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Original Study

New closed form solutions of some nonlinear pseudo-parabolic models via a new extended direct algebraic method

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Abstract

Our investigation delves into a specific category of nonlinear pseudo-parabolic partial differential equations (PDEs) that emerges from physical models. This set of equations includes the one-dimensional (1D) Oskolkov equation, the Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation, the generalized hyperelastic rod wave (HERW) equation, and the Oskolkov Benjamin Bona Mahony Burgers (OBBMB) equation. We employ the new extended direct algebraic (NEDA) method to tackle these equations. The NEDA method serves as a powerful tool for our analysis, enabling us to obtain solutions grounded in various mathematical functions, such as hyperbolic, trigonometric, rational, exponential, and polynomial functions. As we delve into the physical implications of these solutions, we uncover complex structures with well-known characteristics. These include entities like dark, bright, singular, combined dark-bright solitons, dark-singular-combined solitons, solitary wave solutions, and others.

Keywords: Benjamin-Bona-Mahony-Peregrine equation, generalized hyper elastic rod wave equation, pseudo-parabolic class, new extended direct algebraic method.

AMS 2020 codes: 35C07; 35Dxx; 45Kxx; 65Mxx; 45K05.

1 Introduction

Numerous instances of specific models or phenomena are described by nonlinear differential equations (DEs). These equations prove invaluable in capturing and analyzing the dynamic behaviors inherent to the models. Nonlinear PDEs, in particular, hold a remarkable ability to model a wide range of intriguing scenarios. Among these, pseudo-parabolic equations stand out. They incorporate time derivatives within their highest order term and are employed to describe specific physical phenomena. A noteworthy member of this category is the generalized Benjamin Bona Mahony Burgers equation, which

$$v_t - v_{xxt} - \alpha v_{xx} + \gamma v_x + g(v)_x = 0, \quad (1)$$

where $g(v)$ is a C^2 -smooth nonlinear function, α , γ are real constants, $v(x, t)$ indicates velocity of fluid along (horizontal) x -direction. In the special case when we take $g(v)_x = vv_x$ with $\alpha = 0$, $\gamma = 1$ in equation (1), we get

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the well-known KdV equation

$$v_t + v_{xxt} + v_x + vv_x = 0. \quad (2)$$

Equation (2) was proposed by Peregrine [1] and Benjamin et al. [2] and interprets a different regular long-wave in comparison with (1). In the second special case if we take $g(v)_x = \theta vv_x + \beta v_{xxx}$ in (1), we get another well known BBMPB equation

$$v_t - v_{xxt} - \alpha v_{xx} + \gamma v_x + \theta vv_x + \beta v_{xxx} = 0. \quad (3)$$

Next for the third special case we can take $\alpha = \beta = 0$ in (3), we get the well known BBM equation

$$v_t - v_{xxt} + \gamma v_x + \theta vv_x = 0, \quad (4)$$

where $\gamma, \theta \neq 0$ are real parameters. Equation (3) includes different types of the BBM equation, one may refer to [3–8] for more details. For $\beta = 0$ in equation (3), we obtain the OBBMB equation

$$v_t - v_{xxt} - \alpha v_{xx} + \gamma v_x + \theta vv_x = 0. \quad (5)$$

The above equation is a one-dimensional, nonlinear, pseudo-parabolic equation that characterizes the behavior of nonlinear surface waves propagating along the x -axis. The term αv_{xx} represents the viscosity term [9–11]. To uncover multiple soliton solutions for this equation, the inverse scattering method has been effectively employed in [12, 13]. The equation

$$v_t - \lambda v_{xxt} - \alpha v_{xx} + vv_x = 0, \quad (6)$$

is a 1D analog of the Oskolkov system

$$(1 - \kappa \nabla^2)v_t = \alpha \nabla^2 v - (v \cdot \nabla)v - \nabla^2 p + g, \quad \nabla \cdot v = 0. \quad (7)$$

The described system serves as a model for capturing the motion of a viscoelastic, incompressible Kelvin-Voigt fluid. The possibility of the parameter κ taking on negative values is sightable. The generalized HERW is given by [14]

$$v_t - v_{xxt} + \alpha v_x + 2\beta vv_x + 3\theta v^2 v_x - \gamma v_x v_{xx} - vv_{xxx} = 0. \quad (8)$$

The equation given in Eq (8) involves real parameters α , β , θ , and γ . This equation holds relevance in describing numerous noteworthy physical models within the realm of mathematical physics.

When setting $\beta = \frac{3}{2}$, $\theta = 0$, and $\gamma = 2$, the equation transforms into the Camassa Holm equation

$$v_t - v_{xxt} + \alpha v_x + 3vv_x - 2v_x v_{xx} - vv_{xxx} = 0. \quad (9)$$

In this equation, v represents the velocity of the fluid in the x direction, and α is a constant associated with the speed of shallow water waves. Similarly, when $\beta = 2$, $\theta = 0$, and $\gamma = 3$, equation (3) becomes the Degasperis Procesi equation

$$v_t - v_{xxt} + \alpha v_x + 4vv_x - 3v_x v_{xx} - vv_{xxx} = 0. \quad (10)$$

It is worth mentioning that Wazwaz [15] extensively discussed the traveling wave solutions of both the Camassa Holm equation and the Degasperis Procesi equation.

By setting $\alpha = 1$, $\beta = \frac{1}{2}$, $\theta = 0$, and $\gamma = 3$, equation (3) transforms into the Fornberg-Whitham equation

$$v_t - v_{xxt} + v_x + vv_x - 3v_x v_{xx} - vv_{xxx} = 0. \quad (11)$$

The Fornberg-Whitham equation is particularly useful for analyzing wave-breaking phenomena. The solution to this equation, as demonstrated by Fornberg and Whitham, can be expressed as $v(x, t) = Ae^{-\frac{1}{2}|x - \frac{4}{3}t|}$ [16].

The objective of this article is to discover novel families of exact soliton solutions for nonlinear pseudo-parabolic type models in mathematical physics, utilizing the NEDA method. This approach serves as a remarkably potent

tool for addressing soliton solutions in nonlinear PDEs. The NEDA method yields solutions based on hyperbolic, trigonometric, rational, exponential, and polynomial functions, catering to various forms of nonlinear PDEs. However, it is important to note that the current approach has its limitations, especially when seeking more general soliton solutions. To overcome this limitation, we aim to enhance our methodology. Here [17–26] are several established techniques that are effectively applied to handle nonlinear PDEs.

Previous work by Akcagil et al. [27] has reported solutions involving trigonometric, hyperbolic, and rational solitons. We intend to build upon these earlier findings to push the boundaries in our pursuit of a diverse range of new traveling wave solutions. Our research endeavors encompass various types of solitons, including dark, bright, singular, combined dark-bright soliton, dark-singular-combined solitons, solitary wave solutions and many more. The solutions we present in this study are groundbreaking and unique, as they haven't been previously showcased for this specific class of equations.

The organization of this paper is outlined as follows: In Section 2, we introduce the framework of the NEDA method. Moving forward, Sections 3, 4, 5, and 6 are dedicated to the examination of specific equations—namely, the BBMPB, OBBMB equation, the 1D Oskolkov equation and generalized HERW equations, respectively. These sections delve into the application of the NEDA method to extract a variety of solutions. Graphical representations of these solutions are presented in Section 7. Finally, in Section 8, we provide a summary and conclusion, besides suggesting potential avenues for future investigations.

2 Outline of the NEDA method

In this section, we outline the general methodology of the NEDA method [28]. We use the following main steps to find the traveling wave solutions of the nonlinear PDEs.

Step 1. Consider a n th order nonlinear PDE

$$\Theta(\zeta, v, v_1, \dots, v_n) = 0, \quad (12)$$

where $\zeta = \zeta^1, \zeta^2, \dots, \zeta^n$ indicates independent variables, v is dependent variable and v_1 and v_n signifies first and n th order derivatives of v with respect to ϑ , respectively.

Step 2. We use the transformation $v(x, t) = \eta(\xi)$, $\xi = x - \mu t$ to convert (12) into a nonlinear ODE

$$\Lambda(\eta, \eta', \eta'', \dots) = 0. \quad (13)$$

Step 3. The ansatz solution of ODE (13) as a polynomial in $h^i(\xi)$ is given by

$$\eta(\xi) = \sum_{i=0}^M b_i h^i(\xi), \quad (14)$$

where $b_i (0 < i \leq n)$ are the coefficients which can be decided later. Note that equation (14) accepts only a particular class of $h(\xi)$'s that satisfies the ODE condition given as

$$h'(\xi) = (\ln \varphi)[k_1 + k_2 h(\xi) + k_3 h^2(\xi)], \quad \varphi \neq 0, 1, \quad (15)$$

where k_1, k_2, k_3 are nonzero constants. Equation (15) does not influence the solution of ODE (13), it just suggests a proper choice of $h(\xi)$. Next, we substitute (14) and (15) into (13) and get a system of algebraic equations. Solving this system for desired constants leads to the solutions. If $k_2^2 - 4k_1k_3 = \Delta$, then the solutions

to (15) can be written as

(1): If $k_2^2 - 4k_1k_3 < 0$ and $k_3 \neq 0$, then

$$\begin{aligned}
 h_1(x, t) &= -\frac{k_2}{2k_3} + \frac{\sqrt{-\Delta}}{2k_3} \tan_{\varphi} \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\
 h_2(x, t) &= -\frac{k_2}{2k_3} - \frac{\sqrt{-\Delta}}{2k_3} \cot_{\varphi} \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\
 h_3(x, t) &= -\frac{k_2}{2k_3} + \frac{\sqrt{-\Delta}}{2k_3} (\tan_{\varphi}(\sqrt{-\Delta} \xi) \pm \sqrt{pq} \sec_{\varphi}(\sqrt{-\Delta} \xi)), \\
 h_4(x, t) &= -\frac{k_2}{2k_3} - \frac{\sqrt{-\Delta}}{2k_3} (\cot_{\varphi}(\sqrt{-\Delta} \xi) \pm \sqrt{pq} \csc_{\varphi}(\sqrt{-\Delta} \xi)), \\
 h_5(x, t) &= -\frac{k_2}{2k_3} + \frac{\sqrt{-\Delta}}{4k_3} (\tan_{\varphi}(\frac{\sqrt{-\Delta}}{4} \xi) - \cot_{\varphi}(\frac{\sqrt{-\Delta}}{4} \xi)),
 \end{aligned} \tag{16}$$

where the generalized trigonometric and hyperbolic functions are defined in equation (28), and p, q are real constants known as deformation parameters.

(2): If $k_2^2 - 4k_1k_3 > 0$ and $k_3 \neq 0$, then

$$\begin{aligned}
 h_6(x, t) &= -\frac{k_2}{2k_3} - \frac{\sqrt{\Delta}}{2k_3} \tanh_{\varphi} \left(\frac{\sqrt{\Delta}}{2} \xi \right), \\
 h_7(x, t) &= -\frac{k_2}{2k_3} - \frac{\sqrt{\Delta}}{2k_3} \coth_{\varphi} \left(\frac{\sqrt{\Delta}}{2} \xi \right), \\
 h_8(x, t) &= -\frac{k_2}{2k_3} + \frac{\sqrt{\Delta}}{2k_3} (\tanh_{\varphi}(\sqrt{\Delta} \xi) \pm i\sqrt{pq} \operatorname{sech}_{\varphi}(\sqrt{\Delta} \xi)), \\
 h_9(x, t) &= -\frac{k_2}{2k_3} - \frac{\sqrt{\Delta}}{2k_3} (\coth_{\varphi}(\sqrt{\Delta} \xi) \pm \sqrt{pq} \operatorname{csch}_{\varphi}(\sqrt{\Delta} \xi)), \\
 h_{10}(x, t) &= -\frac{k_2}{2k_3} - \frac{\sqrt{\Delta}}{4k_3} (\tanh_{\varphi}(\frac{\sqrt{\Delta}}{4} \xi) + \coth_{\varphi}(\frac{\sqrt{\Delta}}{4} \xi)).
 \end{aligned} \tag{17}$$

(3): If $k_1k_3 > 0$ and $k_2 = 0$, then

$$\begin{aligned}
 h_{11}(x, t) &= \sqrt{\frac{k_1}{k_3}} \tan_{\varphi}(\sqrt{k_1k_3} \xi), \\
 h_{12}(x, t) &= -\sqrt{\frac{k_1}{k_3}} \cot_{\varphi}(\sqrt{k_1k_3} \xi), \\
 h_{13}(x, t) &= \sqrt{\frac{k_1}{k_3}} (\tan_{\varphi}(2\sqrt{k_1k_3} \xi) \pm \sqrt{pq} \sec_{\varphi}(2\sqrt{k_1k_3} \xi)), \\
 h_{14}(x, t) &= \sqrt{\frac{k_1}{k_3}} (-\cot_{\varphi}(2\sqrt{k_1k_3} \xi) \pm \sqrt{pq} \csc_{\varphi}(2\sqrt{k_1k_3} \xi)), \\
 h_{15}(x, t) &= \frac{1}{2} \sqrt{\frac{k_1}{k_3}} (\tan_{\varphi}(\frac{\sqrt{k_1k_3}}{2} \xi) - \cot_{\varphi}(\frac{\sqrt{k_1k_3}}{2} \xi)).
 \end{aligned} \tag{18}$$

(4): If $k_1 k_3 < 0$ and $k_2 = 0$, then

$$\begin{aligned}
 h_{16}(x, t) &= -\sqrt{-\frac{k_1}{k_3}} \tanh_{\varphi}(\sqrt{-k_1 k_3} \xi), \\
 h_{17}(x, t) &= -\sqrt{-\frac{k_1}{k_3}} \coth_{\varphi}(\sqrt{-k_1 k_3} \xi), \\
 h_{18}(x, t) &= -\sqrt{-\frac{k_1}{k_3}} (\tanh_{\varphi}(2\sqrt{-k_1 k_3} \xi) \pm i\sqrt{pq} \operatorname{sech}_{\varphi}(2\sqrt{-k_1 k_3} \xi)), \\
 h_{19}(x, t) &= -\sqrt{-\frac{k_1}{k_3}} (\coth_{\varphi}(2\sqrt{-k_1 k_3} \xi) \pm \sqrt{pq} \operatorname{csch}_{\varphi}(2\sqrt{-k_1 k_3} \xi)), \\
 h_{20}(x, t) &= -\frac{1}{2} \sqrt{-\frac{k_1}{k_3}} \left(\tanh_{\varphi}\left(\frac{\sqrt{-k_1 k_3}}{2} \xi\right) + \coth_{\varphi}\left(\frac{\sqrt{-k_1 k_3}}{2} \xi\right) \right).
 \end{aligned} \tag{19}$$

(5): If $a_1 = a_3$ and $a_2 = 0$, then

$$\begin{aligned}
 h_{21}(x, t) &= \tan_{\varphi}(k_1 \xi), \\
 h_{22}(x, t) &= -\cot_{\varphi}(k_1 \xi), \\
 h_{23}(x, t) &= \tan_{\varphi}(2k_1 \xi) \pm \sqrt{pq} \sec_{\varphi}(2k_1 \xi), \\
 h_{24}(x, t) &= -\cot_{\varphi}(2k_1 \xi) \pm \sqrt{pq} \csc_{\varphi}(2k_1 \xi), \\
 h_{25}(x, t) &= \frac{1}{2} \left(\tan_{\varphi}\left(\frac{k_1}{2} \xi\right) - \cot_{\varphi}\left(\frac{k_1}{2} \xi\right) \right).
 \end{aligned} \tag{20}$$

(6): If $k_3 = -k_1$ and $k_2 = 0$, then

$$\begin{aligned}
 h_{26}(x, t) &= -\tanh_{\varphi}(k_1 \xi), \\
 h_{27}(x, t) &= -\coth_{\varphi}(k_1 \xi), \\
 h_{28}(x, t) &= -\tanh_{\varphi}(2k_1 \xi) \pm i\sqrt{pq} \operatorname{sech}_{\varphi}(2k_1 \xi), \\
 h_{29}(x, t) &= -\coth_{\varphi}(2k_1 \xi) \pm \sqrt{pq} \operatorname{csch}_{\varphi}(2k_1 \xi), \\
 h_{30}(x, t) &= -\frac{1}{2} \left(\tanh_{\varphi}\left(\frac{k_1}{2} \xi\right) - \coth_{\varphi}\left(\frac{k_1}{2} \xi\right) \right).
 \end{aligned} \tag{21}$$

(7): If $k_2^2 = 4k_1 k_3$, then

$$h_{31}(x, t) = -\frac{2k_1(k_2 \ln(\varphi) \xi + 2)}{k_2^2 \ln(\varphi) \xi}. \tag{22}$$

(8): If $k_2 = k$, $k_1 = mk$ ($m \neq 0$), and $k_3 = 0$, then

$$h_{32}(x, t) = \varphi^{k\xi} - m. \tag{23}$$

(9): If $k_2 = k_3 = 0$, then

$$h_{33}(x, t) = k_1 \xi \ln(\varphi). \tag{24}$$

(10): If $k_1 = k_2 = 0$, then

$$h_{34}(x, t) = \frac{-1}{k_3 \xi \ln(\varphi)}. \tag{25}$$

(11): If $k_1 = 0$ and $k_2 \neq 0$, then

$$\begin{aligned} h_{35}(x, t) &= -\frac{k_2 p}{k_3 (\cosh_\varphi(k_2 \xi) - \sinh_\varphi(k_2 \xi) + p)}, \\ h_{36}(x, t) &= -\frac{k_2 (\cosh_\varphi(k_2 \xi) + \sinh_\varphi(k_2 \xi))}{k_3 (\cosh_\varphi(k_2 \xi) + \sinh_\varphi(k_2 \xi) + q)}. \end{aligned} \quad (26)$$

(12): If $k_1 = 0$, $k_2 = k$ and $k_3 = mk (m \neq 0)$, then

$$h_{37}(x, t) = \frac{p \varphi^{k\xi}}{p - mq \varphi^{k\xi}}. \quad (27)$$

The generalized hyperbolic and triangular functions appearing in the solutions (16-27) are defined as [6]

$$\begin{aligned} \sin_\varphi(\xi) &= \frac{p \varphi^{i\xi} - q \varphi^{-i\xi}}{2i}, & \cos_\varphi(\xi) &= \frac{p \varphi^{i\xi} + q \varphi^{-i\xi}}{2}, \\ \tan_\varphi(\xi) &= -i \frac{p \varphi^{i\xi} - q \varphi^{-i\xi}}{p \varphi^{i\xi} + q \varphi^{-i\xi}}, & \cot_\varphi(\xi) &= i \frac{p \varphi^{i\xi} + q \varphi^{-i\xi}}{p \varphi^{i\xi} - q \varphi^{-i\xi}}, \\ \csc_\varphi(\xi) &= \frac{2i}{p \varphi^{i\xi} - q \varphi^{-i\xi}}, & \sec_\varphi(\xi) &= \frac{2}{p \varphi^{i\xi} + q \varphi^{-i\xi}}, \\ \sinh_\varphi(\xi) &= \frac{p \varphi^\xi - q \varphi^{-\xi}}{2}, & \cosh_\varphi(\xi) &= \frac{p \varphi^\xi + q \varphi^{-\xi}}{2}, \\ \tanh_\varphi(\xi) &= \frac{p \varphi^\xi - q \varphi^{-\xi}}{p \varphi^\xi + q \varphi^{-\xi}}, & \coth_\varphi(\xi) &= \frac{p \varphi^\xi + q \varphi^{-\xi}}{p \varphi^\xi - q \varphi^{-\xi}}, \\ \operatorname{csch}_\varphi(\xi) &= \frac{2}{p \varphi^\xi - q \varphi^{-\xi}}, & \operatorname{sech}_\varphi(\xi) &= \frac{2}{p \varphi^\xi + q \varphi^{-\xi}}, \end{aligned} \quad (28)$$

where the nonzero parameters p, q are called deformation parameters and ξ indicates an independent variable.

3 The BBMPB equation

The BBMPB equation (3) is given by

$$v_t - v_{xxt} - \alpha v_x + \gamma v_x + \theta v v_x + \beta v_{xxx} = 0,$$

where θ, β are nonzero real constants and α is a positive constant. We use the transformation $v(x, t) = U(\xi)$, $\xi = x - \mu t$ in (3) and integrate this equation to get

$$(-\mu + \gamma)U - \alpha U' + \frac{\theta}{2}U^2 + (\mu + \beta)U'' = 0. \quad (29)$$

Now, we will discuss the traveling wave structure of the BBMPB equation by using (29) with the help of the NEDA method. Comparing the linear term U'' with the nonlinear term U^2 , we have the solution of the form:

$$\eta(\xi) = b_0 + b_1 h(\xi) + b_2 h^2(\xi). \quad (30)$$

Let us suppose that $h(\xi)$ is the solution of the equation, then

$$h'(\xi) = (\ln \phi)[\omega_1 + \omega_2 h(\xi) + \omega_3 h^2(\xi)]. \quad (31)$$

Substituting (30) and (31) into the equation (29), and then comparing the coefficients of different powers of $h(\xi)$, we arrive at a system of algebraic expressions. Solving this system for $\omega_1, \omega_2, b_1, b_2$ we get the set mentioned below

$$\omega_1 = \frac{\theta b_0(\theta b_0 + 2b + 2c)}{4b^2\omega_3(\ln(\phi))^2}, \quad \omega_2 = -\frac{\theta b_0 + b + c}{b\ln(\phi)}, \quad b_1 = -\frac{2(\ln(\phi))b\omega_3}{\theta}, \quad b_2 = 0, \quad a = -b. \quad (32)$$

Case 1: When $\Delta < 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned} v_1(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} + \frac{\sqrt{-\Delta}}{2\omega_3} \tan_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_2(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{-\Delta}}{2\omega_3} \cot_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_3(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{-\Delta}}{2\omega_3} (\tan_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{rs} \sec_\phi(\sqrt{-\Delta}\xi)), \\ v_4(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{-\Delta}}{2\omega_3} (\cot_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{rs} \csc_\phi(\sqrt{-\Delta}\xi)), \\ v_5(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} + \frac{\sqrt{-\Delta}}{4\omega_3} (\tan_\phi(\frac{\sqrt{-\Delta}}{4}\xi) - \cot_\phi(\frac{\sqrt{-\Delta}}{4}\xi)). \end{aligned} \quad (33)$$

where $\Delta = \omega_2^2 - \omega_1\omega_3$, and r, s are constants greater than zero and called deformation parameters.

Case 2: When $\Delta > 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned} v_6(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{\Delta}}{2\omega_3} \tanh_\phi \left(\frac{\sqrt{\Delta}}{2} \xi \right), \\ v_7(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{\Delta}}{2\omega_3} \coth_\phi \left(\frac{\sqrt{\Delta}}{2} \xi \right), \end{aligned}$$

$$\begin{aligned} v_8(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{\Delta}}{2\omega_3} (\tanh_\phi(\sqrt{\Delta}\xi) \pm i\sqrt{rs} \operatorname{sech}_\phi(\sqrt{\Delta}\xi)), \\ v_9(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{\Delta}}{2\omega_3} (\coth_\phi(\sqrt{\Delta}\xi) \pm \sqrt{pq} \operatorname{csch}_\phi(\sqrt{\Delta}\xi)), \\ v_{10}(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{\Delta}}{4\omega_3} (\tanh_\phi(\frac{\sqrt{\Delta}}{4}\xi) + \coth_\phi(\frac{\sqrt{\Delta}}{4}\xi)). \end{aligned} \quad (34)$$

Case 3: When $\omega_1 \omega_3 > 0$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{11}(x, t) &= -\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \tan_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))} \xi \right), \\
 v_{12}(x, t) &= -\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \cot_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))} \xi \right), \\
 v_{13}(x, t) &= \frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \left[\tan_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \sec_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right], \\
 v_{14}(x, t) &= \frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \left[-\cot_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \csc_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right], \\
 v_{15}(x, t) &= \frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{4b(\ln(\phi))\omega_3} \left[\tan_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{4b(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. - \cot_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2b + 2c)}}{4b(\ln(\phi))} \xi \right) \right].
 \end{aligned} \tag{35}$$

Case 4: When $\omega_1 \omega_3 < 0$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{16}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \tanh_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))} \xi \right), \\
 v_{17}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \coth_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))} \xi \right), \\
 v_{18}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \left[\tanh_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. \pm i\sqrt{rs} \operatorname{sech}_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right], \\
 v_{19}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \left[\coth_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \operatorname{csch}_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{b(\ln(\phi))} \xi \right) \right], \\
 v_{20}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{2b(\ln(\phi))\omega_3} \left[\tanh_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{4b(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. + \coth_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2b + 2c)}}{4b(\ln(\phi))} \xi \right) \right].
 \end{aligned} \tag{36}$$

Case 5: If $\omega_1 = \omega_3$ and $\omega_2 = 0$, then

$$\begin{aligned} v_{21}(x, t) &= \tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{4b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{22}(x, t) &= -\cot_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{4b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{23}(x, t) &= \tan_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right) \pm i\sqrt{rs} \sec_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{24}(x, t) &= -\cot_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right) \pm \sqrt{rs} \csc_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{25}(x, t) &= \frac{1}{2} \left[\tan_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{8b^2(\ln(\phi))^2 \omega_3} \xi \right) - \cot_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right) \right]. \end{aligned} \quad (37)$$

Case 6: When $\omega_3 = -\omega_1$ and $\omega_2 = 0$, then

$$\begin{aligned} v_{26}(x, t) &= -\tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{4b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{27}(x, t) &= -\coth_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{4b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{28}(x, t) &= -\tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right) \pm i\sqrt{pq} \operatorname{sech}_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{29}(x, t) &= -\coth_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right) \pm \sqrt{pq} \operatorname{csch}_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi))^2 \omega_3} \xi \right), \\ v_{30}(x, t) &= -\frac{1}{2} \left[\tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{8b^2(\ln(\phi))^2 \omega_3} \xi \right) - \coth_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2b + 2c)}{8b^2(\ln(\phi))^2 \omega_3} \xi \right) \right]. \end{aligned} \quad (38)$$

Case 7: When $\omega_2^2 = 4\omega_1\omega_3$, then

$$v_{31}(x, t) = -\frac{\theta b_0(\theta b_0 + 2c + 2b)[(\theta b_0 + c + b)\xi](\ln(\phi))}{2b(\theta b_0 + b + c)^2 \xi (\ln(\phi))^2 \omega_3}. \quad (39)$$

Case 8: When $\omega_2 = k$, $\omega_1 = mk$ ($m \neq 0$), and $\omega_3 = 0$, then

$$v_{32}(x, t) = \phi^{k\xi} - p. \quad (40)$$

Case 9: When $\omega_2 = \omega_3 = 0$, then

$$v_{33}(x, t) = \frac{\theta b_0(\theta b_0 + 2b + 2c)}{2b^2(\ln(\phi)) \omega_3} \xi. \quad (41)$$

Case 10: When $\omega_1 = \omega_2 = 0$, then

$$v_{34}(x, t) = \frac{-1}{\omega_3 \rho \ln(\phi)}. \quad (42)$$

Case 11: When $\omega_1 = 0$ and $\omega_2 \neq 0$, then

$$\begin{aligned} v_{35}(x, t) &= \frac{\frac{r(\theta b_0 + b + c)}{b\omega_3(\ln(\phi))}}{\cosh_{\phi} \left(-\frac{(\theta b_0 + b + c)}{b(\ln(\phi))} \xi \right) + \sinh_{\phi} \left(-\frac{(\theta b_0 + b + c)}{b(\ln(\phi))} \xi \right) + r}, \\ v_{36}(x, t) &= \frac{\frac{(\theta b_0 + b + c)}{b\omega_3(\ln(\phi))} (\cosh_{\phi} \left(-\frac{(\theta b_0 + b + c)}{b(\ln(\phi))} \xi \right) + \sinh_{\phi} \left(-\frac{(\theta b_0 + b + c)}{b(\ln(\phi))} \xi \right))}{\cosh_{\phi} \left(-\frac{(\theta b_0 + b + c)}{b(\ln(\phi))} \xi \right) + \sinh_{\phi} \left(-\frac{(\theta b_0 + b + c)}{b(\ln(\phi))} \xi \right) + s}, \end{aligned} \quad (43)$$

Case 12: When $\omega_1 = 0$, $\omega_2 = k$ and $\omega_3 = mk (m \neq 0)$, then

$$v_{37}(x, t) = \frac{r\phi^{k\xi}}{s - pr\phi^{k\xi}}, \quad (44)$$

where $\xi = x - \mu t$.

4 The OBBMB equation

The OBBMB equation is given by

$$v_t - v_{xxt} - \alpha v_{xx} + \gamma v_x + \theta v v_x = 0.$$

In the above equation, the parameters α and θ are real. We use the transformation $v(x, t) = U(\xi)$, $\xi = x - \mu t$ in (4) then simplifying we obtain

$$(-\mu + \gamma)U - \alpha U' + \frac{\theta}{2}U^2 + \mu U'' = 0. \quad (45)$$

Now we will discuss the traveling wave structure of the OBBMB equation by using equation (45) with the help of the NEDA method. Comparing U^2 with the last term U'' , we have the solution of the form

$$\eta(\xi) = b_0 + b_1 h(\xi) + b_2 h^2(\xi). \quad (46)$$

Let us suppose that $h(\xi)$ is the solution of the equation, then

$$h'(\xi) = (\ln \phi)[\omega_1 + \omega_2 h(\xi) + \omega_3 h^2(\xi)]. \quad (47)$$

Substituting (46) and (47) into the (45), and then comparing the different powers of $h(\xi)$, we obtain a system of algebraic expressions. Solving this system for ω_1 , ω_2 , b_1 , b_2 we get the set mentioned below

$$\omega_1 = \frac{\theta b_0(\theta b_0 + 2\gamma)}{4\alpha^2 \omega_3 (\ln(\phi))^2}, \quad \omega_2 = -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi)}, \quad b_1 = -\frac{2(\ln(\phi))\alpha \omega_3}{\theta}, \quad b_2 = 0, \quad (48)$$

Case 1: When $\Delta < 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned} v_1(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} + \frac{\sqrt{-\Delta}}{2\omega_3} \tan_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_2(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} - \frac{\sqrt{-\Delta}}{2\omega_3} \cot_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_3(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} + \frac{\sqrt{-\Delta}}{2\omega_3} (\tan_\phi(\sqrt{-\Delta} \xi) \pm \sqrt{rs} \sec_\phi(\sqrt{-\Delta} \xi)), \\ v_4(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} - \frac{\sqrt{-\Delta}}{2\omega_3} (\cot_\phi(\sqrt{-\Delta} \xi) \pm \sqrt{rs} \csc_\phi(\sqrt{-\Delta} \xi)), \\ v_5(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} + \frac{\sqrt{-\Delta}}{4\omega_3} (\tan_\phi(\frac{\sqrt{-\Delta}}{4} \xi) - \cot_\phi(\frac{\sqrt{-\Delta}}{4} \xi)), \end{aligned} \quad (49)$$

where $\Delta = \omega_2^2 - \omega_1 \omega_3$, and r, s are constants greater than zero and called deformation parameters.

Case 2: When $\Delta > 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned}
 v_6(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} - \frac{\sqrt{-\Delta}}{2\omega_3} \tanh_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\
 v_7(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} - \frac{\sqrt{-\Delta}}{2\omega_3} \coth_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\
 v_8(x, t) &= -\frac{\theta b_0 + \gamma}{\alpha \ln(\phi) \omega_3} - \frac{\sqrt{-\Delta}}{2\omega_3} (\tanh_\phi(\sqrt{-\Delta} \xi) \pm i\sqrt{rs} \operatorname{sech}_\phi(\sqrt{-\Delta} \xi)), \\
 v_9(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{-\Delta}}{2\omega_3} (\coth_\phi(\sqrt{-\Delta} \xi) \pm \sqrt{pq} \operatorname{csch}_\phi(\sqrt{-\Delta} \xi)), \\
 v_{10}(x, t) &= \frac{\theta b_0 + b + c}{2b\omega_3 \ln(\phi)} - \frac{\sqrt{-\Delta}}{4\omega_3} (\tanh_\phi(\frac{\sqrt{-\Delta}}{4} \xi) + \coth_\phi(\frac{\sqrt{-\Delta}}{4} \xi)).
 \end{aligned} \tag{50}$$

Case 3: When $\omega_1 \omega_3 > 0$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{11}(x, t) &= \frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))\omega_3} \tan_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))} \xi \right), \\
 v_{12}(x, t) &= -\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))\omega_3} \cot_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))} \xi \right), \\
 v_{13}(x, t) &= \frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))\omega_3} \left[\tan_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \pm \sqrt{rs} \sec_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \right], \\
 v_{14}(x, t) &= \frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))\omega_3} \left[-\cot_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \pm \sqrt{rs} \csc_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \right], \\
 v_{15}(x, t) &= \frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{4\alpha(\ln(\phi))\omega_3} \left[\tan_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{4\alpha(\ln(\phi))} \xi \right) - \cot_\phi \left(\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{4\alpha(\ln(\phi))} \xi \right) \right].
 \end{aligned} \tag{51}$$

Case 4: When $\omega_1 \omega_3 < 0$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{16}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))\omega_3} \tanh_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))} \xi \right), \\
 v_{17}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))\omega_3} \coth_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha(\ln(\phi))} \xi \right), \\
 v_{18}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{2\alpha \ln(\phi) \omega_3} \left[\tanh_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. \pm i\sqrt{rs} \operatorname{sech}_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \right], \\
 v_{19}(x, t) &= -\frac{\sqrt{\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))\omega_3} \left[\coth_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \operatorname{csch}_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))} \xi \right) \right], \\
 v_{20}(x, t) &= -\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{\alpha(\ln(\phi))\omega_3} \left[\tanh_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{4\alpha(\ln(\phi))} \xi \right) \right. \\
 &\quad \left. + \coth_\phi \left(\frac{\sqrt{-\theta b_0(\theta b_0 + 2\gamma)}}{4\alpha(\ln(\phi))} \xi \right) \right].
 \end{aligned} \tag{52}$$

Case 5: If $\omega_1 = \omega_3$ and $\omega_2 = 0$, then

$$\begin{aligned} v_{21}(x, t) &= \tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{4\alpha^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{22}(x, t) &= -\cot_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{4\alpha^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{23}(x, t) &= \tan_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right) \pm i\sqrt{rs} \sec_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{24}(x, t) &= -\cot_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right) \pm \sqrt{rs} \csc_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{25}(x, t) &= \frac{1}{2} \left[\tan_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{8\alpha^2(\ln(\phi))^2\omega_3} \xi \right) - \cot_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right) \right]. \end{aligned} \quad (53)$$

Case 6: When $\omega_3 = -\omega_1$ and $\omega_2 = 0$, then

$$\begin{aligned} v_{26}(x, t) &= -\tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{\alpha b^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{27}(x, t) &= -\coth_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{4\alpha^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{28}(x, t) &= -\tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right) \pm i\sqrt{pq} \operatorname{sech}_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{29}(x, t) &= -\coth_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right) \pm \sqrt{pq} \operatorname{csch}_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{2\alpha^2(\ln(\phi))^2\omega_3} \xi \right), \\ v_{30}(x, t) &= -\frac{1}{2} \left[\tanh_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{8\alpha^2(\ln(\phi))^2\omega_3} \xi \right) + \coth_{\phi} \left(\frac{\theta b_0(\theta b_0 + 2\gamma)}{8\alpha^2(\ln(\phi))^2\omega_3} \xi \right) \right]. \end{aligned} \quad (54)$$

Case 7: When $\omega_2^2 = 4\omega_1\omega_3$, then

$$v_{31}(x, t) = -\frac{\theta b_0(\theta b_0 + 2\gamma)(\theta b_0 + \gamma)\xi + 2\alpha)}{2\alpha(\theta b_0 + \gamma)^2 \ln(\phi) \xi \omega_3}. \quad (55)$$

Case 8: When $\omega_2 = k$, $\omega_1 = mk(m \neq 0)$, and $\omega_3 = 0$, then

$$v_{32}(x, t) = \phi^{k\xi} - p. \quad (56)$$

Case 9: When $\omega_2 = \omega_3 = 0$, then

$$v_{33}(x, t) = \frac{\theta b_0(\theta b_0 + \gamma)}{4\alpha^2(\ln(\phi))\omega_3} \xi. \quad (57)$$

Case 10: When $\omega_1 = \omega_2 = 0$, then

$$v_{34}(x, t) = \frac{-1}{\omega_3 \ln(\phi) \xi}. \quad (58)$$

Case 11: When $\omega_1 = 0$ and $\omega_2 \neq 0$, then

$$\begin{aligned} v_{35}(x, t) &= \frac{\frac{r(\theta b_0 + \gamma)}{\alpha \omega_3 (\ln(\phi))}}{\cosh_{\phi} \left(\frac{(\theta b_0 + \gamma)}{\alpha (\ln(\phi))} \xi \right) - \sinh_{\phi} \left(\frac{(\theta b_0 + \gamma)}{\alpha (\ln(\phi))} \xi \right) + r}, \\ v_{36}(x, t) &= \frac{\frac{r(\theta b_0 + \gamma)}{\alpha \omega_3 (\ln(\phi))} (\cosh_{\phi} \left(\frac{(\theta b_0 + \gamma)}{\alpha (\ln(\phi))} \xi \right) + \sinh_{\phi} \left(\frac{(\theta b_0 + \gamma)}{\alpha (\ln(\phi))} \xi \right))}{\cosh_{\phi} \left(\frac{(\theta b_0 + \gamma)}{\alpha (\ln(\phi))} \xi \right) + \sinh_{\phi} \left(\frac{(\theta b_0 + \gamma)}{\alpha (\ln(\phi))} \xi \right) + s}. \end{aligned} \quad (59)$$

Case 12: When $\omega_1 = 0$, $\omega_2 = k$ and $\omega_3 = mk (m \neq 0)$, then

$$v_{37}(x, t) = \frac{r\phi^{k\xi}}{s - pr\phi^{k\xi}}, \quad (60)$$

where $\xi = x - \mu t$.

5 The 1D Oskolkov equation

The 1D Oskolkov equation is given by

$$v_t - \lambda v_{xxt} - \alpha v_{xx} + v v_x = 0.$$

We are going to investigate this problem for $\lambda \neq 0$ and $\alpha \in \mathbb{R}$ are the constants of the parameter and assume that θ is a nonzero real number. We use the transformation $v(x, t) = U(\xi)$, $\xi = x - \mu t$ in (5) then simplify it and we obtain

$$-\mu U + \lambda U'' - \alpha U' + \frac{U^2}{2} = 0. \quad (61)$$

Comparing U^2 with the U'' in equation (61), we have the solution of the form

$$\eta(\xi) = b_0 + b_1 h(\xi) + b_2 h^2(\xi). \quad (62)$$

Let us suppose that $h(\xi)$ is the solution, then

$$h'(\xi) = (\ln \phi) [\omega_1 + \omega_2 h(\xi) + \omega_3 h^2(\xi)]. \quad (63)$$

Substituting (62) and (63) into the equation (61), and then comparing the different powers of $h(\xi)$, we obtain a system of algebraic expressions. Solving this system for ω_1 , ω_2 , b_1 , b_2 we get the set mentioned below:

$$\begin{aligned} \alpha &= \frac{5(12\lambda a_2 \ln(\phi) \sqrt{\frac{-b_2}{12\lambda}} + b_1)}{12\sqrt{\frac{-b_2}{12\lambda}}}, \quad \omega_1 = \frac{b_1(-2b_2 a_2^2 \ln(\phi) + b_1 \sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2 \ln(\phi)}, \\ \mu &= \frac{(-12\lambda b_2 a_1^2 (\ln(\phi))^2 + 24\lambda b_1 a_2 \ln(\phi) \sqrt{\frac{-b_2}{12\lambda}} + b_1^2)}{4b_2^2 \ln(\phi)}, \quad \omega_3 = \sqrt{\frac{-b_2}{12\lambda}} \frac{1}{\ln(\phi)}. \end{aligned} \quad (64)$$

Case 1: When $\Delta < 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned} v_1(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} + \ln(\phi) \sqrt{\frac{3\Delta\lambda}{b_2}} \tan_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_2(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} - \ln(\phi) \sqrt{\frac{3\Delta\lambda}{b_2}} \cot_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_3(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} + \ln(\phi) \sqrt{\frac{3\Delta\lambda}{b_2}} (\tan_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{rs} \sec_\phi(\sqrt{-\Delta}\xi)), \\ v_4(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} - \ln(\phi) \sqrt{\frac{3\Delta\lambda}{b_2}} (\cot_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{rs} \csc_\phi(\sqrt{-\Delta}\xi)), \\ v_5(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} + \ln(\phi) \sqrt{\frac{3\Delta\lambda}{b_2}} (\tan_\phi(\frac{\sqrt{-\Delta}}{4}\xi) - \cot_\phi(\frac{\sqrt{-\Delta}}{4}\xi)), \end{aligned} \quad (65)$$

where $\Delta = \omega_2^2 - \omega_1 \omega_3$, and r, s are constants greater than zero and called deformation parameters.

Case 2: When $\Delta > 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned}
 v_6(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} - \ln(\phi) \sqrt{\frac{3\Delta\lambda}{-b_2}} \tanh_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\
 v_7(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} - \ln(\phi) \sqrt{\frac{3\Delta\lambda}{-b_2}} \coth_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\
 v_8(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} - \ln(\phi) \sqrt{\frac{3\Delta\lambda}{-b_2}} (\tanh_\phi(\sqrt{-\Delta}\xi) \pm i\sqrt{rs} \operatorname{sech}_\phi(\sqrt{-\Delta}\xi)), \\
 v_9(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} - \ln(\phi) \sqrt{\frac{3\Delta\lambda}{-b_2}} (\coth_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{pq} \operatorname{csch}_\phi(\sqrt{-\Delta}\xi)), \\
 v_{10}(x, t) &= -\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} - \ln(\phi) \sqrt{\frac{3\Delta\lambda}{-b_2}} (\tanh_\phi(\frac{\sqrt{-\Delta}}{4}\xi) + \coth_\phi(\frac{\sqrt{-\Delta}}{4}\xi)).
 \end{aligned} \tag{66}$$

Case 3: When $\omega_1 \omega_3 > 0$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{11}(x, t) &= \frac{\sqrt{b_1(1 - 4b_2a_2^2\ln(\phi)\sqrt{\frac{3\lambda}{-b_2}})}}{2b_2} \tan_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{2b_2\ln(\phi)} \xi \right), \\
 v_{12}(x, t) &= -\frac{\sqrt{b_1(1 - 4b_2a_2^2\ln(\phi)\sqrt{\frac{3\lambda}{-b_2}})}}{2b_2} \cot_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{2b_2\ln(\phi)} \xi \right), \\
 v_{13}(x, t) &= \frac{\sqrt{b_1(1 - 4b_2a_2^2\ln(\phi)\sqrt{\frac{3\lambda}{-b_2}})}}{2b_2} \left[\tan_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{b_2\ln(\phi)} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \sec_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{b_2\ln(\phi)} \xi \right) \right], \\
 v_{14}(x, t) &= \frac{\sqrt{b_1(1 - 4b_2a_2^2\ln(\phi)\sqrt{\frac{3\lambda}{-b_2}})}}{2b_2} \left[-\cot_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{b_2\ln(\phi)} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \csc_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{b_2\ln(\phi)} \xi \right) \right], \\
 v_{15}(x, t) &= \frac{\sqrt{b_1(1 - 4b_2a_2^2\ln(\phi)\sqrt{\frac{3\lambda}{-b_2}})}}{4b_2} \left[\tan_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{4b_2\ln(\phi)} \xi \right) \right. \\
 &\quad \left. - \cot_\phi \left(\frac{\sqrt{-b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi)\sqrt{-\frac{-b_2}{3\lambda}})}}{4b_2\ln(\phi)} \xi \right) \right].
 \end{aligned} \tag{67}$$

Case 4: When $\omega_1 \omega_3 < 0$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{16}(x, t) &= -\frac{\sqrt{-b_1(1-4b_2a_2^2\ln(\phi))\sqrt{\frac{3\lambda}{-b_2}}}}{2b_2} \tanh_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{2b_2\ln(\phi)} \xi \right), \\
 v_{17}(x, t) &= -\frac{\sqrt{-b_1(1-4b_2a_2^2\ln(\phi))\sqrt{\frac{3\lambda}{-b_2}}}}{2b_2} \coth_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{2b_2\ln(\phi)} \xi \right), \\
 v_{18}(x, t) &= -\frac{\sqrt{-b_1(1-4b_2a_2^2\ln(\phi))\sqrt{\frac{3\lambda}{-b_2}}}}{2b_2} \left[\tanh_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{b_2\ln(\phi)} \xi \right) \right. \\
 &\quad \left. \pm i\sqrt{rs} \operatorname{sech}_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{b_2\ln(\phi)} \xi \right) \right], \\
 v_{19}(x, t) &= -\frac{\sqrt{-b_1(1-4b_2a_2^2\ln(\phi))\sqrt{\frac{3\lambda}{-b_2}}}}{b_2} \left[\coth_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{b_2\ln(\phi)} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \operatorname{csch}_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{2b_2\ln(\phi)} \xi \right) \right], \\
 v_{20}(x, t) &= -\frac{\sqrt{-b_1(1-4b_2a_2^2\ln(\phi))\sqrt{\frac{3\lambda}{-b_2}}}}{2b_2} \left[\tanh_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{4b_2\ln(\phi)} \xi \right) \right. \\
 &\quad \left. + \coth_{\phi} \left(\frac{\sqrt{b_1(\frac{b_2^2}{12\lambda} + b_2a_2^2\ln(\phi))\sqrt{-\frac{-b_2}{3\lambda}}}}{4b_2\ln(\phi)} \xi \right) \right].
 \end{aligned} \tag{68}$$

Case 5: If $\omega_1 = \omega_3$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{21}(x, t) &= \tan_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right), \\
 v_{22}(x, t) &= -\cot_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right), \\
 v_{23}(x, t) &= \tan_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{2b_2^2\ln(\phi)} \xi \right) \\
 &\quad \pm i\sqrt{rs} \sec_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{2b_2^2\ln(\phi)} \xi \right),
 \end{aligned} \tag{69}$$

$$\begin{aligned}
v_{24}(x,t) &= -\cot_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right) \\
&\quad \pm \sqrt{rs} \csc_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right), \\
v_{25}(x,t) &= \frac{1}{2} \left[\tan_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \right) \right. \\
&\quad \left. - \cot_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right) \right].
\end{aligned}$$

Case 6: When $\omega_3 = -\omega_1$ and $\omega_2 = 0$, then

$$\begin{aligned}
v_{26}(x,t) &= -\tanh_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right), \\
v_{27}(x,t) &= -\coth_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right), \\
v_{28}(x,t) &= -\tanh_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right) \\
&\quad \pm i\sqrt{pq} \operatorname{sech}_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right), \\
v_{29}(x,t) &= -\coth_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right) \\
&\quad \pm \sqrt{pq} \operatorname{csch}_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right), \\
v_{30}(x,t) &= -\frac{1}{2} \left[\tanh_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right) \right. \\
&\quad \left. + \coth_{\phi} \left(\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2\ln(\phi)} \xi \right) \right].
\end{aligned} \tag{70}$$

Case 7: When $\omega_2^2 = 4\omega_1\omega_3$, then

$$v_{31}(x,t) = -\frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{2\omega_2^2b_2^2\xi} (\omega_2\xi\ln(\phi) + 2). \tag{71}$$

Case 8: When $\omega_2 = k$, $\omega_1 = mk$ ($m \neq 0$), and $\omega_3 = 0$, then

$$v_{32}(x,t) = \phi^{k\xi} - p. \tag{72}$$

Case 9: When $\omega_2 = \omega_3 = 0$, then

$$v_{33}(x,t) = \frac{b_1(-2b_2a_2^2\ln(\phi) + b_1\sqrt{\frac{-b_2}{12\lambda}})}{4b_2^2} \xi. \tag{73}$$

Case 10: When $\omega_1 = \omega_2 = 0$, then

$$v_{34}(x, t) = \frac{-1}{\omega_3 \xi \ln(\phi)}. \quad (74)$$

Case 11: When $\omega_1 = 0$ and $\omega_2 \neq 0$, then

$$\begin{aligned} v_{35}(x, t) &= \frac{-2r\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}}}{\cosh_\phi(\omega_2 \xi) - \sinh_\phi(\omega_2 \xi) + r}, \\ v_{36}(x, t) &= \frac{-2\omega_2 \ln(\phi) \sqrt{\frac{3\lambda}{-b_2}} (\cosh_\phi(\omega_2 \xi) + \sinh_\phi(\omega_2 \xi))}{\cosh_\phi(\omega_2 \xi) + \sinh_\phi(\omega_2 \xi) + s}, \end{aligned} \quad (75)$$

Case 12: When $\omega_1 = 0$, $\omega_2 = k$ and $\omega_3 = mk (m \neq 0)$, then

$$v_{37}(x, t) = \frac{r\phi^{k\xi}}{s - pr\phi^{k\xi}}, \quad (76)$$

where $\xi = x - \mu t$.

6 Generalized HERW equation

The generalized HERW equation is given by

$$v_t - v_{xxt} + \alpha v_x + 2\beta v v_x + 3\theta v^2 v_x - \gamma v_x v_{xx} - v v_{xxx} = 0.$$

In the above equation, the parameters α , β , θ , and γ are all real constant. Furthermore, we assume that θ is a nonzero real number. We use the transformation $v(x, t) = U(\xi)$, $\xi = x - \mu t$ in the generalized HERW equation and integrate this equation and obtain

$$(-\mu + \alpha)U + \mu U'' + \beta U^2 + \theta U^3 - \frac{\gamma - 1}{2}(U')^2 - UU'' = 0. \quad (77)$$

We use homogeneous balancing principle for U^3 with UU'' in (77). We have the solution of the form

$$\eta(\xi) = b_0 + b_1 h(\xi) + b_2 h^2(\xi). \quad (78)$$

Let us suppose that $h(\xi)$ is the solution of the equation, then

$$h'(\xi) = (\ln \phi)[\omega_1 + \omega_2 h(\xi) + \omega_3 h^2(\xi)]. \quad (79)$$

Substituting (79) and (78) into the equation (77), and then comparing the different powers of $h(\xi)$, we obtain a system of algebraic expressions. Solving this system for ω_1 , ω_2 , b_1 , b_2 we get the set mentioned below

$$\begin{aligned} \omega_1 &= \frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2}, \quad \omega_2 = \frac{\omega_3b_1}{b_1}, \quad \theta = \frac{6(\ln(\phi))^2\omega_3^2}{b_2}, \quad \lambda = 1, \\ \alpha &= -\mu \frac{(12b_0b_1\omega_3^2(\ln(\phi))^2 - 3b_1\omega_3^2(\ln(\phi))^2 - b_2^2)}{b_2^2}, \quad \beta = -3(\ln(\phi))^2\omega_3^2 \frac{(2\mu b_2 - 4b_0b_2 + b_1^2)}{b_2^2}. \end{aligned} \quad (80)$$

Case 1: When $\Delta < 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned} v_1(x, t) &= -\frac{b_1}{2b_2} + \frac{\sqrt{-\Delta}}{2\omega_3} \tan_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_2(x, t) &= -\frac{b_1}{2b_2} - \frac{\sqrt{-\Delta}}{2\omega_3} \cot_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_3(x, t) &= -\frac{b_1}{2b_2} + \frac{\sqrt{-\Delta}}{2\omega_3} (\tan_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{rs} \sec_\phi(\sqrt{-\Delta}\xi)), \\ v_4(x, t) &= -\frac{b_1}{2b_2} - \frac{\sqrt{-\Delta}}{2\omega_3} (\cot_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{rs} \csc_\phi(\sqrt{-\Delta}\xi)), \\ v_5(x, t) &= -\frac{b_1}{2b_2} + \frac{\sqrt{-\Delta}}{4\omega_3} (\tan_\phi(\frac{\sqrt{-\Delta}}{4}\xi) - \cot_\phi(\frac{\sqrt{-\Delta}}{4}\xi)), \end{aligned} \quad (81)$$

where $\Delta = \omega_2^2 - \omega_1\omega_3$, and r, s are constants greater than zero and called deformation parameters.

Case 2: When $\Delta > 0$ and $\omega_3 \neq 0$, then

$$\begin{aligned} v_6(x, t) &= -\frac{b_1}{2b_2} - \frac{\sqrt{-\Delta}}{2\omega_3} \tanh_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_7(x, t) &= -\frac{b_1}{2b_2} - \frac{\sqrt{-\Delta}}{2\omega_3} \coth_\phi \left(\frac{\sqrt{-\Delta}}{2} \xi \right), \\ v_8(x, t) &= -\frac{b_1}{2b_2} - \frac{\sqrt{-\Delta}}{2\omega_3} (\tanh_\phi(\sqrt{-\Delta}\xi) \pm i\sqrt{rs} \operatorname{sech}_\phi(\sqrt{-\Delta}\xi)), \\ v_9(x, t) &= -\frac{b_1}{2b_2} - \frac{\sqrt{-\Delta}}{2\omega_3} (\coth_\phi(\sqrt{-\Delta}\xi) \pm \sqrt{pq} \operatorname{csch}_\phi(\sqrt{-\Delta}\xi)), \\ v_{10}(x, t) &= \frac{b_1}{2b_2} - \frac{\sqrt{-\Delta}}{4\omega_3} (\tanh_\phi(\frac{\sqrt{-\Delta}}{4}\xi) + \coth_\phi(\frac{\sqrt{-\Delta}}{4}\xi)). \end{aligned} \quad (82)$$

Case 3: When $\omega_1\omega_3 > 0$ and $\omega_2 = 0$, then

$$\begin{aligned} v_{11}(x, t) &= \sqrt{\frac{6b_0b_2 - b_1^2}{2b_2^2}} \tan_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{2b_2^2}} \xi \right), \\ v_{12}(x, t) &= -\sqrt{\frac{6b_0b_2 - b_1^2}{2b_2^2}} \cot_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{2b_2^2}} \xi \right), \\ v_{13}(x, t) &= \sqrt{\frac{6b_0b_2 - b_1^2}{2b_2^2}} \left[\tan_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{b_2^2}} \xi \right) \pm \sqrt{rs} \sec_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{b_2^2}} \xi \right) \right], \\ v_{14}(x, t) &= \sqrt{\frac{6b_0b_2 - b_1^2}{2b_2^2}} \left[-\cot_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{b_2^2}} \xi \right) \pm \sqrt{rs} \csc_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{b_2^2}} \xi \right) \right], \\ v_{15}(x, t) &= \sqrt{\frac{6b_0b_2 - b_1^2}{8b_2^2}} \left[\tan_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{8b_2^2}} \xi \right) - \cot_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{8b_2^2}} \xi \right) \right]. \end{aligned} \quad (83)$$

Case 4: When $\omega_1 \omega_3 < 0$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{16}(x, t) &= -\sqrt{\frac{6b_0b_2 - b_1^2}{-2b_2^2}} \tanh_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{-2b_2^2}} \xi \right), \\
 v_{17}(x, t) &= -\sqrt{\frac{6b_0b_2 - b_1^2}{-2b_2^2}} \coth_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{-2b_2^2}} \xi \right), \\
 v_{18}(x, t) &= -\sqrt{\frac{6b_0b_2 - b_1^2}{-2b_2^2}} \left[\tanh_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{-b_2^2}} \xi \right) \right. \\
 &\quad \left. \pm i\sqrt{rs} \operatorname{sech}_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{-b_2^2}} \xi \right) \right], \\
 v_{19}(x, t) &= -\sqrt{\frac{6b_0b_2 - b_1^2}{-2b_2^2}} \left[\coth_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{-b_2^2}} \xi \right) \right. \\
 &\quad \left. \pm \sqrt{rs} \operatorname{csch}_\phi \left(\sqrt{\frac{2\omega_3^2(6b_0b_2 - b_1^2)}{-b_2^2}} \xi \right) \right], \\
 v_{20}(x, t) &= -\sqrt{\frac{6b_0b_2 - b_1^2}{-8b_2^2}} \left[\tanh_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{-8b_2^2}} \xi \right) \right. \\
 &\quad \left. + \coth_\phi \left(\sqrt{\frac{\omega_3^2(6b_0b_2 - b_1^2)}{-8b_2^2}} \xi \right) \right].
 \end{aligned} \tag{84}$$

Case 5: If $\omega_1 = \omega_3$ and $\omega_2 = 0$, then

$$\begin{aligned}
 v_{21}(x, t) &= \tanh_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right), \\
 v_{22}(x, t) &= -\cot_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right), \\
 v_{23}(x, t) &= \tan_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right) \pm i\sqrt{rs} \sec_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right), \\
 v_{24}(x, t) &= -\cot_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right) \pm \sqrt{rs} \csc_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right), \\
 v_{25}(x, t) &= \frac{1}{2} \left[\tan_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{4b_2^2} \xi \right) - \cot_\phi \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{4b_2^2} \xi \right) \right].
 \end{aligned} \tag{85}$$

Case 6: When $\omega_3 = -\omega_1$ and $\omega_2 = 0$, then

$$\begin{aligned} v_{26}(x, t) &= -\tanh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right), \\ v_{27}(x, t) &= -\coth_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right), \\ v_{28}(x, t) &= -\tanh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right) \pm i\sqrt{pq}\operatorname{sech}_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right), \\ v_{29}(x, t) &= -\coth_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right) \pm \sqrt{pq}\operatorname{csch}_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{b_2^2} \xi \right), \\ v_{30}(x, t) &= -\frac{1}{2} \left[\tanh_{\phi} \left(\frac{\omega_3(6b_0b_4 - b_1^2)}{4b_2^2} \xi \right) + \coth_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right) \right]. \end{aligned} \quad (86)$$

Case 7: When $\omega_2^2 = 4\omega_1\omega_3$, then

$$v_{31}(x, t) = -\frac{-(6b_0b_2 - b_1^2)(\omega_3b_1\xi \ln(\phi) + 2b_2)}{\omega_3b_1^2b_2 \ln(\phi)\xi}. \quad (87)$$

Case 8: When $\omega_2 = k$, $\omega_1 = mk(m \neq 0)$, and $\omega_3 = 0$, then

$$v_{32}(x, t) = \phi^{k\xi} - p. \quad (88)$$

Case 9: When $\omega_2 = \omega_3 = 0$, then

$$v_{33}(x, t) = \frac{\omega_3(\ln(\phi))(6b_0b_2 - b_1^2)}{2b_2^2} \xi. \quad (89)$$

Case 10: When $\omega_1 = \omega_2 = 0$, then

$$v_{34}(x, t) = \frac{-1}{\omega_3 \ln(\phi)\xi}. \quad (90)$$

Case 11: When $\omega_1 = 0$ and $\omega_2 \neq 0$, then

$$\begin{aligned} v_{35}(x, t) &= \frac{\frac{rb_1}{b_2}}{\cosh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right) - \sinh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right) + r}, \\ v_{36}(x, t) &= \frac{\frac{b_1}{b_2} \left[\cosh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right) + \sinh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right) \right]}{\cosh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right) + \sinh_{\phi} \left(\frac{\omega_3(6b_0b_2 - b_1^2)}{2b_2^2} \xi \right) + s}, \end{aligned} \quad (91)$$

Case 12: When $\omega_1 = 0$, $\omega_2 = k$ and $\omega_3 = mk(m \neq 0)$, then

$$v_{37}(x, t) = \frac{r\phi^{k\xi}}{s - pr\phi^{k\xi}}, \quad (92)$$

where $\xi = x - \mu t$.

7 Discussion

In this section, we present graphical illustrations of the results we have obtained, with a specific focus on their conditions of validity. The conditions ensuring the existence of the derived solutions can be stated as follows.

Proposition 1. *For the considered class of equations, the existence of solitary wave profiles denoted by $v(x, t)$ hinges on the requirement that $\mu \neq 0$, where μ represents the wave speed.*

Notably, prior work by authors [29] and [3] had only presented hyperbolic function-based solutions, while solutions in [27] encompassed trigonometric, hyperbolic, and rational functions. These solutions primarily showcased features of dark and bright soliton solutions. However, our findings transcend these earlier results. The solutions we obtained, including dark, bright, singular, combined dark-bright soliton, dark-singular combined solitons, solitary wave solutions and some others, are uniquely groundbreaking. They stand as valid solutions that haven't previously been introduced for this particular class of equations. These precise solutions aptly capture the dynamics of diverse soliton wave shapes. As such, they offer valuable tools for evaluation, comparison, and numerical investigations within this field. The solutions we have derived for the pseudo-parabolic equations span diverse categories of soliton solutions. Notably, solutions like v_6 , v_{16} , and v_{26} correspond to dark solitons, while v_8 , v_{18} , and v_{28} manifest as combined dark-bright solitons. Furthermore, solutions such as v_{10} , v_{20} , and v_{30} represent instances of combined dark-singular solitons. Beyond this, solutions v_9 , v_{19} , and v_{29} fall into the category of singular solutions of both types 1 and 2, and v_7 , v_{17} , and v_{27} are classified as singular solutions of type 2. It is noteworthy that the complex waves v_{31} and v_{33} exhibit a blow-up phenomenon as ξ approaches 0. To provide visual representation for these solutions, we present 2D and 3D graphical depictions showcasing profiles like v_1 , v_6 , v_8 , and v_{36} in Figures (1,2,3,4).

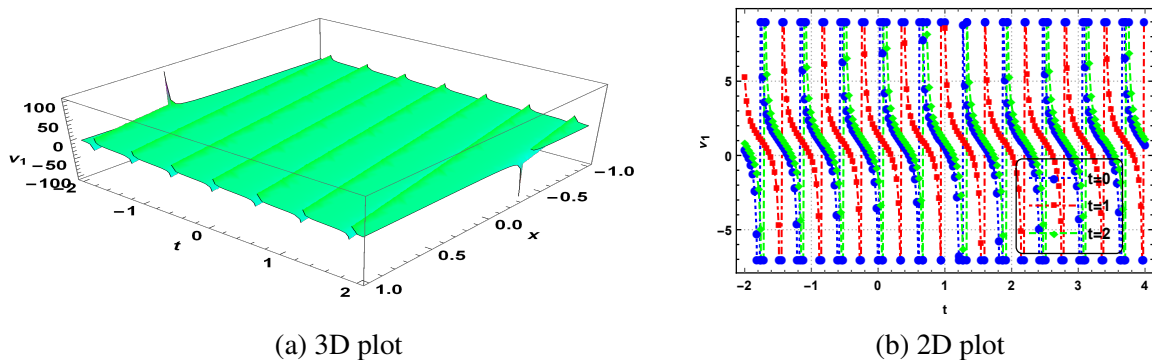


Fig. 1 3D surface of $v_1(x, t)$ given in (49) with $\omega_3 = 1$, $\mu = 3$, $\phi = e$, $b_0 = 1$, $\alpha = \gamma = 1$ and $\theta = -1$.

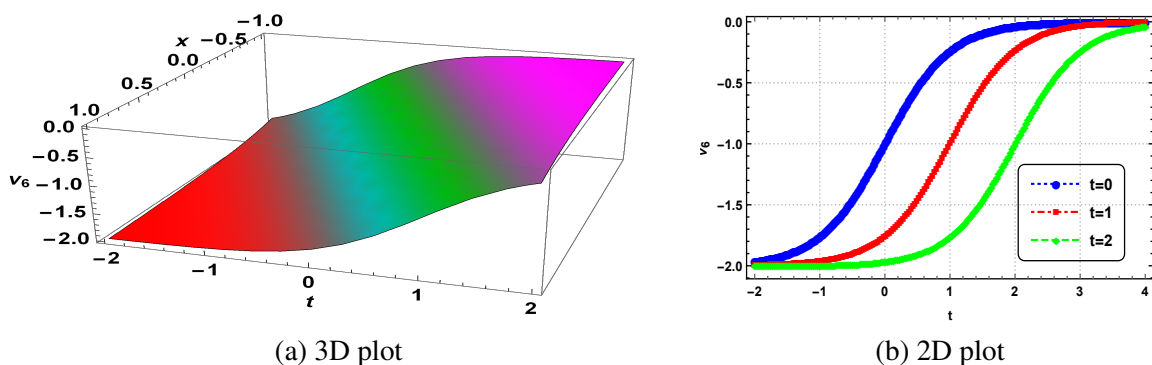


Fig. 2 3D surface of $v_6(x, t)$ given in (66) with $\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 2$, $\mu = 3$, $\phi = e$, $b_2 = -1$ and $\lambda = 1$.

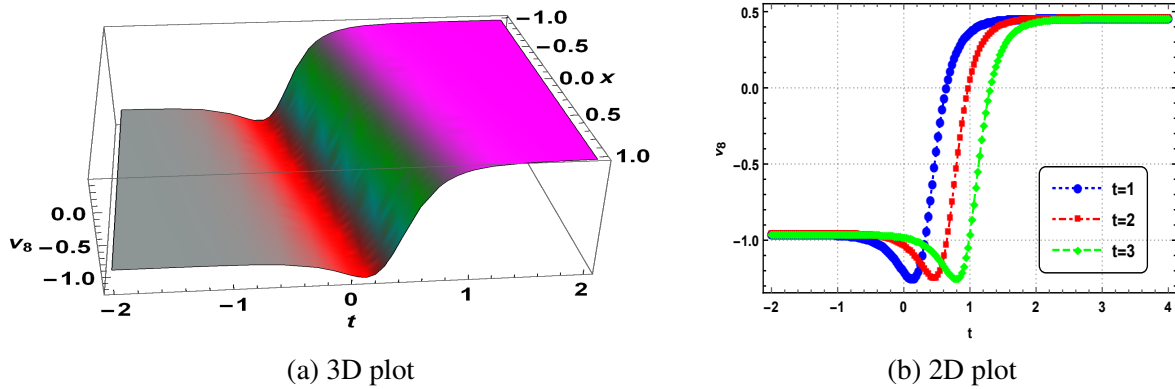


Fig. 3 3D surface of $|v_8(x,t)|$ given in (50) with $\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 2$, $\mu = 3$, $\phi = e$, $b_0 = -1$, $\alpha = 1$, $\theta = 1$, $r = 1$ and $\gamma = 1$.

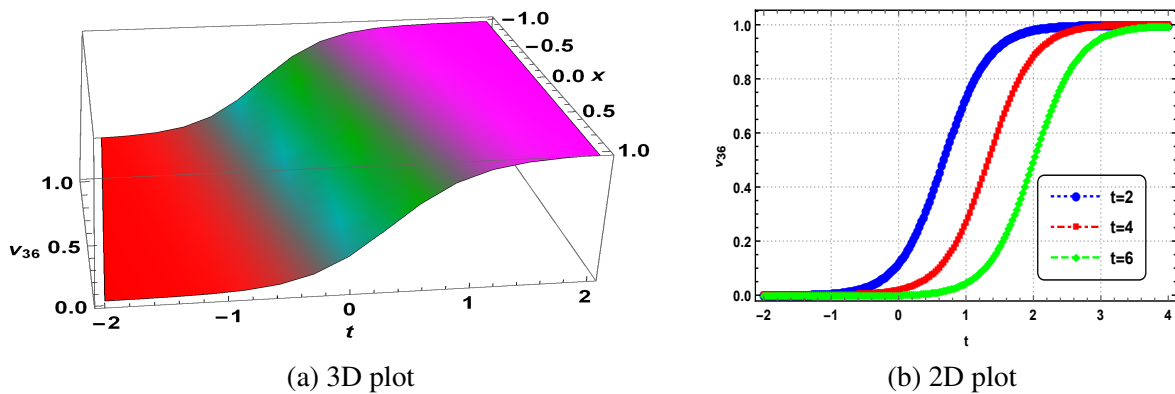


Fig. 4 3D surface of $v_{36}(x,t)$ given in (75) with $\omega_1 = 0$, $b_2 = -1$, $\omega_2 = 1$, $\mu = 3$, $\phi = e$, $s = 1$, and $\lambda_1 = 1$.

8 Conclusion

In this study, we employed a novel application of the NEDA method to integrate pseudo-parabolic type equations. The NEDA method demonstrated its efficacy in providing new analytical solutions for these equations. This technique allowed us to express solutions in terms of hyperbolic, trigonometric, rational, exponential and polynomial functions. The solutions we obtained, including dark, bright, singular, combined dark-bright soliton, dark-singular combined solitons, solitary wave solutions and some others, are uniquely groundbreaking. They stand as valid solutions that haven't previously been introduced for this particular class of equations. Moreover, the methodology employed here holds the potential for broader applications. It is worth highlighting that the solutions we unveil in this study are original and haven't been documented in existing literature. This approach can extend to other mathematical and physics-related problems. Our work underscores the necessity for further exploration to advance fresh and effective analytical techniques for solving PDEs. By harnessing the enhanced capabilities of such techniques, we can more effectively tackle real-world challenges in science and engineering. This progress serves as a compelling incentive for researchers to delve deeper into this remarkable domain of study.

9 Declarations

9.1 Conflict of interests:

According to the authors of this paper, there are no conflicts of interest to report regarding the article that is being presented.

9.2 Author's contributions:

A.H.-Methodology, Writing-Review, Software and Editing, Supervision; H.A.-Resources, Writing-Original Draft, Methodology; F.Z.-Validation, Supervision, Conceptualization, Formal Analysis; N.A.-Data Curation, Investigation, Plotting, Visualization. All authors read and approved the final submitted version of this manuscript.

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9.5 Data availability statement:

All data that support the findings of this study are included within the article.

9.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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