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Original Study

Derivations of the stress-strain relations for viscoanelastic media and the heat equation in irreversible thermodynamic with internal variables

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Abstract

By using a procedure of classical irreversible thermodynamics with internal variable (CIT-IV), some possible interactions among heat conduction and viscous-anelastic flows for rheological media are studied. In particular, we introduce as internal variables a second rank tensor $\varepsilon^{(1)}_{\alpha\beta}$ that is contribution to inelastic strain and a vector ξ_α that influences the thermal transport phenomena and we derive the phenomenological equations for these variables in the anisotropic and isotropic cases. The stress-strain equations, the general flows and the temperature equation in visco-anelastic processes are obtained and when the medium is isotropic, we obtain that the total heat flux $\mathbf{J}^{(q)}$ can be split in two parts: a first contribution $\mathbf{J}^{(0)}$, governed by Fourier law, and a second contribution $\mathbf{J}^{(1)}$, obeying Maxwell-Cattaneo-Vernotte (M-C-V) equation.

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1 Introduction

The classical irreversible thermodynamic with internal variables (CIT-IV) means a general approach to the study of different interactions among irreversible thermodynamic processes using classical exstensive variables. After the pioneering works of Lars Onsager [1, 2], many excellent scientists: Eckart [3–6], Prigogine [7], Groot et al. [8], Meixner et al. [9], Colemann et al. [10], Truesdell [11], Kluitenberg [12–16], Maugin et al. [17, 18] developed the basics of the theory CIT-IV characterized by the introduction of thermodynamic variables, called *internal* (or *hidden*) variables.

The flexibility of the methodologies used in CIT-IV consists in the fact that *"a priori"* the physical meaning of the internal thermodynamical variables is not specified but only their influence on particular types of occurring phenomena in the considered medium is assumed. For this reason, in the following discussion, we will call these thermodynamical variables: *dynamical variables* since they have also been used, successfully, for the study of problems in dielectric and magnetic relaxation phenomena [19–22], diffusion and anelastic deformation [23–32]. It's important to note that the many theoretical results have been confirmed by the experimental data [33–40].

Parallel approaches were worked out in the frame of *Extended Irreversible Thermodynamics* (EIT) [41] and in *Rational Extended Thermodynamics* (RET) [42, 43] which, in consideration of suggestions from the kinetic

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theory, considers an entropy depending on the fluxes as well as on the classical variables. It was shown [29–31] that by using the usual procedure of CIV-IV it is possible to describe the relaxation of thermal phenomena thus obtaining results which are generally justified by additional hypothesis suggested by kinetic theory.

In this paper, we show that thermoconduction phenomena in visco-anelastic media may be studied by using the systematic procedure of CIT-IV. In order to study the behavior of rheological media, in the presence of viscous and anelastic phenomena, besides determining the stress-strain relationships it is important to obtain the heat propagation equation. For this purpose the tensor $\varepsilon^{(1)}_{\alpha\beta}$ (inelastic strain) and the vector $\boldsymbol{\xi}$ are introduced as internal variables which charaterize both stress-strain relationships and which generalize the Fourier heat equation (parabolic type equation) [44] and Maxwell-Cattaneo-Vernotte (hyperbolic type equation) [45–47].

In Section 2, introducing a vectorial dynamical variable $\mathbf{j}$ and a stress field $\tau^{(eq)}_{\alpha\beta}$ which is of a thermoelastic nature, an explicit form for the entropy production is derived.

In Sections 3-7, the phenomenological equations, which generalize the Kelvin, Jeffreys and Poynting-Thomson bodies are obtained.

In Section 8, by virtue of the dynamical variable, influencing the thermal transport phenomena, the heat equation is obtained. In particular, in the isotropic case, when the medium has symmetry properties that, under orthogonal transformations, are invariant with respect to all rotations and inversions of the frame of axes, it is obtained that the heat flux can be split in two parts: a first contribution $\mathbf{J}^{(0)}$, governed by Fourier law and a second contribution $\mathbf{J}^{(1)}$, obeying the Maxwell-Cattaneo-Vernotte equation (MCV) [45–48] in which a relaxation time is present.

Finally, in Section 9, we obtain a general temperature equation which generalizes the analogous equations of Fourier and MCV.

2 The balance equation of entropy

In the contest of irreversible processes an important role is played by the flow of heat which, classically, is not considered to be a state variable. Therefore, we will suppose that the specific entropy s, depends not only on the specific internal energy u and the total strain $\varepsilon_{\alpha\beta}$, the tensor $\varepsilon^{(1)}_{\alpha\beta}$ describing the inelastic strain and also on a vectorial dynamic variable, $\boldsymbol{\xi}$ that is an odd function of microscopic particles velocities that have influence on the propagation phenomena which occur in the medium.

$$s = s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{1}$$

where ξ_α ($\alpha = 1,2,3$) is the α-component of the vector $\boldsymbol{\xi}$.

Theorem 1 (Gibbs relation). *By using (1) we can obtain the following Gibbs relation*

$$T\,ds = du - \nu\,\tau^{(eq)}_{\alpha\beta}\,d\varepsilon_{\alpha\beta} + \nu\,\tau^{(1)}_{\alpha\beta}\,d\varepsilon^{(1)}_{\alpha\beta} + \nu\,j_\alpha\,d\xi_\alpha\,, \tag{2}$$

where ν is the specific volume, $\tau^{(eq)}_{\alpha\beta}$ is the equilibrium stress tensor, $\tau^{(1)}_{\alpha\beta}$ is the affinity-stress conjugate to $\varepsilon^{(1)}_{\alpha\beta}$, T is the absolute temperature and j_α is the α-component of the vector $\mathbf{j}$ conjugate of $d\xi_\alpha/dt$.

Proof. We define the *absolute temperature*

$$T^{-1} \overset{\text{def}}{=} \frac{\partial}{\partial u} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{3}$$

the *equilibrium-stress tensor*

$$\tau^{(eq)}_{\alpha\beta} \overset{\text{def}}{=} -\rho T \frac{\partial}{\partial \varepsilon_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{4}$$

the *affinity-stress* conjugate to $\varepsilon^{(1)}_{\alpha\beta}$

$$\tau^{(1)}_{\alpha\beta} \overset{\text{def}}{=} \rho T \frac{\partial}{\partial \varepsilon^{(1)}_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{5}$$

and the vector $\mathbf{j}$ conjugate to the internal vector variable $\boldsymbol{\xi}$

$$j_\alpha \overset{\text{def}}{=} \rho T \frac{\partial}{\partial \xi_\alpha} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{6}$$

where $\rho \overset{\text{def}}{=} \nu^{-1}$ is the mass density. By using equations (3)-(6) from (2), we obtain the differential ds of s:

$$T\,ds = du - \nu\,\tau^{(eq)}_{\alpha\beta}\,d\varepsilon_{\alpha\beta} + \nu\,\tau^{(1)}_{\alpha\beta}\,d\varepsilon^{(1)}_{\alpha\beta} + \nu\,j_\alpha\,d\xi_\alpha\,. \tag{7}$$

The relation (7) is usually called *Gibbs relation*, in which the usual summation convection for dummy is used.

From (7), we have:

$$\rho T \frac{ds}{dt} = \rho \frac{du}{dt} - \tau^{(eq)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} + \tau^{(1)}_{\alpha\beta} \frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt}\,, \tag{8}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla, \tag{9}$$

is the substantial derivative respect to time and $\mathbf{v}$ is the velocity field and ∇ is the gradient.
To analyze phenomena due to viscous flows (analogous to those which occur during flows in ordinary viscous liquids and gases) we introduce the following viscous stress tensor

$$\tau^{(vi)}_{\alpha\beta} \overset{\text{def}}{=} \tau_{\alpha\beta} - \tau^{(eq)}_{\alpha\beta}\,, \tag{10}$$

where $\tau_{\alpha\beta}$ is the mechanical stress tensor which occurs in the equation of motion

$$\rho \frac{d\mathrm{v}_\alpha}{dt} = \rho F_\alpha + \frac{\partial \tau_{\alpha\beta}}{\partial x^\beta}, \tag{11}$$

and in the first law of thermodynamics

$$\rho \frac{du}{dt} = -\nabla\cdot\mathbf{J}^{(q)} + \tau_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt}. \tag{12}$$

In (11) the force F_α is the volume force per unit of mass and in (12) the vector $\mathbf{J}^{(q)}$ is the heat flux.

Theorem 2 (Entropy production). *By using the first law of thermodynamics* (12) *the balance equation of the entropy can be obtained*

$$\rho \frac{ds}{dt} = -\nabla\cdot\left(\frac{\boldsymbol{J}^{(q)}}{T}\right) + \sigma^{(s)}, \tag{13}$$

and the entropy production

$$\sigma^{(s)} = T^{-1}\left[\boldsymbol{J}^{(q)}\cdot\left(-T^{-1}\nabla T\right) + \tau^{(vi)}_{\alpha\beta}\frac{d\,\varepsilon_{\alpha\beta}}{dt} + \tau^{(1)}_{\alpha\beta}\frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} + j_\alpha\frac{d\xi_\alpha}{dt}\right] \geq 0. \tag{14}$$

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Proof. The equation (12), by virtue of (10) becomes

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{J}^{(q)} + \left(\tau_{\alpha\beta}^{(eq)} + \tau_{\alpha\beta}^{(vi)}\right) \frac{d\varepsilon_{\alpha\beta}}{dt}, \tag{15}$$

which substituted in the equation (8) gives

$$\rho \frac{ds}{dt} = T^{-1} \left(-\nabla \cdot \mathbf{J}^{(q)} + \tau_{\alpha\beta}^{(vi)} \frac{d\varepsilon_{\alpha\beta}}{dt} + \tau_{\alpha\beta}^{(1)} \frac{d\varepsilon_{\alpha\beta}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt}\right). \tag{16}$$

Using the following identity

$$T^{-1} \nabla \cdot \mathbf{J}^{(q)} = \nabla \cdot \left(\frac{\mathbf{J}^{(q)}}{T}\right) + T^{-2} \mathbf{J}^{(q)} \cdot \nabla T, \tag{17}$$

the equation (16) gives (13) (balance equation of the entropy) and (14) (entropy production of entropy).

It is observed that the entropy production is due to three types of phenomena: the first term on the right-hand of (14) gives the contribution of the heat conduction phenomena, the second sum is the contribution of viscous phenomena, the last sum is the contribution of the variation of the dynamical variable. Each term is a production of flux ($\mathbf{J}^{(q)}$, $\tau_{\alpha\beta}^{(vi)}$, $d\varepsilon_{\alpha\beta}^{(1)}/dt$, j_α of the process and affinities conjugated to them: $T^{-1}\nabla T$, $d\,\varepsilon_{\alpha\beta}/dt$, $\tau_{\alpha\beta}^{(1)}$, $d\xi_\alpha/dt$, respectively.

3 Phenomenological equations

According to the usual procedure of non-equilibrium thermodynamics, by virtue of the form (14) for the entropy production, as anisotropic media, we have the following phenomenological equations:

$$J_\alpha^{(q)} = L_{\alpha\beta}^{(q)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\beta}\right) + L_{\alpha(\mu\nu)}^{(q)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{\alpha(\mu\nu)}^{(q)(1)} \tau_{\mu\nu}^{(1)} + L_{\alpha\mu}^{(q)(\xi)} \frac{d\xi_\mu}{dt}, \tag{18}$$

$$\tau_{\alpha\beta}^{(vi)} = L_{(\alpha\beta)\mu}^{(0)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\mu}\right) + L_{(\alpha\beta)(\mu\nu)}^{(0)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{(\alpha\beta)(\mu\nu)}^{(0)(1)} \tau_{\mu\nu}^{(1)} + L_{(\alpha\beta)\mu}^{(0)(\xi)} \frac{d\xi_\mu}{dt}, \tag{19}$$

$$\frac{d\varepsilon_{\alpha\beta}^{(1)}}{dt} = L_{(\alpha\beta)\mu}^{(1)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\mu}\right) + L_{(\alpha\beta)(\mu\nu)}^{(1)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{(\alpha\beta)(\mu\nu)}^{(1)(1)} \tau_{\mu\nu}^{(1)} + L_{(\alpha\beta)\mu}^{(1)(\xi)} \frac{d\xi_\mu}{dt}, \tag{20}$$

$$j_\alpha = L_{\alpha\beta}^{(\xi)(q)} \left(-T^{-1} \frac{\partial T}{\partial x^\beta}\right) + L_{\alpha(\mu\nu)}^{(\xi)(0)} \frac{d\varepsilon_{\mu\nu}}{dt} + L_{\alpha(\mu\nu)}^{(\xi)(1)} \tau_{\mu\nu}^{(\xi)(1)} + L_{\alpha\beta}^{(\xi)(\xi)} \frac{d\xi_\beta}{dt}. \tag{21}$$

The tensors L are called phenomenological tensors and the indices of these tensors enclosed in round brackets mean that they are symmetrical because the tensors $\varepsilon_{\mu\nu}$, $\varepsilon_{\alpha\beta}^{(1)}$, $\tau_{\alpha\beta}^{(1)}$ and $\tau_{\alpha\beta}^{(vi)}$ are symmetric. The first of these equations may be regarded as a generalization of Fourier's law. Equation (19) describes the viscous flow phenomenon and it may be considered to be a generalization of Stockes-Navier's law. Finally, the equations (20) and (21) are the phenomenological equations for the irreversible process of the dynamic degrees of freedom. Substituting (18)-(21) into (14) one has

$$\begin{aligned} T\sigma^{(s)} =& L_{\alpha\beta}^{(q)(q)} \left(T^{-2} \frac{\partial T}{\partial x^\alpha} \frac{\partial T}{\partial x^\beta}\right) + \left[L_{\alpha\beta}^{(\xi)(q)} + L_{\alpha\beta}^{(q)(\xi)}\right] \frac{d\xi_\alpha}{dt} \left(-T^{-1} \frac{\partial T}{\partial x^\beta}\right) + \\ &+ L_{(\alpha\beta)(\mu\nu)}^{(0)(0)} \frac{d\varepsilon_{\alpha\beta}}{dt} \frac{d\varepsilon_{\mu\nu}}{dt} + \\ &+ \left[L_{(\alpha\beta)(\mu\nu)}^{(1)(0)} + L_{(\mu\nu)(\alpha\beta)}^{(0)(1)}\right] \tau_{\alpha\beta}^{(1)} \frac{d\varepsilon_{\mu\nu}}{dt} + \\ &+ L_{(\alpha\beta)(\mu\nu)}^{(1)(1)} \tau_{\alpha\beta}^{(1)} \tau_{\mu\nu}^{(1)} + \\ &+ L_{\alpha\beta}^{(\xi)(\xi)} \frac{d\xi_\alpha}{dt} \frac{d\xi_\beta}{dt}. \end{aligned} \tag{22}$$

This form for the entropy production ($s^{(s)} \geq 0$) is a useful detail if isotropic media are considered in Section 5.

4 Symmetric relations

Since the time derivatives $d\varepsilon_{\alpha\beta}/dt$, $d\varepsilon^{(1)}_{\alpha\beta}/dt$, the heat flow $\mathbf{J}^{(q)}$ and $\mathbf{j}$ vector conjugate to the internal variable ξ_α are odd functions of the microscopic particle velocities and the stress $\tau^{(vi)}_{\alpha\beta}$, $\tau^{(1)}_{\alpha\beta}$, $d\xi_\alpha/dt$ and the temperature gradient $\partial T/\partial t$ are even functions of these velocities, the Onsager-Casimir reciprocity relation reads [8]:

$$L^{(q)(q)}_{\alpha\beta} = L^{(q)(q)}_{\beta\alpha}\ ,\ L^{(\xi)(q)}_{\alpha\beta} = L^{(\xi)(q)}_{\beta\alpha}\ ,\ L^{(q)(\xi)}_{\alpha\beta} = L^{(q)(\xi)}_{\beta\alpha}\ ,\ L^{(\xi)(\xi)}_{\alpha\beta} = L^{(\xi)(\xi)}_{\beta\alpha}\ , \tag{23}$$

$$L^{(0)(0)}_{(\alpha\beta)(\mu\nu)} = L^{(0)(0)}_{(\mu\nu)(\alpha\beta)}\ ,\ L^{(1)(1)}_{(\alpha\beta)(\mu\nu)} = L^{(1)(1)}_{(\mu\nu)(\alpha\beta)}\ , \tag{24}$$

$$L^{(1)(0)}_{(\alpha\beta)(\mu\nu)} = L^{(1)(0)}_{(\mu\nu)(\alpha\beta)}\ ,\ L^{(0)(1)}_{(\alpha\beta)(\mu\nu)} = L^{(0)(1)}_{(\mu\nu)(\alpha\beta)}\ , \tag{25}$$

$$L^{(1)(0)}_{(\alpha\beta)(\mu\nu)} = -L^{(0)(1)}_{(\mu\nu)(\alpha\beta)}\ . \tag{26}$$

The equations (23)-(26) reduce the number of indipendent components of the phenomenological tensors.

5 Phenomenological equations for isotropic media

In this section, we consider perfect isotropic media for which the symmetry properties are invariant under orthogonal transformations with respect to all rotations and to inversion of the frame of axes.
In this case, it can be shown [8, 49] that the phenomenological tensors of odd order vanish while even tensors can have the following forms:

$$L^{(q)(q)}_{\alpha\beta} = T\lambda^{(q,q)}\delta_{\alpha\beta}\quad ,\quad L^{(q)(\xi)}_{\alpha\beta} = \lambda^{(q,\xi)}\delta_{\alpha\beta}\ , \tag{27}$$

$$L^{(\xi)(q)}_{\alpha\beta} = T\lambda^{(\xi,q)}\delta_{\alpha\beta}\quad ,\quad L^{(\xi)(\xi)} = \lambda^{(\xi,\xi)}\delta_{\alpha\beta}\ , \tag{28}$$

$$L^{(i)(k)}_{(\alpha\beta)(\mu\nu)} = \frac{1}{2}\eta^{(i,k)}_s\left(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu}\right)+\frac{1}{3}\left(\eta^{(i,k)}_v-\eta^{(i,k)}_s\right)\delta_{\alpha\beta}\delta_{\mu\nu}\quad (i,k=0,1). \tag{29}$$

By using (27)-(29), the equations (18)-(21) become

$$J^{(q)}_\alpha = -\lambda^{(q,q)}\frac{\partial T}{\partial x^\alpha}+\lambda^{(q,\xi)}\frac{d\xi_\alpha}{dt}\ , \tag{30}$$

$$\tau^{(vi)}_{\alpha\beta} = \eta^{(0,0)}_s\frac{d\varepsilon_{\alpha\beta}}{dt}+(\eta^{(0,0)}_v-\eta^{(0,0)}_s)\frac{d\varepsilon}{dt}\delta_{\alpha\beta}+\eta^{(0,1)}_s\tau^{(1)}_{\alpha\beta}+(\eta^{(0,1)}_v-\eta^{(0,1)}_s)\tau^{(1)}\delta_{\alpha\beta}\ , \tag{31}$$

$$\frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} = \eta^{(1,0)}_s\frac{d\varepsilon_{\alpha\beta}}{dt}+(\eta^{(1,0)}_v-\eta^{(1,0)}_s)\frac{d\varepsilon}{dt}\delta_{\alpha\beta}+\eta^{(1,1)}_s\tau^{(1)}_{\alpha\beta}+(\eta^{(1,1)}_v-\eta^{(1,1)}_s)\tau^{(1)}\delta_{\alpha\beta}\ , \tag{32}$$

$$j_\alpha = -\lambda^{(\xi,q)}\frac{\partial T}{\partial x^\alpha}+\lambda^{(\xi,\xi)}\frac{d\xi_\alpha}{dt}\ . \tag{33}$$

The scalar quantities λ, $\eta^{(0,0)}_s$ and $\eta^{(0,0)}_v$ are called phenomenological coefficients. In particular $\eta^{(0,0)}_s$ and $\eta^{(0,0)}_v$ may be called the shear viscosity and the volume viscosity, respectively. These coefficients also occur in the theory of ordinary (Stokes-Navier) viscous fluids. Equation (22), by virtue of (30)-(33), becomes

$$\begin{aligned} T\sigma^{(s)} = {}&\lambda^{(q,q)}\left(\frac{\partial T}{\partial x^\alpha}\right)^2+\eta^{(0,0)}_s\left(\frac{d\varepsilon_{\alpha\beta}}{dt}\right)^2+3\left(\eta^{(0,0)}_v-\eta^{(0,0)}_s\right)\left(\frac{d\varepsilon}{dt}\right)^2+\\ &+\eta^{(1,1)}_s(\tau^{(1)})^2+3\left(\eta^{(1,1)}_v-\eta^{(1,1)}_s\right)(\tau^{(1)})^2+\lambda^{(\xi,\xi)}\left(\frac{d\xi_\alpha}{dt}\right)^2. \end{aligned} \tag{34}$$

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By virtue of the positive definite character of the entropy production, several inequalities for the phenomenological coefficients may be derived.
For example, we have

$$\lambda^{(q,q)} \geq 0 \quad , \quad \eta_s^{(0,0)} \geq 0 \quad , \quad \eta_v^{(0,0)} \geq \eta_s^{(0,0)} , \tag{35}$$

$$\eta_s^{(1,1)} \geq 0 \quad , \quad \eta_v^{(1,1)} \geq \eta_s^{(1,1)} \quad , \quad \lambda^{(\xi,\xi)} \geq 0 . \tag{36}$$

The coefficients $\lambda^{(q,\xi)}$ and $\lambda^{(\xi,q)}$ represent possible cross effects between the heat flux and the irreversible process of the dynamic degrees of freedom.

6 Linear equations of state

Let f the specific free energy of the medium

$$f = u - T\,s. \tag{37}$$

With the aid of Gibbs relation (7), we have:

$$df = \nu\,\tau_{\alpha\beta}^{(eq)} d\varepsilon_{\alpha\beta} - \nu\tau_{\alpha\beta}^{(1)} d\varepsilon_{\alpha\beta}^{(1)} - \nu\, j_\alpha\, d\xi_\alpha - s dT\,, \tag{38}$$

where

$$\nu\,\tau_{\alpha\beta}^{(eq)} = \frac{\partial f}{\partial \varepsilon_{\alpha\beta}}, \quad \nu\,\tau_{\alpha\beta}^{(1)} = -\frac{\partial f}{\partial \varepsilon_{\alpha\beta}^{(1)}}, \quad \nu\, j_\alpha = -\frac{\partial f}{\partial \xi_\alpha}, \quad s = -\frac{\partial f}{\partial T}. \tag{39}$$

Let us choose a configuration $\Sigma^{(0)}$ with uniform temperature T_0 and in which s_0 is the specific entropy, v_0 is the specific volume and $(\tau_{\alpha\beta}^{(eq)})_0$, $(\tau_{\alpha\beta}^{(1)})_0$, $(\tau_{\alpha\beta})_0$, $(\varepsilon_{\alpha\beta})_0$, $(\varepsilon_{\alpha\beta}^{(1)})_0$ and $(j_\alpha)_0$ are zero.
Let us suppose that the deviation from $\Sigma^{(0)}$ is sufficiently small (with $\rho \approx \rho_0$) and the free energy for isotropic medium can be written in the following form

$$\begin{aligned} f = \nu_0 \Big\{ & \frac{1}{2}\Big[a_{\alpha\beta\mu\nu}^{(0)(0)}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\mu\nu} + a_{\alpha\beta\mu\nu}^{(1)(1)}\,\varepsilon_{\alpha\beta}^{(1)}\,\varepsilon_{\mu\nu}^{(1)} + a_{\alpha\beta}^{(\xi)(\xi)}\,\xi_\alpha\,\xi_\beta\Big] \\ & + a_{\alpha\beta\mu\nu}^{(0)(1)}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\mu\nu}^{(1)} + a_{\alpha\beta\mu}^{(0)(\xi)}\,\varepsilon_{\alpha\beta}^{(1)}\,\xi_\mu + a_{\alpha\beta\mu}^{(1)(\xi)}\,\varepsilon_{\alpha\beta}^{(1)}\,\xi_\mu \\ & + (T-T_0)\Big[a_{\alpha\beta}^{(0)(T)}\varepsilon_{\alpha\beta} + a_{\alpha\beta}^{(1)(T)}\,\varepsilon_{\alpha\beta}^{(1)} + a_{\alpha}^{(\xi)(T)}\xi_\alpha\Big]\Big\} - \Psi(T)\,. \end{aligned} \tag{40}$$

where Ψ is some function of the temperature. In (40), the tensors "a" and the vector $a_\alpha^{(\xi)(T)}$, are constants (i.e. they do not depend on temperature and the strains) and are determined by the physical properties of the medium in the reference state.
From $(39)_4$ we have

$$s = -\nu_0\Big(a_{\alpha\beta}^{(0)(T)}\varepsilon_{\alpha\beta} + a_{\alpha\beta}^{(1)(T)}\varepsilon_{\alpha\beta}^{(1)} + a_{\alpha}^{(\xi)(T)}\,\xi_\alpha\Big) + \frac{d\Psi}{dT}\,, \tag{41}$$

and as $u = f + T\,s$ we obtain

$$\begin{aligned} u = \nu_0 \Big\{ & \frac{1}{2}\Big[a_{\alpha\beta\mu\nu}^{(0)(0)}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\mu\nu} + a_{\alpha\beta\mu\nu}^{(1)(1)}\,\varepsilon_{\alpha\beta}^{(1)}\,\varepsilon_{\mu\nu}^{(1)} + a_{\alpha\beta}^{(\xi)(\xi)}\,\xi_\alpha\,\xi_\beta\Big] \\ & - T_0\Big[a_{\alpha\beta}^{(0)(T)}\varepsilon_{\alpha\beta} + a_{\alpha\beta}^{(1)(T)}\,\varepsilon_{\alpha\beta}^{(1)} + a_{\alpha}^{(\xi)(T)}\,\xi_\alpha\Big]\Big\} + T\,\frac{d\Psi}{dT} - \Psi(T)\,. \end{aligned} \tag{42}$$

The specific heat at constant deformation, $c_{(\varepsilon)}$, may be defined by

$$c_{(\varepsilon)} = \frac{\partial u}{\partial T}\,, \tag{43}$$

and from (42) it obtains

$$c_{(\varepsilon)} = T\,\frac{d^2\Psi}{dT^2}\,. \tag{44}$$

By integrating the equation (44), one has

$$\Psi = c_{(\varepsilon)}T\log\left(\frac{T}{T_0}\right) + s_0, T - c_{(\varepsilon)}(T-T_0) - u_0\,. \tag{45}$$

If $c_{(\varepsilon)}$ is constant where s_0 and u_0 are integration constants that are the specific entropy and the specific energy in the reference state, respectively.
From $(39)_{1,2,3}$ we obtain

$$\tau^{(eq)}_{\alpha\beta} = a^{(0)(0)}_{\alpha\beta\mu\nu}\,\varepsilon_{\mu\nu} + a^{(0)(1)}_{\alpha\beta\mu\nu}\,\varepsilon^{(1)}_{\mu\nu} + a^{(0)(\xi)}_{\alpha\beta\mu}\,\xi_\mu + a^{(0)(T)}_{\alpha\beta}\,(T-T_0)\,, \tag{46}$$

$$-\tau^{(1)}_{\alpha\beta} = a^{(1)(1)}_{\alpha\beta\mu\nu}\,\varepsilon^{(1)}_{\mu\nu} + a^{(0)(1)}_{\mu\nu\alpha\beta}\,\varepsilon_{\mu\nu} + a^{(1)(\xi)}_{\alpha\beta\mu}\,\xi_\mu + a^{(1)(T)}_{\alpha\beta}\,(T-T_0), \tag{47}$$

$$-j_\alpha = a^{(\xi)(\xi)}_{\alpha\beta}\,\xi_\beta + a^{(0)(\xi)}_{\alpha\mu\nu}\,\varepsilon_{\mu\nu} + a^{(1)(\xi)}_{\alpha\mu\nu}\,\varepsilon^{(1)}_{\mu\nu} + a^{(\xi)(T)}_{\alpha}\,(T-T_0)\,. \tag{48}$$

We consider that the medium is isotropic and the even order tensors have the following forms

$$a^{(0)(0)}_{\alpha\beta\mu\nu} = \frac{1}{2}a^{(0,0)}(\delta_{\alpha\mu}\,\delta_{\beta\nu} + \delta_{\alpha\nu}\,\delta_{\beta\mu}) + \frac{1}{3}(b^{(0,0)} - a^{(0,0)})\delta_{\alpha\beta}\,\delta_{\mu\nu}, \tag{49}$$

$$a^{(0)(1)}_{\alpha\beta\mu\nu} = \frac{1}{2}a^{(0,1)}(\delta_{\alpha\mu}\,\delta_{\beta\nu} + \delta_{\alpha\nu}\,\delta_{\beta\mu}) + \frac{1}{3}(b^{(0,1)} - a^{(0,1)})\delta_{\alpha\beta}\,\delta_{\mu\nu}, \tag{50}$$

$$a^{(1)(1)}_{\alpha\beta\mu\nu} = \frac{1}{2}a^{(1,1)}(\delta_{\alpha\mu}\,\delta_{\beta\nu} + \delta_{\alpha\nu}\,\delta_{\beta\mu}) + \frac{1}{3}(b^{(1,1)} - a^{(1,1)})\delta_{\alpha\beta}\,\delta_{\mu\nu}, \tag{51}$$

$$a^{(0)(1)}_{\mu\nu\alpha\beta} = \frac{1}{2}a^{(0,1)}(\delta_{\mu\alpha}\,\delta_{\nu\beta} + \delta_{\mu\beta}\,\delta_{\nu\alpha}) + \frac{1}{3}(b^{(0,1)} - a^{(0,1)})\delta_{\mu\nu}\,\delta_{\alpha\beta}, \tag{52}$$

$$a^{(\xi)(\xi)}_{\alpha\beta} = a^{(\xi,\xi)}\delta_{\alpha\beta}, \quad a^{(0)(T)}_{\alpha\beta} = a^{(0,T)}\delta_{\alpha\beta}, \quad a^{(1)(T)}_{\alpha\beta} = a^{(1,T)}\delta_{\alpha\beta}\,, \tag{53}$$

and the odd tensors are zero. Using (49)-(53), the equations (46)-(48) become

$$\tau^{(eq)}_{\alpha\beta} = a^{(0,0)}\,\varepsilon_{\alpha\beta} + (b^{(0,0)} - a^{(0,0)})\,\varepsilon\,\delta_{\alpha\beta} + a^{(0,1)}\varepsilon^{(1)}_{\alpha\beta} + (b^{(0,1)} - a^{(0,1)})\,\varepsilon^{(1)}\delta_{\alpha\beta} + a^{(0,T)}\,(T-T_0)\,\delta_{\alpha\beta}\,. \tag{54}$$

$$\tau^{(1)}_{\alpha\beta} = a^{(1,1)}\,\varepsilon^{(1)}_{\alpha\beta} + (b^{(1,1)} - a^{(1,1)})\,\varepsilon^{(1)}\,\delta_{\alpha\beta} + a^{(0,1)}\varepsilon_{\alpha\beta} + (b^{(0,1)} - a^{(0,1)})\,\varepsilon\,\delta_{\alpha\beta} + a^{(1,T)}\,(T-T_0)\,\delta_{\alpha\beta}\,, \tag{55}$$

$$j_\alpha = -a^{(\xi,\xi)}\,\xi_\alpha\,. \tag{56}$$

The equation (54) is the generalization of the Duhamel-Neumann law for thermoelastic phenomena in isotropic media. If $a^{(0,T)}$ vanishes or if one considers isothermal processes at the temperature T_0 is analogous to Hooke's law for isotropic media [50]. The equation of state (55) and (56) are specific for the theory developed in this paper.

7 Stress-strain equations

In the following the deviator $\tilde{A}_{\alpha\beta}$ and the scalar part A of arbitrary tensor field $A_{\alpha\beta}$ are defined by

$$\begin{cases} \tilde{A}_{\alpha\beta} = A_{\alpha\beta} - A\,\delta_{\alpha\beta}\,, \\ A = \frac{1}{3}\,\Sigma_{\gamma=1}^{3} A_{\gamma\gamma}\,. \end{cases} \tag{57}$$

Using (54), (55), (31) and (32), we have

$$\tilde{\tau}_{\alpha\beta}^{(eq)} = a^{(0,0)}\,\tilde{\varepsilon}_{\alpha\beta} + a^{(0,1)}\,\tilde{\varepsilon}_{\alpha\beta}^{(1)}\,, \tag{58}$$

$$\tilde{\tau}_{\alpha\beta}^{(1)} = a^{(1,1)}\,\tilde{\varepsilon}_{\alpha\beta}^{(1)} + a^{(0,1)}\,\tilde{\varepsilon}_{\alpha\beta}\,. \tag{59}$$

$$\tilde{\tau}_{\alpha\beta}^{(vi)} = \eta_s^{(0,0)}\,\frac{d\,\tilde{\varepsilon}_{\alpha\beta}}{dt} + \eta_s^{(0,1)}\,\tilde{\tau}_{\alpha\beta}^{(1)}\,, \tag{60}$$

$$\frac{d\tilde{\varepsilon}_{\alpha\beta}^{(1)}}{dt} = \eta_s^{(1,0)}\,\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt} + \eta_s^{(1,1)}\,\tilde{\tau}_{\alpha\beta}^{(1)}\,. \tag{61}$$

The equations (58) and (59) are generalization of the Duhamel-Neuman law for thermoviscoanelastic phenomena in isotropic media. From (10), using (58) and (60), we have

$$\tilde{\tau}_{\alpha\beta} = \chi_s^{(1)}\,\tilde{\varepsilon}_{\alpha\beta} + \chi_s^{(2)}\,\tilde{\varepsilon}_{\alpha\beta}^{(1)} + \eta_s^{(0,0)}\,\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}\,, \tag{62}$$

where

$$\chi_s^{(1)} = a^{(0,0)} - a^{(0,1)}\,\eta_s^{(0,1)}\,, \tag{63}$$

$$\chi_s^{(2)} = a^{(0,1)} - a^{(1,1)}\,\eta_s^{(0,1)}\,. \tag{64}$$

Eliminating $\tau_{\alpha\beta}^{(1)}$ between (61) and (59), we obtain

$$\frac{d\tilde{\varepsilon}_{\alpha\beta}^{(1)}}{dt} = \eta_s^{(1,0)}\,\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt} - \eta_s^{(1,1)}\,a^{(1,1)}\,\tilde{\varepsilon}_{\alpha\beta}^{(1)} - \eta_s^{(1,1)}\,a^{(0,1)}\,\tilde{\varepsilon}_{\alpha\beta}\,, \tag{65}$$

The equation (65), using $\tilde{\varepsilon}_{\alpha\beta}^{(1)}$ from (62), becomes

$$\frac{d\tilde{\tau}_{\alpha\beta}}{dt} + R_{(d)0}^{(\tau)}\,\tilde{\tau}_{\alpha\beta} = R_{(d)2}^{(\varepsilon)}\,\frac{d^2\tilde{\varepsilon}_{\alpha\beta}}{dt^2} + R_{(d)1}^{(\varepsilon)}\,\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt} + R_{(d)0}^{(\varepsilon)}\,\tilde{\varepsilon}_{\alpha\beta}\,, \tag{66}$$

where

$$R_{(d)0}^{(\tau)} = \eta_s^{(1,1)}\,a^{(1,1)}, \tag{67}$$

$$R_{(d)2}^{(\varepsilon)} = \eta_s^{(0,0)}, \tag{68}$$

$$R_{(d)1}^{(\varepsilon)} = \chi_s^{(1)} + \eta_s^{(1,0)}\,\chi_s^{(2)} + \eta_s^{(0,0)}\,\eta_s^{(1,1)}\,a^{(1,1)}\,, \tag{69}$$

$$R_{(d)0}^{(\varepsilon)} = \eta_s^{(1,1)}(\chi_s^{(1)}\,a^{(1,1)} - a^{(0,1)}\,\chi_s^{(2)}). \tag{70}$$

Let's examine the scalar parts of the tensors. Using (54), (55), (31) and (32) we have

$$\tau^{(eq)} = b^{(0,0)}\,\varepsilon + b^{(0,1)}\,\varepsilon^{(1)} + a^{(0,T)}(T - T_0) \tag{71}$$

$$-\tau^{(1)} = b^{(1,1)}\,\varepsilon^{(1)} + b^{(0,1)}\,\varepsilon + a^{(1,T)}\,(T - T_0)\,, \tag{72}$$

$$\tau^{(vi)} = \eta_v^{(0,0)}\,\frac{d\,\varepsilon}{dt} + \eta_v^{(0,1)}\,\tau^{(1)}\,, \tag{73}$$

$$\frac{d\varepsilon^{(1)}}{dt} = \eta_v^{(1,0)}\,\frac{d\varepsilon}{dt} + \eta_v^{(1,1)}\,\tau^{(1)}\,. \tag{74}$$

As $\tau = \tau^{(eq)} + \tau^{(vi)}$, using (71), (73) and (72) we have:

$$\tau = \chi_v^{(1)}\,\varepsilon + \chi_v^{(2)}\,\varepsilon^{(1)} + \eta_v^{(0,0)}\,\frac{d\varepsilon}{dt} + \chi_v^{(T)}\,(T - T_0)\,, \tag{75}$$

with

$$\chi_v^{(1)} = b^{(0,0)} - \eta_v^{(0,1)}\,b^{(0,1)}\,, \tag{76}$$

$$\chi_v^{(2)} = b^{(0,1)} - \eta_v^{(0,1)}\,b^{(1,1)}\,, \tag{77}$$

$$\chi_v^{(T)} = a^{(0,T)} - \eta_v^{(0,1)}\,a^{(1,T)}\,. \tag{78}$$

Using (72), the equation (74) becomes

$$\frac{d\,\varepsilon^{(1)}}{dt} = \eta_v^{(1,0)}\,\frac{d\,\varepsilon}{dt} - \eta_v^{(1,1)}\,b^{(1,1)}\,\varepsilon^{(1)} - \eta_v^{(1,1)}\,b^{(0,1)}\,\varepsilon - \eta_v^{(1,1)}\,a^{(0,T)}\,(T - T_0)\,. \tag{79}$$

From (75), we get

$$\eta_v^{(2)}\,\varepsilon^{(1)} = \tau - \chi_v^{(1)}\,\varepsilon - \eta_v^{(0,0)}\,\frac{d\varepsilon}{dt} - \chi_v^{(T)}\,(T - T_0)\,, \tag{80}$$

and replacing in (79) we have

$$\frac{d\,\tau}{dt} + R_{(v)0}^{(\tau)}\,\tau = R_{(v)2}^{(\varepsilon)}\,\frac{d^2\varepsilon}{dt^2} + R_{(v)1}^{(\varepsilon)}\,\frac{d\varepsilon}{dt} + R_{(v)0}^{(\varepsilon)}\,\varepsilon + R_{(v)1}^{(T)}\,\frac{d\,T}{dt} + R_{(v)0}^{(T)}\,(T - T_0) \tag{81}$$

where

$$R_{(v)0}^{(\tau)} = b^{(1,1)}\,\eta_v^{(1,1)}, \tag{82}$$

$$R_{(v)2}^{(\varepsilon)} = \eta_v^{(0,0)}, \tag{83}$$

$$R_{(v)1}^{(\varepsilon)} = \chi_v^{(1)} + \eta_v^{(1,0)}\,\chi_v^{(2)} + \eta_v^{(1,1)}\,b^{(1,1)}\,\eta_v^{(0,0)}\,, \tag{84}$$

$$R_{(v)0}^{(\varepsilon)} = \eta_v^{(1,1)}\,(b^{(1,1)}\,\chi_v^{(1)} - b^{(0,0)}\,\chi_v^{(2)}), \tag{85}$$

$$R_{(v)1}^{(T)} = \chi_v^{(T)}\,, \tag{86}$$

$$R_{(v)0}^{(T)} = \eta_v^{(1,1)}\,(b^{(1,1)}\chi_v^{(T)} - a^{(1,T)}\chi_v^{(2)}). \tag{87}$$

The equations (66) and (81) generalize the stress-strain relations for ordinary viscous fluids, for elastic media end for Maxwell, Kelvin, Jeffreys and Poynting-Thomson bodies.

8 Heat flows in visco-anelastic processes

Using the phenomenological equations (Section 5) and the linear equations of state (Section 6) we obtain the following results:

Theorem 3 (Decomposition of the heat flow). *By using equation (56) and (33) the heat flow $\boldsymbol{J}^{(q)}$ can be split in two parts*

$$\boldsymbol{J}^{(q)} = \boldsymbol{J}^{(0)} + \boldsymbol{J}^{(1)}, \tag{88}$$

where $\boldsymbol{J}^{(0)}$ satisfies the Fourier's law and $\boldsymbol{J}^{(1)}$ obeys the Maxwell-Cattaneo-Vernotte law.

Proof. Assuming $\lambda^{(\xi)(\xi)} \neq 0$, by virtue of equation (56), from the equation (33) we obtain

$$\frac{d\xi_\alpha}{dt} = -\frac{a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\xi_\alpha + \frac{\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\frac{\partial T}{\partial x^\alpha}. \tag{89}$$

Substituting the equation (89) into equation (30), one obtains

$$\mathbf{J}^{(q)} = \mathbf{J}^{(0)} + \mathbf{J}^{(1)}, \tag{90}$$

where

$$\mathbf{J}^{(0)} = -\lambda^{(0)}\nabla T, \tag{91}$$

with

$$\lambda^{(0)} = \lambda^{(q,q)} - \frac{\lambda^{(q,\xi)}\lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}, \tag{92}$$

and

$$\mathbf{J}^{(1)} = -\frac{\lambda^{(q,\xi)}a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\boldsymbol{\xi}. \tag{93}$$

By virtue of equation (89), we see that $\mathbf{J}^{(1)}$ satisfies the following equation:

$$t_r\frac{d\mathbf{J}^{(1)}}{dt} + \mathbf{J}^{(1)} = -\frac{\lambda^{(\xi,q)}\lambda^{(q,\xi)}}{\lambda^{(\xi,\xi)}}\nabla T, \tag{94}$$

where

$$t_r = \frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}, \tag{95}$$

is the relaxation time.

From (90) we can see that the heat current is split into two parts: the first one, $\mathbf{J}^{(0)}$, is governed by Fourier's law (91) while the second part, $\mathbf{J}^{(1)}$, is governed by the MCV type constitutive equation (94).

9 Temperature equation

We consider an undeformable medium at rest, i.e.

$$\frac{d}{dt} = \frac{\partial}{\partial t}, \tag{96}$$

and we assume that the specific internal energy u is related to the temperature by

$$du = c_{(v)}dT, \tag{97}$$

being $c_{(v)}$ the heat capacity per unit of mass at constant volume.

Theorem 4 (Generalization of temperature equation). *By using (91) and the first law of thermodynamics we obtain the following temperature equation*

$$t_r\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha'\triangle T + t_r\eta'\frac{\partial}{\partial t}\triangle T, \tag{98}$$

where α' and η' are parameters expressed through the phenomenological and state coefficients.

Proof. By virtue of (96) and (97) the first law of thermostatic (12) becomes

$$\rho\, c_{(v)} \frac{\partial T}{\partial t} + \nabla \cdot \left(\mathbf{J}^{(0)} + \mathbf{J}^{(1)} \right) = 0, \tag{99}$$

and from (99) we obtain

$$\rho\, c_{(v)}\, t_r \frac{\partial^2 T}{\partial t^2} + t_r \nabla \cdot \left[\frac{\partial \mathbf{J}^{(0)}}{\partial t} + \frac{\partial \mathbf{J}^{(1)}}{\partial t} \right] = 0\,. \tag{100}$$

Summing (99) and (100), by using (91) and (94), we have the following temperature equation

$$t_r \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha' \triangle T + t_r\, \eta' \frac{\partial \triangle T}{\partial t}, \tag{101}$$

where

$$\begin{cases} \alpha' = \dfrac{\lambda^{(0)} \lambda^{(\xi,\xi)} + \lambda^{(\xi,q)} \lambda^{(q,\xi)}}{\rho\, c_{(v)}\, \lambda^{(\xi,\xi)}} \\ \eta' = \dfrac{\lambda^{(0)}}{\rho\, c_{(v)}}\,. \end{cases} \tag{102}$$

The equation (101) generalizes the temperature equation with analogous equations of Fourier and Maxwell-Cattaneo-Vernotte.

9.1 Special cases

- If the relaxation time t_r (95) can be neglected with respect to the heat propagation time t ($t_r \ll t$) the equation (101) becomes

$$\frac{\partial T}{\partial t} = \alpha' \triangle T\,. \tag{103}$$

  i.e. the Fourier equation.

- If the phenomenological coefficients satisfy the following condition

$$\lambda^{(q,q)} = \frac{\lambda^{(q,\xi)}, \lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}, \tag{104}$$

  from (92) one has: $\lambda^{(0)} = 0$ and the equations (102) become

$$\begin{cases} \alpha' = \dfrac{\lambda^{(\xi,q)} \lambda^{(q,\xi)}}{\rho\, c_{(v)}\, \lambda^{(\xi,\xi)}} \\ \eta' = 0 \end{cases}, \tag{105}$$

and the equation (101) becomes

$$t_r \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \alpha' \triangle T = 0\,, \tag{106}$$

being the Maxwell-Cattaneo-Vernotte.

10 Remarks and conclusion

- The Fourier equation (103) is a parabolic equation and therefore the heat flux can be considered of infinite speed. Several scientists view this result as a paradox. In 1992, Fichera [51] states that this claim is unfounded as Fourier's theory is not correctly interpreted. In our opinion the Fourier result is justified by the value of the phenomenological parameters of the medium and therefore it cannot be said that Fourier's theory leads to a paradox.
- The Maxwell-Cattaneo-Vernotte law leads to a hyperbolic equation obtained with intuitive hypotheses but without justifying it in the context of a general theory. Various authors, for instance [41–43], have proposed several theories suggested by kinetic theory in order to justify the Maxwell-Cattaneo-Vernotte equation by assuming that entropy depends on fluxes instead of state variables.

In this paper, without *a priori* hypotheses and the important role of phenomenological and state coefficients, we obtained the stress-strain relations for viscoanelastic media and an equation for the temperature which generalizes the Fourier and MCV temperature equations.

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All data that support the findings of this study are included within the article.

11.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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