namely magnetic, heat source, prandtl, heat flux, suction and stretching/shrinking. After choosing appropriate values for $f''(0) = t$ (some initial guess using initial guess algorithms e.g Bisection, Secant, Newton methods etc) which satisfies the condition $f'(\infty) = 0$ and $\theta(0) = s$ (*some initial guess*) which satisfies the condition $\theta(\infty) = 0$ and restricting the domain i.e. $f'(\infty) = 0$ arbitrarily to $f'(10) = 0$ and $\theta(\infty) = 0$ to $\theta(10) = 0$, the resulting IVP is integrated using RK-4 method. The best possible guess leads to fast convergence to the adjoining boundary conditions. For this, we refine our initial guess using proposed higher order Algorithms 2 and 3 in comparison to the Newton method.
We assign various values to the physical parameters occurring in our chosen problem namely magnetic parameter M^2, heat source parameter B, Prandtl number Pr, heat flux parameter n, suction parameter S and stretching/shrinking parameter ε to obtain the numerical values of the solution given by the follows

$$M^2 = 2, \ \ B = 0.05, \ \ Pr = 0.71, \ \ n = 2, \ \ S = 2, \ \ \varepsilon = -1.$$

Based on convergence criteria, different initial guesses were made for different values of the parameters and the suitable initial guesses are taken as $t_0 = 2.414214$ and $s_0 = 1.350447$ which are revealed by the numerical solutions in literature [27]. Then, efficient and accurate fifth and sixth order Algorithms 2 and 3 for obtaining initial guesses (c.f. Section 3.4) are applied and the results are compared with second order algorithm 1 (c.f. Section 3.4). The initial guesses say t_1 and s_1 are given below.

Algorithm 1 gives $t_1 = 2.414213562379738$ and $s_1 = 1.350433988415789$,
Algorithm 2 gives $t_1 = 2.414213558978872$ and $s_1 = 1.350423606035475$,
Algorithm 3 gives $t_1 = 2.414213562373149$ and $s_1 = 1.350401422589242$.

We observed that Algorithm 3 approached the target $f'(10) = 0$ in just one iteration. Also, after 12 iteration of shooting method, Algorithm 3 converged for $\theta(10) = 0$ and its results were compared with second order and fifth order algorithms (c.f. Section 3.4). The initial guesses say t_{12} and s_{12} are given below;

Algorithm 1 gives $t_{12} = 2.414213562373132$ and $s_{12} = 1.350383812070473$,
Algorithm 2 gives $t_{12} = 2.414213562373139$ and $s_{12} = 1.350376073781895$,
Algorithm 3 gives $t_{12} = 2.414213562373149$ and $s_{12} = 1.350375652724329$.

Using these aforementioned initial guesses, the RK-4 method suggested in Section 3.4 is applied to the mentioned test problem. The results acquired from the chosen algorithms together with absolute percentage errors and number of iterations taken to reach the right boundary $f'(10) = 0$ and $\theta(10) = 0$ are presented in Tables 3 and 4.

It is clear from the table that sixth order Algorithm 3 is converged in 1 iteration for $f'(10) = 0$ and in 12 iterations for $\theta(10) = 0$ and reached the right boundary rapidly while Algorithm 1 took 4 iterations to reach $f'(10) = 0$ and 69 iterations to reach $\theta(10) = 0$ and Algorithm 2 hit $f'(10) = 0$ in 2 iterations and $\theta(10) = 0$

in 32 iterations. The simulation time (c.f. Table 3) of applying Algorithms 2 and 3 is much less than Algorithm 1 (which is the conventional Newton method). Results reveal that sixth order Algorithm 3 is more accurate and efficient. This is also quantitatively analyzed from absolute percentage error in Table 3.

Table 3 Comparison of Algorithms 1, 2 and 3 and their absolute percentage errors and number of iterations (IT) in hitting conditions $f'(10) = 0$.

Algo	t	s	f'	IT	Abs err (%) f'
1	2.414213562379738	1.350433988415789	0.0000000059	4	5.90000000×10^{-7}
2	2.414213558978872	1.350423606035475	-0.0000030464	2	3.0464000×10^{-4}
3	2.414213562373149	1.350401422589242	0.0000000000	1	0.0000000000

Table 4 Comparison of Algorithms 1, 2 and 3 and their absolute percentage error and number of iterations in hitting conditions $\theta(10) = 0$.

Algo	t	s	θ	IT	CPU time [s]	Abs err (%) θ
1	2.414213562373132	1.350383812070473	0.0000031736	69	4.173	3.1736×10^{-4}
2	2.414213562373139	1.350376073781895	0.0000001639	32	2.039	1.6390×10^{-5}
3	2.414213562373149	1.350375652724329	0.0000000000	12	1.113	0.0000000000

The solution of coupled BVP is presented in Figures (4,5,6) by using the step size $h = 0.01$. It can be easily seen from figures that the results obtained by using Algorithms 2 and 3 are more accurate compared to the rest of algorithms. As Algorithm 1 and Algorithm 2 did not hit the target in the first iteration, we repeated the procedure until the right boundary condition was met. This makes clear that Algorithms 1 and 2 require more computational time as compared to Algorithm 3. Therefore, Algorithm 3 is proven a more suitable algorithm to acquire initial guess in shooting method for solving BVP. On the basis of these results, one can infer that the inclusion of better initial approximation algorithms with shooting method produce more accurate results and converges fast.

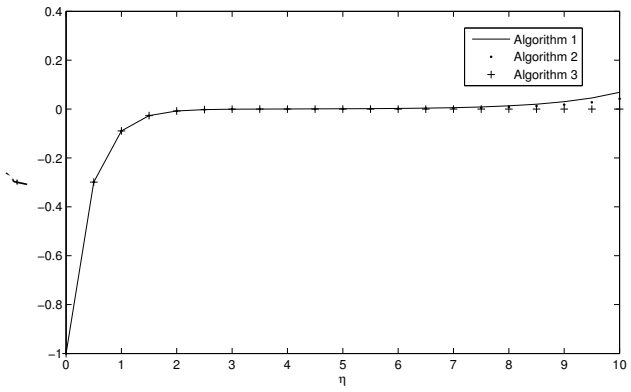

Fig. 4 Problem 2: Comparison of Algorithm 1, 2 and 3 in approaching $f'(10) = 0$ after 1 time recursive iterations with $h = 0.01$.

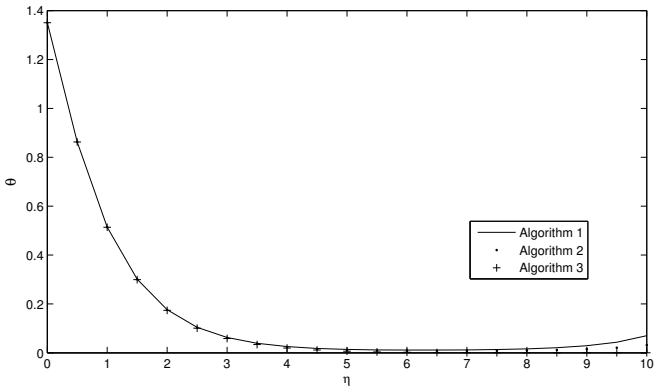

Fig. 5 Problem 2: Comparison of Algorithm 1, 2 and 3 in approaching $\theta(10) = 0$ after 12 times recursive iterations with $h = 0.01$.

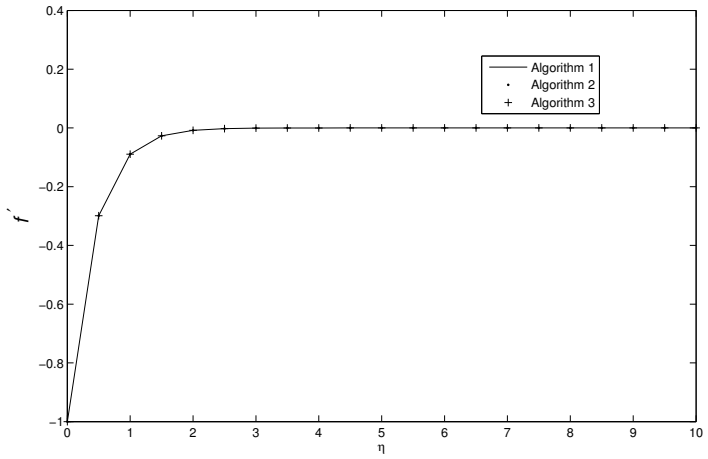

Fig. 6 Problem 2:Comparison of Algorithm 1, 2 and 3 in approaching $f'(10) = 0$ after 4 times recursive iterations with $h = 0.01$.

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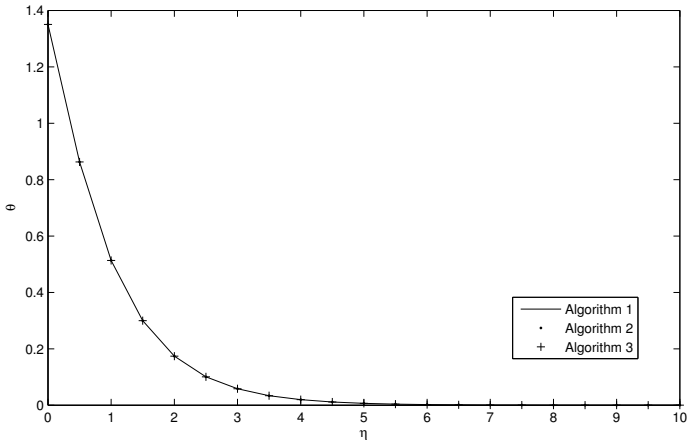

Fig. 7 Problem 2: Comparison of Algorithm 1, 2 and 3 in approaching $\theta(10) = 0$ after 69 times recursive iterations with $h = 0.01$.

5 Concluding remarks

This paper proposed the idea of using a family of higher order initial approximation algorithms to refine initial guesses for a shooting method to solve BVPs. Faster convergence was achieved and solutions approached the right end boundary rapidly by implementing the aforementioned algorithms. In each case, the given BVP was reformulated as an IVP with one unknown initial condition. We applied three different algorithms to better approximate initial guesses instead of using conventional methods, such as Newton's method. Then, the procedure of the shooting method was applied to solve the non-linear higher order coupled BVPs. On the basis of results, we concluded that the suggested Algorithms 2 and 3 accurately reached the target in almost half number of iterations as compared to Algorithm 1. Thus, the proposed shooting technique with Algorithms 2 and 3 was proven more efficient and accurate than Algorithms 1. Furthermore, the results of the proposed method were accurate as compared to the build in shooting method solvers. Generally, the build-in shooting method solvers use Bisection or Newton methods for obtaining initial guess, while, in this study, we used the higher order algorithms (5th and 6th orders) which enhance the accuracy and efficiency. Results prove that the suggested method is more efficient and accurate as compared to build-in-shooting method solvers. The current study recommends the inclusion of using better initial approximation algorithms when using shooting methods for solving BVPs. The proposed method can be implemented to solve different non-linear higher order coupled problems. Future research consists of improving shooting method for solving Non-linear singular BVPs [29].

6 Declarations

6.1 Conflict of interest:

According to the authors of this paper, there are no conflicts of interest to report regarding the article that is being presented.

6.2 Author's contributions:

S.J.-Conceptualization, Methodology, Formal analysis, Writing-Review and Editing, Supervision. E.H.-Resources, Writing-Original Draft, Methodology, Validation. The paper has been submitted with the knowledge and consent of all authors.

6.3 Funding:

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6.5 Data availability statement:

All data that support the findings of this study are included within the article.

6.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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