



Rapid Communication

Fuzzy fractional factors in fuzzy graphs-II

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Communicated by Oscar Castillo; Received: 07.09.2023; Accepted: 25.10.2023; Online: 10.01.2024

Abstract

The fractional factor implicates the characteristics of fractional flow in network data transmission, and it is a crucial tool for analyzing network information transfer. When there is uncertain information in the network, the corresponding network graph should be characterized by fuzzy graphs, in which the vertex membership function (MF) describes the uncertainty of sites, and the edge membership reveals the uncertainty of channels. The previous work introduced the concept of fuzzy fractional factor (FFF) on fuzzy graphs, but the correlated concepts are still open on other fuzzy graph classes. In order to overcome this defect, in this contribution, the concept of fuzzy fractional factor is extended to intuitionistic fuzzy graph, Pythagorean fuzzy graph, and picture fuzzy graph. Sign-alternating walk and increasing walk are extended to the corresponding settings, and the transformation operations are redefined in light of various situations. By means of constructive approaches, the corresponding theoretical results are further generalized in these settings, which characterize the existence of (resp. maximum) fuzzy fractional factors in different kinds of fuzzy graphs.

Keywords: Intuitionistic fuzzy graph, Pythagorean fuzzy graph, picture fuzzy graph, fuzzy fractional factor.

AMS 2020 codes: 05C72.

1 Introduction

In data storage, graphs are frequently utilized to represent structured or irregular data. In the computer network, the graph structure stores the topological characteristics of the network and is the basis for the analysis of the entire network connectivity, data flow and service flow. When the data to be represented has uncertainties, or the sites and channels of the network itself have uncertainties, then the membership functions are employed to describe such uncertainties, thereby transforming the original vertex set into a fuzzy set, and the edge set is a binary fuzzy relation set. In this way, the entire graph is termed as a fuzzy graph.

The study on fuzzy graphs is the focused topic fuzzy mathematics and graph theory in recent years, involving two fields of mathematical theory and computer algorithm. Akram et al. [1] discussed the threshold graphs involving Pythagorean fuzzy information. Yuan and Wang [2] raised a spectral rotation system with fuzzy anchor. Raut and Pal [3] studied the coloring number and perfectness of fuzzy graphs. Perumal [4] proposed a file clustering approach in terms of fuzzy association rule generation trick. Ullah et al. [5] gave the theoretical analysis on complex q -rung orthopair fuzzy competition graphs. Xue et al. [6] introduced distributed infinity

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fuzzy filtering algorithm. Josy et al. [7] researched the neighborhood connectivity index in fuzzy graph setting. Prakash et al. [8] considered lifetime prolongation in specific fuzzy graph environments.

In computer networks, the feasibility of data transmission can be modeled by network flow which is characterized by the existence of fractional flows. Formally, the fractional flow is represented by a fractional factor in a specific graph framework, and in this way, the existence of the fractional factor in a particular setting can be used to describe the characteristics of data transmission in the network.

In a real scenario, realistic factors such as bandwidth, throughput, uplink and downlink capacity, delay, and model capacity need to be considered, resulting in a large number of uncertain qualities in the network. These uncertainties can be characterized by the uncertainty of the sites and the uncertainty of the channels, which are stated as follows.

- **Site uncertainty:** Each site has a different storage and model capacity, resulting in differences in computing ability and throughput. Hence, it inevitably leads to site uncertainty during data transmission. In addition, human factors, such as geolocation of the site, scientific staff configuration, and leadership management capabilities, will also bring uncertainties to sites.
- **Channel uncertainty:** The bandwidth, delay, quality of inlets at both site ends, and link capacity of the channel bring the uncertainty of data transmission. Human factors, such as the degree of connection and cooperation between the stations at both ends of the channel, the response speed and work efficiency of maintenance personnel, also contribute to channels uncertainties.

In order to describe the uncertainty of sites and channels in the network graph, the entire network is represented by a fuzzy graph, and the uncertainty of sites and channels is described by the membership function (MF) of vertices and edges. Furthermore, the data transmission on the fuzzy graph needs to be characterized by the fractional factor in the corresponding fuzzy graph. To this purpose, Gao et al. [9] first introduced the fuzzy fractional factor (FFF) in fuzzy graph (FG) setting, and the necessary and sufficient condition and correlated theoretical properties for the existence of FFFs are determined. However, from the prospect of representation theory, there are obvious flaws in FGs to characterize the uncertainty of graphs.

It is well acknowledged that an important difference between fuzzy sets and universe sets lies in that MF does not satisfy the law of complementarity. For example: let $A = \{\text{apple, banana, orange}\}$, the MF μ_1 represents the fruit that Jane likes, and assume that A is fuzzified by $A = \{(\text{apple}, 0.9), (\text{banana}, 0.6), (\text{orange}, 0.3)\}$. Let μ_2 be the MF representing the fruit that Jane does not like, but $A = \{(\text{apple}, 0.1), (\text{banana}, 0.4), (\text{orange}, 0.7)\}$ can't be deduced by μ_1 . This is one of the essential differences between probability functions and MFs, and in fuzzy setting, the negative MF values can't be deduced from positive MF values. Correspondingly, in the fuzzy graph setting, neither the MF of the vertex nor the MF of the edge satisfy the law of complementarity.

To solve the aforementioned problem, multiple MFs are often defined to describe the uncertainty from multiple different angles. Frequently used fuzzy sets are summarized as follows.

- **Intuitionistic fuzzy set (IFS):** $S = \{(x, \mu(x), \nu(x)) : x \in X\}$, where $\mu, \nu : X \rightarrow [0, 1]$ satisfies $\mu(x) + \nu(x) \leq 1$ for any $x \in X$.
- **Pythagorean fuzzy set (PFS):** $S = \{(x, \mu(x), \nu(x)) : x \in X\}$, where $\mu, \nu : X \rightarrow [0, 1]$ satisfies $\mu^2(x) + \nu^2(x) \leq 1$ for any $x \in X$.
- **Picture fuzzy set (PTFS):** $S = \{(x, \mu(x), \nu(x), \xi(x)) : x \in X\}$, where $\mu, \nu, \xi : X \rightarrow [0, 1]$ satisfies $\mu(x) + \nu(x) + \xi(x) \leq 1$ for any $x \in X$.

It is noteworthy that in IFS and PFS, μ and ν represent MF and non-MF respectively, and accordingly $\mu(x)$ and $\nu(x)$ denote the degree of MF and degree of non-MF for element $x \in X$ respectively. Analogously, in PTFS, μ, ν and ξ represent yes (agree), abstain, no (disagree). It can be seen that IFS, PFS, and PTFS all describe uncertainties in various aspects from different angles by defining multiple MFs. The fuzzy sets defined in terms of this trick can effectively make up for the problem that the traditional fuzzy sets do not satisfy the complementarity law.

When it comes to FG setting, a single vertex MF cannot fully describe the vertex uncertainty, and a sin-

gle edge MF cannot globally measure edge uncertainty. Hence, Parvathi and Karunambigai [10] introduced intuitionistic fuzzy graph (IFG), Naz et al. [11] raised the idea of Pythagorean fuzzy graph (PFG), and Zuo et al. [12] defined picture fuzzy graph (PTFG), where multiple vertex (edge) MFs describe the uncertainty of vertices (edges) from different angles. More fractional studies can be referred to Ata and Kıymaz [13], [14] and [15], Ata [16], Dokuyucu et al. [17] and Veerasha et al. [18]. This motivates us to extend the corresponding FFF to IFG, PFG and PTFG.

In this contribution, the relevant concepts for FFF in IFG, PFG and PTFG settings are defined respectively which included sign-alternating walk, transformation operation and increasing walk, and the corresponding necessary and sufficient conditions are determined by means of alternating path. Moreover, the features of the maximum fuzzy fraction factor (MFFF) are characterized.

The organization of reminder parts are arranged as follows. Firstly, the notations and terminologies are manifested in the next section. Secondly, the new concepts related to FFF in associated settings are defined. Thirdly, we analyze the characterizes of FFF from a theoretical prospective. Finally, two open problems are raised at the end of article.

2 Preliminaries

The aim of this section is to present the terminologies in fuzzy graph setting.

2.1 Fuzzy graph

Let V be a universal set (vertex set), and $G = (V(G), E(G))$ be a crisp graph (without MF defined on V and E). A FG $G = (V, A, B)$ is a graph with fuzzy set $A = \{(v_1, \mu_A(v_1)), (v_2, \mu_A(v_2)), \dots, (v_n, \mu_A(v_n))\}$ and $B = \{(v_i, v_j, \mu_B(v_i, v_j)) : v_i, v_j \in V, v_i \neq v_j\}$, where $\mu_A : V \rightarrow [0, 1]$ is the MF on V and $\mu_B : V \times V \rightarrow [0, 1]$ is the binary fuzzy relation on $V \times V$ (binary MF on V^2). There is a restriction on μ_A and μ_B : $\mu_B(v_i, v_j) \leq \min\{\mu_A(v_i), \mu_A(v_j)\}$ for any $v_i, v_j \in V$. Furthermore, $\mu_B(v_i, v_j) = 0$ for any $v_i v_j \notin E(G)$. If $\mu_B(v_i, v_j) = \min\{\mu_A(v_i), \mu_A(v_j)\}$ for each pair of $(v_i, v_j) \in V \times V$, then $G = (V, A, B)$ is called a complete MF.

2.2 Intuitionistic fuzzy graph

A fuzzy graph $G = (V, A, B)$ is an IFG with IFSs $A = \{(v_i, \mu_A(v_i), \nu_A(v_i)) | i \in \{1, 2, \dots, |V|\}\}$ and $B = \{(v_i, v_j, \mu_B(v_i, v_j), \nu_B(v_i, v_j)) : v_i, v_j \in V, v_i \neq v_j\}$, where $\mu_A, \nu_A : V \rightarrow [0, 1]$ are the MFs on V satisfying $\mu_A(v) + \nu_A(v) \leq 1$ for any $v \in V$, and $\mu_B, \nu_B : V \times V \rightarrow [0, 1]$ are the binary MFs on $V \times V$ satisfying $\mu_B(v, v') + \nu_B(v, v') \leq 1$ for any $(v, v') \in V^2$. The restriction on μ_A and μ_B is formulated by

$$\mu_B(v_i, v_j) \leq \min\{\mu_A(v_i), \mu_A(v_j)\} \quad (1)$$

for any $v_i, v_j \in V$. While the restriction on ν_A and ν_B is stated by

$$\nu_B(v_i, v_j) \geq \max\{\nu_A(v_i), \nu_A(v_j)\} \quad (2)$$

for any $v_i, v_j \in V$. Furthermore,

$$\mu_B(v_i, v_j) = \nu_B(v_i, v_j) = 0 \quad (3)$$

for any $v_i v_j \notin E(G)$.

2.3 Pythagorean fuzzy graph

A FG $G = (V, A, B)$ is a PFG with PFSs $A = \{(v_i, \mu_A(v_i), \nu_A(v_i)) | i \in \{1, 2, \dots, |V|\}\}$ and $B = \{(v_i, v_j, \mu_B(v_i, v_j), \nu_B(v_i, v_j)) : v_i, v_j \in V, v_i \neq v_j\}$, where $\mu_A, \nu_A : V \rightarrow [0, 1]$ are the MFs on V satisfying $\mu_A^2(v) + \nu_A^2(v) \leq 1$ for any $v \in V$, and $\mu_B, \nu_B : V \times V \rightarrow [0, 1]$ are the binary MFs on $V \times V$ satisfying $\mu_B^2(v, v') + \nu_B^2(v, v') \leq 1$ for any

$(v, v') \in V^2$. The restriction on μ_A and μ_B is described by (1) for any $v_i, v_j \in V$, the restriction on v_A and v_B stated by (2) for any $v_i, v_j \in V$, and (3) holds for any $v_i v_j \notin E(G)$.

2.4 Picture fuzzy graph

A fuzzy graph $G = (V, A, B)$ is a PTFG with PTFSSs $A = \{(v_i, \mu_A(v_i), v_A(v_i), \xi_A(v_i)) | i \in \{1, 2, \dots, |V|\}\}$ and $B = \{(v_i, v_j, \mu_B(v_i, v_j), v_B(v_i, v_j), \xi_B(v_i, v_j)) : v_i, v_j \in V, v_i \neq v_j\}$, where $\mu_A, v_A, \xi_A : V \rightarrow [0, 1]$ are the MFs on V satisfying $\mu_A(v) + v_A(v) + \xi_A(v) \leq 1$ for any $v \in V$, and $\mu_B, v_B, \xi_B : V \times V \rightarrow [0, 1]$ are the binary MFs on $V \times V$ satisfying $\mu_B(v, v') + v_B(v, v') + \xi_B(v, v') \leq 1$ for any $(v, v') \in V^2$. The relations between μ_A and μ_B , v_A and v_B , and ξ_A and ξ_B are formulated by

$$\mu_B(v_i, v_j) \leq \min\{\mu_A(v_i), \mu_A(v_j)\},$$

$$v_B(v_i, v_j) \leq \min\{v_A(v_i), v_A(v_j)\}$$

and

$$\xi_B(v_i, v_j) \geq \max\{\xi_A(v_i), \xi_A(v_j)\}$$

for any $v_i, v_j \in V$. Furthermore, $\mu_B(v_i, v_j) = v_B(v_i, v_j) = \xi_B(v_i, v_j) = 0$ for any $v_i v_j \notin E(G)$.

3 Definitions

3.1 Fuzzy factor and fuzzy fractional factor

The fuzzy factor (FF) and FFF in FG setting are suggested by Gao et al. [9] which are formulated by the following definitions respectively.

Definition 1. (Gao et al. [9]) Let $G = (V, A, B)$ be a FG with MFs μ_A and μ_B and $g, f : \rightarrow \mathbb{R}^+ \cup \{0\}$ be two functions satisfying $g(v) \leq f(v)$ for any $v \in V$. We say a FG G admits a fuzzy (g, f) -factor if there is a fuzzy indicator function (FIF) $h : E \rightarrow \{0, \mu_B\}$ satisfying $g(v) \leq \sum_{e \sim v} h(e) \leq f(v)$ for any $v \in V$, where $e \sim v$ implies edge e is incident with vertex v . The fuzzy (g, f) -factor w.r.t. FIF h is a fuzzy spanning subgraph induced by $E_h = \{e \in E : h(e) > 0\}$.

Definition 2. (Gao et al. [9]) Let $G = (V, A, B)$ be a FG with MFs μ_A and μ_B and $g, f : \rightarrow \mathbb{R}^+ \cup \{0\}$ be two functions satisfying $g(v) \leq f(v)$ for any $v \in V$. We say a fuzzy graph G admits a fuzzy fractional (g, f) -factor if there is a fuzzy fractional indicator function (FFIF) $h : E \rightarrow [0, \mu_B]$ satisfying $g(v) \leq \sum_{e \sim v} h(e) \leq f(v)$ for any $v \in V$. The fuzzy fractional (g, f) -factor w.r.t. FFIF h denoted by $\mathcal{F}_h$ is a fuzzy spanning subgraph induced by $E_h = \{e \in E : h(e) > 0\}$.

Now, we raise the concepts of FF and FFF in IFGs.

Definition 3. (FF in IFG) Let $G = (V, A, B)$ be an IFG with MFs μ_A, v_A, μ_B and v_B , and $g_\mu, g_v, f_\mu, f_v : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$ and $g_v(v) \leq f_v(v)$ for any $v \in V$. We say an IFG G admits a fuzzy (g_μ, g_v, f_μ, f_v) -factor if there exist FIFs $h_\mu, h_v : E \rightarrow \{0, \mu_B\}$ satisfying $g_\mu(v) \leq \sum_{e \sim v} h_\mu(e) \leq f_\mu(v)$ and $g_v(v) \leq \sum_{e \sim v} h_v(e) \leq f_v(v)$ for any $v \in V$. The fuzzy (g_μ, g_v, f_μ, f_v) -factor w.r.t. FIF (h_μ, h_v) is a fuzzy spanning subgraph induced by $E_{h_\mu, h_v} = \{e \in E : h_\mu(e) > 0 \text{ or } h_v(e) > 0\}$.

Definition 4. (FFF in IFG) Let $G = (V, A, B)$ be an IFG with MFs μ_A, v_A, μ_B and v_B , and $g_\mu, g_v, f_\mu, f_v : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$ and $g_v(v) \leq f_v(v)$ for any $v \in V$. We say an IFG G admits a fuzzy fractional (g_μ, g_v, f_μ, f_v) -factor if there exist FFIFs $h_\mu, h_v : E \rightarrow [0, \mu_B]$ satisfying $g_\mu(v) \leq \sum_{e \sim v} h_\mu(e) \leq f_\mu(v)$ and $g_v(v) \leq \sum_{e \sim v} h_v(e) \leq f_v(v)$ for any $v \in V$. The fuzzy fractional (g_μ, g_v, f_μ, f_v) -factor w.r.t. FFIFs (h_μ, h_v) denoted by $\mathcal{F}_{h_\mu, h_v}$ is a fuzzy spanning subgraph induced by $E_{h_\mu, h_v} = \{e \in E : h_\mu(e) > 0 \text{ or } h_v(e) > 0\}$.

Analogously, the FF and FFF in PFG can be defined using the same fashion. For completeness, we list the specific definitions below.

Definition 5. (FF in PFG) Let $G = (V, A, B)$ be a PFG with MFs μ_A, ν_A, μ_B and ν_B , and $g_\mu, g_\nu, f_\mu, f_\nu : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$ and $g_\nu(v) \leq f_\nu(v)$ for any $v \in V$. We say a PFG G admits a fuzzy $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factor if there exist FIFs $h_\mu, h_\nu : E \rightarrow \{0, \mu_B\}$ satisfying $g_\mu(v) \leq \sum_{e \sim v} h_\mu(e) \leq f_\mu(v)$ and $g_\nu(v) \leq \sum_{e \sim v} h_\nu(e) \leq f_\nu(v)$ for any $v \in V$. The fuzzy $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factor w.r.t. FIFs (h_μ, h_ν) is a fuzzy spanning subgraph induced by $E_{h_\mu, h_\nu} = \{e \in E : h_\mu(e) > 0 \text{ or } h_\nu(e) > 0\}$.

Definition 6. (FFF in PFG) Let $G = (V, A, B)$ be a PFG with MFs μ_A, ν_A, μ_B and ν_B , and $g_\mu, g_\nu, f_\mu, f_\nu : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$ and $g_\nu(v) \leq f_\nu(v)$ for any $v \in V$. We say a PFG G admits a fuzzy fractional $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factor if there exist FFIFs $h_\mu, h_\nu : E \rightarrow [0, \mu_B]$ satisfying $g_\mu(v) \leq \sum_{e \sim v} h_\mu(e) \leq f_\mu(v)$ and $g_\nu(v) \leq \sum_{e \sim v} h_\nu(e) \leq f_\nu(v)$ for any $v \in V$. The fuzzy fractional $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factor w.r.t. FFIFs (h_μ, h_ν) denoted by $\mathcal{F}_{h_\mu, h_\nu}$ is a fuzzy spanning subgraph induced by $E_{h_\mu, h_\nu} = \{e \in E : h_\mu(e) > 0 \text{ or } h_\nu(e) > 0\}$.

By extending the number of MFs to three, and using the same trick, the definitions of FF and FFF in PTFGs are obtained.

Definition 7. (FF in PTFG) Let $G = (V, A, B)$ be a PTFG with MFs $\mu_A, \nu_A, \xi_A, \mu_B, \nu_B$ and ξ_B , and $g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$, $g_\nu(v) \leq f_\nu(v)$ and $g_\xi(v) \leq f_\xi(v)$ for any $v \in V$. We say a PTFG G admits a fuzzy $(g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi)$ -factor if there exist FIFs $h_\mu, h_\nu, h_\xi : E \rightarrow \{0, \mu_B\}$ satisfying $g_\mu(v) \leq \sum_{e \sim v} h_\mu(e) \leq f_\mu(v)$, $g_\nu(v) \leq \sum_{e \sim v} h_\nu(e) \leq f_\nu(v)$ and $g_\xi(v) \leq \sum_{e \sim v} h_\xi(e) \leq f_\xi(v)$ for any $v \in V$. The fuzzy $(g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi)$ -factor w.r.t. FIFs (h_μ, h_ν, h_ξ) is a fuzzy spanning subgraph induced by $E_{h_\mu, h_\nu, h_\xi} = \{e \in E : h_\mu(e) > 0 \text{ or } h_\nu(e) > 0 \text{ or } h_\xi(e) > 0\}$.

Definition 8. (FFF in PTFG) Let $G = (V, A, B)$ be a PTFG with MFs $\mu_A, \nu_A, \xi_A, \mu_B, \nu_B$ and ξ_B , and $g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$, $g_\nu(v) \leq f_\nu(v)$ and $g_\xi(v) \leq f_\xi(v)$ for any $v \in V$. We say a PTFG G admits a fuzzy fractional $(g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi)$ -factor if there exist FFIFs $h_\mu, h_\nu, h_\xi : E \rightarrow [0, \mu_B]$ satisfying $g_\mu(v) \leq \sum_{e \sim v} h_\mu(e) \leq f_\mu(v)$, $g_\nu(v) \leq \sum_{e \sim v} h_\nu(e) \leq f_\nu(v)$ and $g_\xi(v) \leq \sum_{e \sim v} h_\xi(e) \leq f_\xi(v)$ for any $v \in V$. The fuzzy fractional $(g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi)$ -factor w.r.t. FFIFs (h_μ, h_ν, h_ξ) denoted by $\mathcal{F}_{h_\mu, h_\nu, h_\xi}$ is a fuzzy spanning subgraph induced by $E_{h_\mu, h_\nu, h_\xi} = \{e \in E : h_\mu(e) > 0 \text{ or } h_\nu(e) > 0 \text{ or } h_\xi(e) > 0\}$.

3.2 Crucial concepts in three types of fuzzy graphs

In order to obtain the corresponding theoretical results, we need to introduce corresponding concepts on the three types of fuzzy graphs.

3.2.1 Notations in intuitionistic fuzzy graph and Pythagorean fuzzy graph

In this part, we expend terminologies raised in Gao et al. [9] to IFG and PFG setting, and all graphs described in this subsection are considered to be IFG or PFG with MF $(\mu_A, \nu_A, \mu_B, \nu_B)$ such that (μ_A, ν_A) are IFs (resp. PFSs) defined on V and (μ_B, ν_B) are IFs (resp. PFSs) defined on $V \times V$.

Let $\mathcal{F}_{h_{\mu_1}, h_{\nu_1}}$ and $\mathcal{F}_{h_{\mu_2}, h_{\nu_2}}$ be two fuzzy fractional $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factors w.r.t. FFIFs (h_{μ_1}, h_{ν_1}) and (h_{μ_2}, h_{ν_2}) , respectively. Set

$$E_{h_{\mu_1}, h_{\mu_2}}^+ = \{e \in E(G) : h_{\mu_1}(e) > h_{\mu_2}(e)\},$$

$$E_{h_{\mu_1}, h_{\mu_2}}^- = \{e \in E(G) : h_{\mu_1}(e) < h_{\mu_2}(e)\},$$

$$E_{h_{\nu_1}, h_{\nu_2}}^+ = \{e \in E(G) : h_{\nu_1}(e) > h_{\nu_2}(e)\}$$

and

$$E_{h_{\nu_1}, h_{\nu_2}}^- = \{e \in E(G) : h_{\nu_1}(e) < h_{\nu_2}(e)\}.$$

Definition 9. (Sign-alternating walk) A sign-alternating walk (SAW) (each edge is allowed to appear at most twice) of IFG (resp. PFG) G w.r.t. FFIFs (h_{μ_1}, h_{ν_1}) and (h_{μ_2}, h_{ν_2}) is a closed walk with even length, its edges alternately belong to $E_{h_{\mu_1}, h_{\mu_2}}^+$ and $E_{h_{\mu_1}, h_{\mu_2}}^-$, and edges also alternately belong to $E_{h_{\nu_1}, h_{\nu_2}}^+$ and $E_{h_{\nu_1}, h_{\nu_2}}^-$.

Definition 10. (Transformation operation) Let $W = e_1 e_2 \cdots e_{2m}$ be an edge sequence representation of a SAW w.r.t. FFIFs (h_{μ_1}, h_{ν_1}) and (h_{μ_2}, h_{ν_2}) , and set $\Delta_{h_{\mu_1}, h_{\mu_2}} = h_{\mu_1} - h_{\mu_2}$ and $\Delta_{h_{\nu_1}, h_{\nu_2}} = h_{\nu_1} - h_{\nu_2}$. If an edge e appears twice, then we regard it as two parallel edges e' and e'' , and assign $(\frac{\Delta_{h_{\mu_1}}(e)}{2}, \frac{\Delta_{h_{\nu_1}}(e)}{2})$ to these two parallel edges respectively. Set

$$\varepsilon_{\mu} = \min_{e \in E(C)} \{|\Delta_{h_{\mu_1}, h_{\mu_2}}(e)|\}$$

and

$$\varepsilon_{\nu} = \min_{e \in E(C)} \{|\Delta_{h_{\nu_1}, h_{\nu_2}}(e)|\}.$$

• If the signs of $\Delta_{h_{\mu_1}, h_{\mu_2}}$ and $\Delta_{h_{\nu_1}, h_{\nu_2}}$ are opposite, W.L.O.G., we assume that $\Delta_{h_{\mu_1}, h_{\mu_2}}(e_i) > 0$ and $\Delta_{h_{\nu_1}, h_{\nu_2}}(e_i) < 0$ if $i \equiv 1(\text{mod}2)$; and $\Delta_{h_{\mu_1}, h_{\mu_2}}(e_i) < 0$ and $\Delta_{h_{\nu_1}, h_{\nu_2}}(e_i) > 0$ if $i \equiv 0(\text{mod}2)$. Define new FFIFs (h_{μ_3}, h_{ν_3}) as follows: $h_{\mu_3}(e_i) = h_{\mu_1}(e_i) - \varepsilon_{\mu}$ and $h_{\nu_3}(e_i) = h_{\nu_1}(e_i) + \varepsilon_{\nu}$ if $i \equiv 1(\text{mod}2)$; $h_{\mu_3}(e_i) = h_{\mu_1}(e_i) + \varepsilon_{\mu}$ and $h_{\nu_3}(e_i) = h_{\nu_1}(e_i) - \varepsilon_{\nu}$ if $i \equiv 0(\text{mod}2)$; $h_{\mu_3}(e) = h_{\mu_1}(e)$ and $h_{\nu_3}(e) = h_{\nu_1}(e)$ if $e \notin W$.

• If the signs of $\Delta_{h_{\mu_1}, h_{\mu_2}}$ and $\Delta_{h_{\nu_1}, h_{\nu_2}}$ are consistent, W.L.O.G., we assume that $\Delta_{h_{\mu_1}, h_{\mu_2}}(e_i) > 0$ and $\Delta_{h_{\nu_1}, h_{\nu_2}}(e_i) > 0$ if $i \equiv 1(\text{mod}2)$; and $\Delta_{h_{\mu_1}, h_{\mu_2}}(e_i) < 0$ and $\Delta_{h_{\nu_1}, h_{\nu_2}}(e_i) < 0$ if $i \equiv 0(\text{mod}2)$. Define new FFIFs (h_{μ_3}, h_{ν_3}) as follows: $h_{\mu_3}(e_i) = h_{\mu_1}(e_i) - \varepsilon_{\mu}$ and $h_{\nu_3}(e_i) = h_{\nu_1}(e_i) - \varepsilon_{\nu}$ if $i \equiv 1(\text{mod}2)$; $h_{\mu_3}(e_i) = h_{\mu_1}(e_i) + \varepsilon_{\mu}$ and $h_{\nu_3}(e_i) = h_{\nu_1}(e_i) + \varepsilon_{\nu}$ if $i \equiv 0(\text{mod}2)$; $h_{\mu_3}(e) = h_{\mu_1}(e)$ and $h_{\nu_3}(e) = h_{\nu_1}(e)$ if $e \notin W$.

After the transformation operation, the parallel edges e' and e'' return to e and set $h_{\mu_3}(e) = h_{\mu_3}(e') + h_{\mu_3}(e'')$ and $h_{\nu_3}(e) = h_{\nu_3}(e') + h_{\nu_3}(e'')$. Note that $\mathcal{F}_{h_{\mu_3}, h_{\nu_3}}$ is a fuzzy fractional $(g_{\mu}, g_{\nu}, f_{\mu}, f_{\nu})$ -factor of IFG (resp. PFG) G w.r.t. FFIFs (h_{μ_3}, h_{ν_3}) , and

$$|E_{h_{\mu_3}, h_{\mu_2}}^+| + |E_{h_{\mu_3}, h_{\mu_2}}^-| < |E_{h_{\mu_1}, h_{\mu_2}}^+| + |E_{h_{\mu_1}, h_{\mu_2}}^-|$$

and

$$|E_{h_{\nu_3}, h_{\nu_2}}^+| + |E_{h_{\nu_3}, h_{\nu_2}}^-| < |E_{h_{\nu_1}, h_{\nu_2}}^+| + |E_{h_{\nu_1}, h_{\nu_2}}^-|.$$

Definition 11. (Maximum fuzzy fractional factor) Let $\mathcal{F}_{h_{\mu}, h_{\nu}}$ be a fuzzy fractional $(g_{\mu}, g_{\nu}, f_{\mu}, f_{\nu})$ -factor w.r.t. FFIFs (h_{μ}, h_{ν}) . If $|E_{h_{\mu}, h_{\nu}}| \geq |E_{h'_{\mu}, h'_{\nu}}|$ holds for any fuzzy fractional $(g_{\mu}, g_{\nu}, f_{\mu}, f_{\nu})$ -factor $\mathcal{F}_{h'_{\mu}, h'_{\nu}}$ w.r.t. FFIFs (h'_{μ}, h'_{ν}) , then $\mathcal{F}_{h_{\mu}, h_{\nu}}$ is called a maximum fuzzy fractional $(g_{\mu}, g_{\nu}, f_{\mu}, f_{\nu})$ -factor which implies the maximum number of non-zero edges w.r.t. FFIFs.

Definition 12. (Increasing walk) The increasing walk W of IFG (resp. PFG) G w.r.t. FFIFs (h_{μ}, h_{ν}) is a closed walk with even length satisfying the following conditions:

- (1) its edge values under h_{μ} are greater than 0 or less than $\mu_B(e)$ alternately, and there is at least one edge in W satisfying $h_{\mu}(e) = 0$;
- (2) its edge values under h_{ν} are greater than 0 or less than $\nu_B(e)$ alternately, and there is at least one edge in W satisfying $h_{\nu}(e) = 0$.

3.2.2 Notations in picture fuzzy graph

In this part, new notations raised in Gao et al. [9] are modified to PTFG setting, and all graphs described in this subsection are considered to be PTFG with MFs $(\mu_A, \nu_A, \xi_A, \mu_B, \nu_B, \xi_B)$ such that (μ_A, ν_A, ξ_A) are PTFSS defined on V and (μ_B, ν_B, ξ_B) are PTFSSs defined on $V \times V$.

Let $\mathcal{F}_{h_{\mu_1}, h_{\nu_1}, h_{\xi_1}}$ and $\mathcal{F}_{h_{\mu_2}, h_{\nu_2}, h_{\xi_2}}$ be two fuzzy fractional $(g_{\mu}, g_{\nu}, g_{\xi}, f_{\mu}, f_{\nu}, f_{\xi})$ -factors w.r.t. FFIFs $(h_{\mu_1}, h_{\nu_1}, h_{\xi_1})$ and $(h_{\mu_2}, h_{\nu_2}, h_{\xi_2})$, respectively. Set

$$E_{h_{\mu_1}, h_{\mu_2}}^+ = \{e \in E(G) : h_{\mu_1}(e) > h_{\mu_2}(e)\},$$

$$E_{h_{\mu_1}, h_{\mu_2}}^- = \{e \in E(G) : h_{\mu_1}(e) < h_{\mu_2}(e)\},$$

$$E_{h_{\nu_1}, h_{\nu_2}}^+ = \{e \in E(G) : h_{\nu_1}(e) > h_{\nu_2}(e)\},$$

$$E_{h_{v_1}, h_{v_2}}^- = \{e \in E(G) : h_{v_1}(e) < h_{v_2}(e)\},$$

$$E_{h_{\xi_1}, h_{\xi_2}}^+ = \{e \in E(G) : h_{\xi_1}(e) > h_{\xi_2}(e)\}$$

and

$$E_{h_{\xi_1}, h_{\xi_2}}^- = \{e \in E(G) : h_{\xi_1}(e) < h_{\xi_2}(e)\}.$$

Definition 13. (Sign-alternating walk) A sign-alternating walk (SAW) (each edge is allowed to appear at most twice) of picture fuzzy graph G w.r.t. FFIFs $(h_{\mu_1}, h_{v_1}, h_{\xi_1})$ and $(h_{\mu_2}, h_{v_2}, h_{\xi_2})$ is a closed walk with even length and its edges alternately belong to

- 1) $E_{h_{\mu_1}, h_{\mu_2}}^+$ and $E_{h_{\mu_1}, h_{\mu_2}}^-$;
- 2) $E_{h_{v_1}, h_{v_2}}^+$ and $E_{h_{v_1}, h_{v_2}}^-$;
- 3) $E_{h_{\xi_1}, h_{\xi_2}}^+$ and $E_{h_{\xi_1}, h_{\xi_2}}^-$.

Definition 14. (Transformation operation) Let $W = e_1 e_2 \cdots e_{2m}$ be an edge sequence representation of a SAW w.r.t. FFIFs $(h_{\mu_1}, h_{v_1}, h_{\xi_1})$ and $(h_{\mu_2}, h_{v_2}, h_{\xi_2})$, and set $\Delta_{h_{\mu_1}, h_{\mu_2}} = h_{\mu_1} - h_{\mu_2}$, $\Delta_{h_{v_1}, h_{v_2}} = h_{v_1} - h_{v_2}$ and $\Delta_{h_{\xi_1}, h_{\xi_2}} = h_{\xi_1} - h_{\xi_2}$. If an edge e appears twice, then we regard it as two parallel edges e' and e'' , and assign $(\frac{\Delta_{h_{\mu}}(e)}{2}, \frac{\Delta_{h_v}(e)}{2}, \frac{\Delta_{h_{\xi}}(e)}{2})$ to these two parallel edges respectively. Set

$$\varepsilon_{\mu} = \min_{e \in E(C)} \{|\Delta_{h_{\mu_1}, h_{\mu_2}}(e)|\},$$

$$\varepsilon_v = \min_{e \in E(C)} \{|\Delta_{h_{v_1}, h_{v_2}}(e)|\}$$

and

$$\varepsilon_{\xi} = \min_{e \in E(C)} \{|\Delta_{h_{\xi_1}, h_{\xi_2}}(e)|\}.$$

- For $\Delta_{h_{\mu_1}, h_{\mu_2}}(e_i)$, W.L.O.G., we assume that $\Delta_{h_{\mu_1}, h_{\mu_2}}(e_i) > 0$ if $i \equiv 1(\text{mod}2)$; and $\Delta_{h_{\mu_1}, h_{\mu_2}}(e_i) < 0$ if $i \equiv 0(\text{mod}2)$. Define new FFIF h_{μ_3} as follows: $h_{\mu_3}(e_i) = h_{\mu_1}(e_i) - \varepsilon_{\mu}$ if $i \equiv 1(\text{mod}2)$; $h_{\mu_3}(e_i) = h_{\mu_1}(e_i) + \varepsilon_{\mu}$ if $i \equiv 0(\text{mod}2)$; $h_{\mu_3}(e) = h_{\mu_1}(e)$ if $e \notin W$.
- For $\Delta_{h_{v_1}, h_{v_2}}(e_i)$, we use the same trick to define new FFIF h_{v_3} .
- For $\Delta_{h_{\xi_1}, h_{\xi_2}}(e_i)$, we use the same approach to define new FFIF h_{ξ_3} .

After the transformation operation, the parallel edges e' and e'' return to e and set $h_{\mu_3}(e) = h_{\mu_3}(e') + h_{\mu_3}(e'')$, $h_{v_3}(e) = h_{v_3}(e') + h_{v_3}(e'')$ and $h_{\xi_3}(e) = h_{\xi_3}(e') + h_{\xi_3}(e'')$. Note that $\mathcal{F}_{h_{\mu_3}, h_{v_3}, h_{\xi_3}}$ is a fuzzy fractional $(g_{\mu}, g_v, g_{\xi}, f_{\mu}, f_v, f_{\xi})$ -factor of PTFG G w.r.t. FFIFs $(h_{\mu_3}, h_{v_3}, h_{\xi_3})$, and

$$|E_{h_{\mu_3}, h_{\mu_2}}^+| + |E_{h_{\mu_3}, h_{\mu_2}}^-| < |E_{h_{\mu_1}, h_{\mu_2}}^+| + |E_{h_{\mu_1}, h_{\mu_2}}^-|,$$

$$|E_{h_{v_3}, h_{v_2}}^+| + |E_{h_{v_3}, h_{v_2}}^-| < |E_{h_{v_1}, h_{v_2}}^+| + |E_{h_{v_1}, h_{v_2}}^-|$$

and

$$|E_{h_{\xi_3}, h_{\xi_2}}^+| + |E_{h_{\xi_3}, h_{\xi_2}}^-| < |E_{h_{\xi_1}, h_{\xi_2}}^+| + |E_{h_{\xi_1}, h_{\xi_2}}^-|.$$

Definition 15. (Maximum fuzzy fractional factor) Let $\mathcal{F}_{h_{\mu}, h_v, h_{\xi}}$ be a fuzzy fractional $(g_{\mu}, g_v, g_{\xi}, f_{\mu}, f_v, f_{\xi})$ -factor w.r.t. FFIFs (h_{μ}, h_v, h_{ξ}) . If $|E_{h_{\mu}, h_v, h_{\xi}}| \geq |E_{h'_{\mu}, h'_{v}, h'_{\xi}}|$ holds for any fuzzy fractional $(g_{\mu}, g_v, g_{\xi}, f_{\mu}, f_v, f_{\xi})$ -factor $\mathcal{F}_{h'_{\mu}, h'_{v}, h'_{\xi}}$ w.r.t. FFIFs $(h'_{\mu}, h'_{v}, h'_{\xi})$, then $\mathcal{F}_{h_{\mu}, h_v, h_{\xi}}$ is called a maximum fuzzy fractional $(g_{\mu}, g_v, g_{\xi}, f_{\mu}, f_v, f_{\xi})$ -factor which implies the maximum number of non-zero edges w.r.t. FFIFs.

Definition 16. (Increasing walk) The increasing walk W of PTFG G w.r.t. FFIFs (h_{μ}, h_v, h_{ξ}) is a closed walk with even length, its edge values under h_{μ} (resp. h_v or h_{ξ}) are greater than 0 or less than $\mu_B(e)$ alternately, and there is at least one edge in W satisfying $h_{\mu}(e) = 0$ (resp. $h_v(e) = 0$ or $h_{\xi}(e) = 0$).

4 Extended conclusions

In this section, we aim to extend the results on the FFF of FGs presented in Gao et al. [9] to IFG, PFG, and PTFG settings.

4.1 Results in IFG and PFG

The tricks to prove the following theorems are analogue to Gao et al. [9], and we will not repeat it here.

Theorem 1. Let $G = (V, A, B)$ be an IFG (resp. a PFG) with MFs $(\mu_A, \nu_A, \mu_B, \nu_B)$ such that (μ_A, ν_A) is IFS (resp. PFS) defined on V and (μ_B, ν_B) is IFS (resp. PFS) defined on $V \times V$. Let $g_\mu, g_\nu, f_\mu, f_\nu : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$ and $g_\nu(v) \leq f_\nu(v)$ for any $v \in V$. Then, G admits a fuzzy fractional $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factor if and only if for any subset S of V ,

$$f_\mu(S) - g_\mu(T_\mu) + \sum_{v \in T} d_{G-S}(v) \geq 0, \quad (4)$$

$$f_\nu(S) - g_\nu(T_\nu) + \sum_{v \in T} d_{G-S}(v) \geq 0, \quad (5)$$

where $T_\mu = \{v : v \in V - S \text{ and } d_{G-S}(v) \leq g_\mu(v)\}$ and $T_\nu = \{v : v \in V - S \text{ and } d_{G-S}(v) \leq g_\nu(v)\}$.

Obviously, Theorem 1 has the following equivalent version.

Theorem 2. Let $G = (V, A, B)$ be an IFG (resp. a PFG) with MFs $(\mu_A, \nu_A, \mu_B, \nu_B)$ such that (μ_A, ν_A) is IFS (resp. PFS) defined on V and (μ_B, ν_B) is IFS (resp. PFS) defined on $V \times V$. Let $g_\mu, g_\nu, f_\mu, f_\nu : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$ and $g_\nu(v) \leq f_\nu(v)$ for any $v \in V$. Then, G admits a fuzzy fractional $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factor if and only if (4) and (5) hold for any subsets S, T of V with $S \cap T = \emptyset$.

Let fuzzy fractional (f_μ, f_ν) -factor be a special fuzzy fractional $(g_\mu, g_\nu, f_\mu, f_\nu)$ -factor with $g_\mu = f_\mu$ and $g_\nu = f_\nu$. We have the following conclusions on fuzzy fractional (f_μ, f_ν) -factor.

Theorem 3. Let $\mathcal{F}_{h_{\mu_1}, h_{\nu_1}}$ and $\mathcal{F}_{h_{\mu_2}, h_{\nu_2}}$ be two fuzzy fractional (f_μ, f_ν) -factors w.r.t. FFIFs (h_{μ_1}, h_{ν_1}) and (h_{μ_2}, h_{ν_2}) , respectively. Then $\mathcal{F}_{h_{\mu_2}, h_{\nu_2}}$ can be derived from $\mathcal{F}_{h_{\mu_1}, h_{\nu_1}}$ by finitely transformation operations.

Theorem 4. Let $\mathcal{F}_{h_\mu, h_\nu}$ be a fuzzy fractional (f_μ, f_ν) -factor w.r.t. FFIFs (h_μ, h_ν) . The $\mathcal{F}_{h_\mu, h_\nu}$ is the maximum fuzzy fractional (f_μ, f_ν) -factor if and only if G has no increasing walk w.r.t. (h_μ, h_ν) .

4.2 Results in picture fuzzy graph

Similarly, the technologies to prove theorems in this section are analogue to Gao et al. [9], we skip the detailed proofs here.

Theorem 5. Let $G = (V, A, B)$ be a PTFG with MFs $(\mu_A, \nu_A, \xi_A, \mu_B, \nu_B, \xi_B)$ such that (μ_A, ν_A, ξ_A) is a PTFS defined on V and (μ_B, ν_B, ξ_B) is a PTFS defined on $V \times V$. Let $g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$, $g_\nu(v) \leq f_\nu(v)$ and $g_\xi(v) \leq f_\xi(v)$ for any $v \in V$. Then, G admits a fuzzy fractional $(g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi)$ -factor if and only if for any subset S of V ,

$$f_\mu(S) - g_\mu(T_\mu) + \sum_{v \in T} d_{G-S}(v) \geq 0, \quad (6)$$

$$f_\nu(S) - g_\nu(T_\nu) + \sum_{v \in T} d_{G-S}(v) \geq 0, \quad (7)$$

$$f_\xi(S) - g_\xi(T_\xi) + \sum_{v \in T} d_{G-S}(v) \geq 0, \quad (8)$$

where $T_\mu = \{v : v \in V - S \text{ and } d_{G-S}(v) \leq g_\mu(v)\}$, $T_\nu = \{v : v \in V - S \text{ and } d_{G-S}(v) \leq g_\nu(v)\}$ and $T_\xi = \{v : v \in V - S \text{ and } d_{G-S}(v) \leq g_\xi(v)\}$.

Obviously, Theorem 5 has the following equivalent version.

Theorem 6. Let $G = (V, A, B)$ be a PTFG with MFs $(\mu_A, \nu_A, \xi_A, \mu_B, \nu_B, \xi_B)$ such that (μ_A, ν_A, ξ_A) is a PTFS defined on V and (μ_B, ν_B, ξ_B) is a PTFS defined on $V \times V$. Let $g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi : \rightarrow \mathbb{R}^+ \cup \{0\}$ be four functions satisfying $g_\mu(v) \leq f_\mu(v)$, $g_\nu(v) \leq f_\nu(v)$ and $g_\xi(v) \leq f_\xi(v)$ for any $v \in V$. Then, G admits a fuzzy fractional $(g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi)$ -factor if and only if (6), (7) and (8) hold for any subsets S, T of V with $S \cap T = \emptyset$.

Let fuzzy fractional (f_μ, f_ν, f_ξ) -factor be a special fuzzy fractional $(g_\mu, g_\nu, g_\xi, f_\mu, f_\nu, f_\xi)$ -factor with $g_\mu = f_\mu$, $g_\nu = f_\nu$ and $g_\xi = f_\xi$. We have the following conclusions on fuzzy fractional (f_μ, f_ν, f_ξ) -factor.

Theorem 7. Let $\mathcal{F}_{h_{\mu_1}, h_{\nu_1}, h_{\xi_1}}$ and $\mathcal{F}_{h_{\mu_2}, h_{\nu_2}, h_{\xi_2}}$ be two fuzzy fractional (f_μ, f_ν, f_ξ) -factors w.r.t. FFIFs $(h_{\mu_1}, h_{\nu_1}, h_{\xi_1})$ and $(h_{\mu_2}, h_{\nu_2}, h_{\xi_2})$, respectively. Then $\mathcal{F}_{h_{\mu_2}, h_{\nu_2}, h_{\xi_2}}$ can be derived from $\mathcal{F}_{h_{\mu_1}, h_{\nu_1}, h_{\xi_1}}$ by finitely transformation operations.

Theorem 8. Let $\mathcal{F}_{h_\mu, h_\nu, h_\xi}$ be a fuzzy fractional (f_μ, f_ν, f_ξ) -factor w.r.t. FFIFs (h_μ, h_ν, h_ξ) . The $\mathcal{F}_{h_\mu, h_\nu, h_\xi}$ is the maximum fuzzy fractional (f_μ, f_ν, h_ξ) -factor if and only if G has no increasing walk w.r.t. (h_μ, h_ν, h_ξ) .

4.3 Technical remark

Due to the similarity of the proof method to Gao et al. [9], the detailed process of the above proof has been omitted. However, we still provide some overview of the overall idea. The proof of the results is algorithmic and can be extended to Algorithm 1 in [9]. The key technology is to start with augmenting path and sign alternating walk, continuously modifying the value of the fractional indicator functions, in order to achieve the effect of transformation. Although multiple MFs appear in the settings of this article, each membership function can be treated similarly to [9] and combined to obtain the results.

5 Conclusion and open problems

In this contribution, we proposed the new concepts of FFF in IFG, PFG and PTFG, respectively. Moreover, the theoretical results involving the necessary and sufficient condition of existence of FFF, and the characteristics of MFFF were determined by means of structure tricks.

Although the concepts and properties of the three types of FGs have been confirmed in this paper, the properties of the FFF in specific FG settings are still open. We believe that the following questions can be the subject of future research.

Open Problem 1. How to define FFF in bipolar fuzzy graphs? What are the theoretical characteristics of FFF in bipolar fuzzy graphs?

Open Problem 2. Can we define FFF in 2-type fuzzy graphs? What is the relationship between footprint of uncertainty and the FFF in 2-type fuzzy graph?

Open Problem 3. How to apply the FFF theory in real world networks with uncertainty fractional data flow?

6 Declarations

6.1 Availability of data and materials:

This paper is a pure theoretical work and has been developed without any data.

6.2 Author's contributions:

H.Z.-Conceptualization, Methodology, Formal analysis, Writing - Original Draft. J.G.-Investigation, Writing-Original Draft. W.G.-Writing, Supervision, Original Draft, Conceptualization, Methodology, Formal analysis, The authors have worked equally when writing this paper. All authors read and approved the final manuscript.

6.3 Funding:

Not applicable.

6.4 Conflict of interests:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

6.5 Acknowledgement:

Many thanks to the reviewers for their constructive comments on revisions to the article. The research is partially supported by NSFC (no. 12161094).

6.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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