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Representation of the solution of the M-Sturm-Liouville problem with natural transform

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Abstract

In this article, we develop the natural transform in terms of M-derivative improving the basic notions for a new interesting version of the fractional transform. The properties and relations of certain functions for the natural transform of the M-derivative are introduced. The natural transform with the M-derivative is the more general version of the natural transform for the conformable operator. Furthermore, this method is successfully applied to find the general solutions of some fractional differential equations with M-derivative. We propose a significant spectral data with boundary conditions under M-Sturm-Liouville problem. We offer the representation of the solution for the M-Sturm-Liouville problem, depending on both the boundary and initial conditions. Our main aim is to extract the solution representation of the M-Sturm-Liouville problem by using the natural transform and observe the problem by supporting the spectral structure of the M-Sturm-Liouville problem with graphs. Finally, these results show that our new approach is simple, effective and accurate.

Keywords: M-derivative, Sturm-Liouville problem, natural transform.
AMS 2020 codes: 26A33; 34B24.

1 Introduction

In engineering and physical sciences, differential equations have been encountered in exhibited problems. Scientific advances have shown that classical derivative and integral notions are unsatisfactory to analyze differential equations. The known notions of derivative and integral have thus to be widened. Scientists have eagerly ventured new derivatives and integral search to remove this problem. After a few studies, L'Hospital has been asked about how to take the derivative if the order of the derivative is rational [1]. The first time, the foundation of the fractional analysis field was laid. After this original idea, the subject of fractional analysis has been debated by Leibniz and Bernoulli [1]. Miller et al. as a result of the exploratories, remarkable fractional derivatives such as Caputo, Riemann-Liouville, Grunwald-Letnikov have been introduced to the literature [2]. Hilfer et al. have made many applications in physics by using fractional calculus [3]. Fractional analysis is a powerful tool for studying the behavior of complex systems. Fractional analysis opens new horizons in science and engineering by being used to investigate the systems that go beyond the limits of classical analysis. Acay et al. have used the fractional derivatives to model the systems with non-uniform behavior [4]. Pinar has arisen the explicit solutions of the fractional Bagley-Torvik equation in engineering [5]. Blohincev et al. have used

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the fractional derivatives in the field of quantum mechanics, producing quite effective studies [6]. Euler has proved the Gamma function, which is a significant notion in the fractional analysis [1]. Right now, the notion of fractional analysis has a significant location in many areas such as physical perusal of the behavior of earthquake fault lines. Levitan et al. have introduced some studies on spectral theory [7]. The Sturm-Liouville problem is one of the considerable problems in physics and mathematics since 1836 [7]. Derakhshan et al. have presented the various studies on the fractional Sturm-Liouville problem [8]. Khalil et al. have discovered a new operator, the conformable derivative [9]. In 2017, Sousa and Oliveira set up a parameter that satisfies the properties of the integer order calculus and an M-derivative that includes the Mittag-Leffler function [10]. In later years, it was the subject of major different publications by a lot of remarkable scientists. Odabasi et al. have developed the unique methods by applying for some nonlinear fractional order differential equations [11]. By Belgacem et al., Natural transform have been introduced to the literature [12]. According to Belgacem et al., it has been showed that the Natural transform (in solving differential equations) is twin of the traditional Laplace transform and the new Sumudu transform [12]. Due to the convergence of the Laplace as well as the Sumudu transforms, the natural transform treats all the problems that Laplace and Sumudu come across with [13]. Khan et al. have contributed to the literature with some properties of natural transform "in the name of N transform" [14].

The rest of this paper is organized as follows: Section 2 introduces the foundation of the M-derivative. In this chapter, we demonstrate some essential tools for our study, such as the M-derivative of specific functions and the natural transform for the M-derivative. This method has turned into an alternative method for solving differential equations. The natural transform of the M-derivative is a transform that can be used in solving the mathematical equations of physics as well as providing great convenience in solving differential equations. We can say that some integrals can be solved by the M-natural transform method and it is a beneficial method in other branches of science in terms of its practicality. Section 3 presents the main results of our study. The representation of the solution for the M-Sturm-Liouville problem is presented with the support of the natural transform of the M-derivative. The inimitableness of the solution has been proven here. Section 4 offers an elaborate discussion supported by graphs for various values of α, γ and s. The final section contains some significant explanations about the novelty of this paper.

2 General properties of the scheme

Some essential definitions, concepts and theorems about the M-derivative that we will use in the applications of this article are here explained. Moreover, we also describe the definition and properties of natural transform for the M-derivative, which forms the main backbone of the article. In addition, we introduce some new practical tools for the M-derivative by taking advantage of the conformable derivative.

Definition 1. [1] The Mittag-Leffler function with two parameter is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{1}$$

with $z \in \mathbb{C}$ and $\alpha, \beta > 0$.

Definition 2. [15] Let $h:[0,\infty) \to \Re$ function be defined. The M-derivative for $0 < \alpha \leq 1$ is defined as follows

$$D_M^{\alpha,\beta} h(t) = \lim_{\varepsilon \to 0} \frac{h\left(tE_\beta(\varepsilon t^{-\alpha})\right) - h(t)}{\varepsilon}. \tag{2}$$

Definition 3. [15] Let h is defined in $(a,t]$ and $0 < \alpha \leq 1$. The M-integral is defined by

$$I_M^{\alpha,\beta} h(t) = \Gamma(\beta+1) \int_a^t \frac{h(x)}{x^{1-\alpha}} dx. \tag{3}$$

Theorem 1. *[15] Let $0 < \alpha \leq 1, \beta > 0$ and h,k are $\alpha-$differentiable functions at a $t > 0$ point. The properties of the M-derivative are as follows:*

- $D_M^{\alpha,\beta}(n) = 0$, *$n$ is a constant.*
- $D_M^{\alpha,\beta}(h.k)(t) = \left[(h(t)D_M^{\alpha,\beta}k(t) + k(t)D_M^{\alpha,\beta}h(t)\right]$.
- $D_M^{\alpha,\beta}\left(\frac{h}{k}\right)(t) = \frac{k(t)D_M^{\alpha,\beta}h(t) - h(t)D_M^{\alpha,\beta}k(t)}{[k(t)]^2}$.
- *(Chain rule)* $D_M^{\alpha,\beta}(h\circ k)(t) = h^{'}(k(t))D_M^{\alpha,\beta}k(t)$.

Presently, we submit a few definitions and theorems obligatory for our application in the next part.

Theorem 2. *[15] M-derivative of a few particular functions are as follows:*

- $D_M^{\alpha,\beta}(\sin(qt)) = \frac{qt^{1-\alpha}\cos(qt)}{\Gamma(\beta+1)}$, $q \in \Re$.
- $D_M^{\alpha,\beta}(\cos(at)) = -\frac{at^{1-\alpha}\sin(at)}{\Gamma(\beta+1)}$, $a \in \Re$.
- $D_M^{\alpha,\beta}(e^{nt}) = \frac{nt^{1-\alpha}e^{nt}}{\Gamma(\beta+1)}$, $n \in \Re$.
- $D_M^{\alpha,\beta}(t^j) = \frac{jt^{j-\alpha}}{\Gamma(\beta+1)}$, $j \in \Re$.

Definition 4. [15] $n-$order linear non-homogeneous differential equation for the M-derivative is

$$c_0 D_M^{n\alpha,\beta}y + c_1 D_M^{(n-1)\alpha,\beta}y + \cdots + c_{n-1}D_M^{\alpha,\beta}y + c_n y = x(t) \tag{4}$$

where $D_M^{n\alpha,\beta}y = D_M^{\alpha,\beta}D_M^{\alpha,\beta}\cdots D_M^{\alpha,\beta}$ and $0 < \alpha < 1$, $c_0, c_1, \cdots, c_n$ are constants. Let the function y be differentiable $n-$times and if $n-$independent solutions for a homogeneous differential equation are $y_1, y_2, \cdots y_n$

$$c_0 D_M^{n\alpha,\beta}y + c_1 D_M^{(n-1)\alpha,\beta}y + \cdots + c_{n-1}D_M^{\alpha,\beta}y + c_n y = 0. \tag{5}$$

The specific functions for the M-derivative are as follows [15]:

- $D_M^{\alpha,\sigma}\left(e^{\frac{\Gamma(\sigma+1)t^\alpha}{\alpha}}\right) = \left(e^{\frac{\Gamma(\sigma+1)t^\alpha}{\alpha}}\right)$.
- $D_M^{\alpha,\sigma}\left(\sin\frac{\Gamma(\sigma+1)t^\alpha}{\alpha}\right) = \left(\cos\frac{\Gamma(\sigma+1)t^\alpha}{\alpha}\right)$.
- $D_M^{\alpha,\sigma}\left(\cos\frac{\Gamma(\sigma+1)t^\alpha}{\alpha}\right) = -\left(\sin\frac{\Gamma(\sigma+1)t^\alpha}{\alpha}\right)$.

Natural transform of M-derivative is a focal point in our study.

Definition 5. Let $f:[a,\infty)\to\Re$, $\gamma>0$, $a\in\Re$ and $0<\alpha\leq 1$. Natural transform of M-derivative is defined as follows

$$\mathcal{N}^{a}_{\alpha,\beta}\{f(t)\}(s)=F^{a}_{\alpha,\beta}(s,u)=\frac{\Gamma(\beta+1)}{u}\int_{a}^{\infty}e^{-s\frac{\Gamma(\beta+1)(t-a)^{\alpha}}{u\alpha}}f(t)d_{\alpha}t \tag{6}$$

where $(t-a)^{\alpha-1}dt=d_{\alpha}t$ and u, s are variables.

Definition 6. [16] Let $f:[a,\infty)\to\Re$, $\gamma>0$ and $0<\alpha\leq 1$. Laplace transform of M-derivative is defined as follows

$$\mathcal{L}^{a}_{\alpha,\gamma}\{f(t)\}(s)=F^{a}_{\alpha,\gamma}(s)=\Gamma(\gamma+1)\int_{a}^{\infty}e^{-s\frac{\Gamma(\gamma+1)(t-a)^{\alpha}}{\alpha}}f(t)(t-a)^{\alpha-1}dt \tag{7}$$

where $d_{\alpha}t=(t-a)^{\alpha-1}dt$.

Theorem 3. *Let $f:[0,\infty)\to\Re$ be a differentiable function and is defined in the range $0<\alpha\leq 1$. In this case*

$$\mathcal{N}_{\alpha,\beta}\{D^{\alpha,\beta}_{M}f(x)\}=\frac{\Gamma(\beta+1)}{u}\left(\frac{s}{u}F_{\alpha,\beta}\left(\frac{s}{u}\right)-f(0)\right) \tag{8}$$

there is equality.

Proof. $\mathcal{N}_{\alpha,\beta}[f(x)]=\frac{\Gamma(\beta+1)}{u}F_{\alpha,\beta}\left(\frac{s}{u}\right)$ and $\mathcal{L}_{\alpha,\beta}\{D^{\alpha,\beta}_{M}f(x)\}=sF_{\alpha,\beta}(s)-f(0)$ using the equations,

$$\mathcal{N}_{\alpha,\beta}\{D^{\alpha,\beta}_{M}f(x)\}=\left(\frac{\Gamma(\beta+1)}{u}\mathcal{L}_{\alpha,\beta}\{D^{\alpha,\beta}_{M}f(x)\}_{s\to\frac{s}{u}}\right)$$

$$=\frac{\Gamma(\beta+1)}{u}[sF_{\alpha,\beta}(s)-f(0)]_{s\to\frac{s}{u}}=\frac{\Gamma(\beta+1)}{u}\left[\frac{s}{u}F_{\alpha,\beta}\left(\frac{s}{u}\right)-f(0)\right]$$

is obtained.

Theorem 4. *Let $f:[0,\infty)\to\Re$ be a n-differentiable function $(n\in N)$ for $0<\alpha\leq 1$ is given by*

$$\mathcal{N}_{\alpha,\beta}\{D^{n\alpha,\beta}_{M}f(x)\}=\frac{s^{n}\Gamma(\beta+1)F_{\alpha,\beta}(s,u)}{u^{n}}-\frac{s^{n-1}f(0)}{u^{n}}-\frac{D^{(n-1)\alpha,\gamma}_{M}y(0,\lambda)}{u^{n-1}}. \tag{9}$$

Theorem 5. *Let $f:[0,\infty)\to\Re$ be a differentiable function. The relation between the M-derivative Laplace transform and the M-derivative N-transform in the range $0<\alpha\leq 1$ is given as*

$$\mathcal{N}_{\alpha,\beta}[f(x)]=\frac{\Gamma(\beta+1)}{u}F_{\alpha,\beta}\left(\frac{s}{u}\right). \tag{10}$$

Proof. Firstly, from the definition of the natural transform with M-derivative

$$F^{a}_{\alpha,\beta}(s,u)=\frac{\Gamma(\beta+1)}{u}\int_{0}^{\infty}e^{-s\frac{\Gamma(\beta+1)(x)^{\alpha}}{u\alpha}}f(x)d_{\alpha}x$$

$v=\frac{(x)^{\alpha}}{\alpha}$ and $dv=x^{\alpha-1}dx$ by changing the variable,

$$F_{\alpha,\beta}(s,u)=\frac{\Gamma(\beta+1)}{u}\int_{0}^{\infty}e^{-s\frac{\Gamma(\beta+1)v}{u}}f(\alpha v)\frac{1}{\alpha}dv$$

$$= \frac{\Gamma(\beta+1)}{u}\int_0^{\infty} e^{-s\frac{\Gamma(\beta+1)x}{u}} f(\alpha x)\frac{1}{\alpha}d_\alpha x$$

$$= \left[\frac{\Gamma(\beta+1)}{u}L_{\alpha,\beta}\left\{(f(\alpha x))\frac{1}{\alpha}\right\}\right]_{s\to\frac{s}{u}}$$

$$= \frac{\Gamma(\beta+1)}{u}F_{\alpha,\beta}\left(\frac{s}{u}\right)$$

is obtained. We use the above formulas in our study. Let $\beta,\alpha \in \Re$ and $0<\alpha\leq 1$. We define the natural transform of a few functions for the derivative M [16]:

- $\mathcal{N}_{\alpha,\beta}\left\{e^{a\Gamma(\beta+1)\frac{x^\alpha}{\alpha}}\right\} = \frac{1}{s-au} \qquad s>au.$
- $\mathcal{N}_{\alpha,\beta}\left\{\sin\left(a\Gamma(\beta+1)\frac{x^\alpha}{\alpha}\right)\right\} = \frac{au}{s^2+a^2u^2} \qquad s>0.$
- $\mathcal{N}_{\alpha,\beta}\left\{\cos\left(a\Gamma(\beta+1)\frac{x^\alpha}{\alpha}\right)\right\} = \frac{s}{s^2+a^2u^2} \qquad s>0.$
- $\mathcal{N}_{\alpha,\beta}\left\{\sinh\left(a\Gamma(\beta+1)\frac{x^\alpha}{\alpha}\right)\right\} = \frac{au}{s^2-a^2u^2} \qquad s>\mid au\mid.$
- $\mathcal{N}_{\alpha,\beta}\left\{\cosh\left(a\Gamma(\beta+1)\frac{x^\alpha}{\alpha}\right)\right\} = \frac{s}{s^2-a^2u^2} \qquad s>\mid au\mid.$

Definition 7. [16] Let $g(t)$ and $f(t)$ are continuous functions and have exponential order, then the convolution of f and g for the M-derivative is defined by

$$(f*g)(t) = \Gamma(\gamma+1)\int_a^t f(\tau)g(a+((t-a)^\alpha-(\tau-a)^\alpha)\frac{1}{\alpha})(\tau-a)^{\alpha-1}d\tau. \tag{11}$$

Differently from various studies on this subject, the M-natural transform is used in this article, thus allowing us to obtain a more general result.

3 Applications

In the present part, our aim is to offer a view of the M-Sturm-Liouville problem to indicate that it is a more general version of the Sturm-Liouville problem in classical analysis. We thus handle the M-derivative taking into account of the Sturm-Liouville problem having

$$-D_M^{\alpha,\gamma}D_M^{\alpha,\gamma}y(x)+[q(x)y(x)] = \lambda y(x) \tag{12}$$

here the function $q(x)$ continuous on the interval $[a,b]$ and $0<\alpha\leq 1$.
We consider (12) with boundary conditions;

$$y(a)\cos(\alpha)+D_M^{\alpha,\gamma}y(a)\sin(\alpha)=0 \tag{13}$$

$$y(b)\cos(\beta)+D_M^{\alpha,\gamma}y(b)\sin(\beta)=0. \tag{14}$$

Change $\cot\beta = H$, $\cot\alpha = -h$ and for $x \in [0,\pi]$, we pay attention here in above boundary conditions as

$$D_M^{\alpha,\gamma} y(0) - hy(0) = 0 \tag{15}$$

$$D_M^{\alpha,\gamma} y(\pi) + Hy(\pi) = 0. \tag{16}$$

It should be noted that changing the interval $[a,b]$ with $[0,\pi]$ does not replace the boundary condition (13) and (14). Thus one can imagine that $a = 0$ and $b = \pi$. Besides, for any λ the boundary value problem present above has a solution $y(x,\lambda)$ [15]. Considering the principal idea above, we offer following significant theorem.

Theorem 6. *We consider the representation of the solution of the M-Sturm-Liouville problem having*

$$-D_M^{\alpha,\gamma} D_M^{\alpha,\gamma} y(x) + [q(x)y(x)] = \lambda y(x) \tag{17}$$

where the initial conditions are $D_M^{\alpha,\gamma} y(0,\lambda) = 1$, $y(0,\lambda) = 0$

$$y(x,\lambda) = \frac{1}{s}\sin\left(s\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right) + \frac{\Gamma(\gamma+1)}{s}\int_0^x \sin\left(s\Gamma(\gamma+1)\left(\frac{x^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha}\right)\right) q(\tau)y(\tau,\lambda)d_\alpha\tau.$$

Proof. Let's get the solution of the problem by using the natural transform (17). If we apply the natural transform for the M-derivative to both sides of the problem (17),

$$-\mathcal{N}^{\alpha,\gamma}\{D_M^{\alpha,\gamma} D_M^{\alpha,\gamma} y(x)\} + \mathcal{N}^{\alpha,\gamma}\{[q(x)y(x)]\} = \mathcal{N}^{\alpha,\gamma}\{\lambda y(x)\} \tag{18}$$

it becomes. We can write $D_M^{\alpha,\gamma} D_M^{\alpha,\gamma} y(x) = D_M^{(2)\alpha,\gamma} y(x)$ from equation $D_M^{n\alpha,\gamma} y = D_M^{\alpha,\gamma} D_M^{\alpha,\gamma} \cdots D_M^{\alpha,\gamma}$.

$$-\mathcal{N}^{\alpha,\gamma}\{D_M^{(2)\alpha,\gamma} y(x)\} = -\left(\frac{s^2}{u^2} Y_{\alpha,\gamma}(s,u) - \frac{s}{u^2}y(0) - \frac{D_M^{\alpha,\gamma} y(0,\lambda)}{u}\right). \tag{19}$$

If we write the initial conditions in equation (19),

$$-\mathcal{N}^{\alpha,\gamma}\{D_M^{(2)\alpha,\gamma} y(x)\} = -\frac{s^2}{u^2} Y_{\alpha,\gamma}(s,u) + \frac{1}{u} \tag{20}$$

is obtained. Using the formula (6), we can write

$$\mathcal{N}_{\alpha,\gamma}^a\{q(x)y(x)\} = \frac{\Gamma(\gamma+1)}{u}\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{u\alpha}} [q(x)y(x)]d_\alpha x. \tag{21}$$

It is written

$$\mathcal{N}^{\alpha,\gamma}\{\lambda y(x)\} = \lambda Y_{\alpha,\gamma}(s,u) \tag{22}$$

in format. Substitute equations (20), (21), (22) in equation (18) to get

$$-\frac{s^2}{u^2} Y_{\alpha,\gamma}(s,u) + \frac{1}{u} + \frac{\Gamma(\gamma+1)}{u}\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{u\alpha}} [q(x)y(x)]d_\alpha x = \lambda Y_{\alpha,\gamma}(s,u)$$

$$\lambda Y_{\alpha,\gamma}(s,u) + \frac{s^2}{u^2} Y_{\alpha,\gamma}(s,u) = \frac{1}{u} + \frac{\Gamma(\gamma+1)}{u}\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{u\alpha}} [q(x)y(x)]d_\alpha x$$

$$Y_{\alpha,\gamma}(s,u)\left[\frac{\lambda u^2 + s^2}{u^2}\right] = \frac{1}{u} + \frac{\Gamma(\gamma+1)}{u}\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{u\alpha}} [q(x)y(x)]d_\alpha x \tag{23}$$

$$Y_{\alpha,\gamma}(s,u)=\frac{u}{\lambda u^2+s^2}+\frac{u\Gamma(\gamma+1)}{\lambda u^2+s^2}\int_0^{\infty}e^{-s\frac{\Gamma(\gamma+1)x^{\alpha}}{u\alpha}}[q(x)y(x)]d_{\alpha}x. \tag{24}$$

Applying the inverse natural transform of the M-derivative to both sides of (24) we get

$$\mathcal{N}_{\alpha,\gamma}^{-1}\{Y_{\alpha,\gamma}(s,u)\}=\mathcal{N}_{\alpha,\gamma}^{-1}\left\{\frac{u}{\lambda u^2+s^2}\right\}+\mathcal{N}_{\alpha,\gamma}^{-1}\left\{\frac{u\Gamma(\gamma+1)}{\lambda u^2+s^2}\int_0^{\infty}e^{-s\frac{\Gamma(\gamma+1)x^{\alpha}}{u\alpha}}[q(x)y(x)]d_{\alpha}x\right\}. \tag{25}$$

We can write $\mathcal{N}_{\alpha,\gamma}^{-1}\{Y_{\alpha,\gamma}(s,u)\}=y(x,\lambda)$. Using the formula of $\mathcal{N}_{\alpha,\gamma}^{-1}\left\{\frac{au}{s^2+a^2u^2}\right\}=\sin\left(a\Gamma(\gamma+1)\frac{x^{\alpha}}{\alpha}\right)$, we obtain

$$\mathcal{N}_{\alpha,\gamma}^{-1}\left\{\frac{u}{\lambda u^2+s^2}\right\}=\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\Gamma(\gamma+1)\frac{x^{\alpha}}{\alpha}\right). \tag{26}$$

$$\mathcal{N}_{\alpha,\gamma}^{-1}\left\{\frac{u\Gamma(\gamma+1)}{\lambda u^2+s^2}\int_0^{\infty}e^{-s\frac{\Gamma(\gamma+1)x^{\alpha}}{u\alpha}}[q(x)y(x)]d_{\alpha}x\right\} \tag{27}$$

expression using the convolution property,

$$f(s)=\frac{u\Gamma(\gamma+1)}{\lambda u^2+s^2}\longrightarrow F(x)=\frac{\Gamma(\gamma+1)}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\Gamma(\gamma+1)\frac{x^{\alpha}}{\alpha}\right)$$

$$g(s)=\int_0^{\infty}e^{-s\frac{\Gamma(\gamma+1)x^{\alpha}}{u\alpha}}[q(x)y(x)]d_{\alpha}x\longrightarrow G(x)=\left[q(x)y(x,\lambda)\right]$$

$$\frac{\Gamma(\gamma+1)}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\Gamma(\gamma+1)\frac{x^{\alpha}}{\alpha}\right)*q(x)y(x,\lambda)$$

$$=\frac{\Gamma(\gamma+1)}{\sqrt{\lambda}}\int_0^{x}\sin\left(\sqrt{\lambda}\Gamma(\gamma+1)\left(\frac{x^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right)q(\tau)y(\tau,\lambda)d_{\alpha}\tau. \tag{28}$$

If we substitute the expressions (26), (28) in equation (25) we get the representation of the solution in form,

$$y(x,\lambda)=\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\Gamma(\gamma+1)\frac{x^{\alpha}}{\alpha}\right)+\frac{\Gamma(\gamma+1)}{\sqrt{\lambda}}\int_0^{x}\sin\left(\sqrt{\lambda}\Gamma(\gamma+1)\left(\frac{x^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right)q(\tau)y(\tau,\lambda)d_{\alpha}\tau. \tag{29}$$

Taking $\lambda=s^2$ we obtain

$$y(x,\lambda)=\frac{1}{s}\sin\left(s\Gamma(\gamma+1)\frac{x^{\alpha}}{\alpha}\right)+\frac{\Gamma(\gamma+1)}{s}\int_0^{x}\sin\left(s\Gamma(\gamma+1)\left(\frac{x^{\alpha}}{\alpha}-\frac{\tau^{\alpha}}{\alpha}\right)\right)q(\tau)y(\tau,\lambda)d_{\alpha}\tau. \tag{30}$$

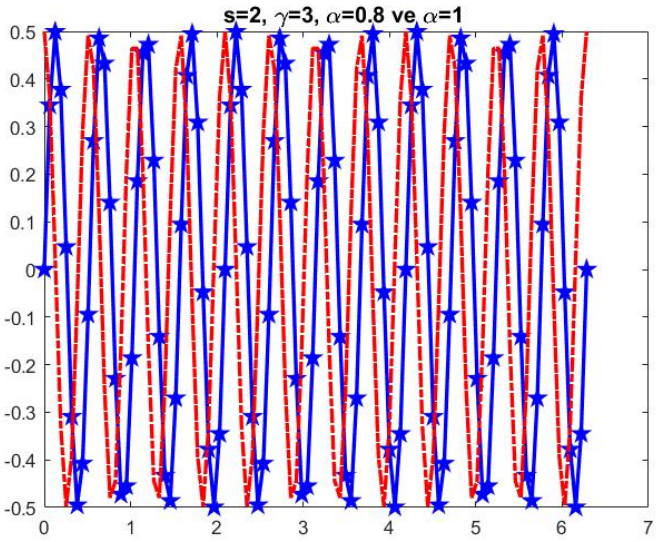


Fig. 1 Approach to classical analysis for various alpha values

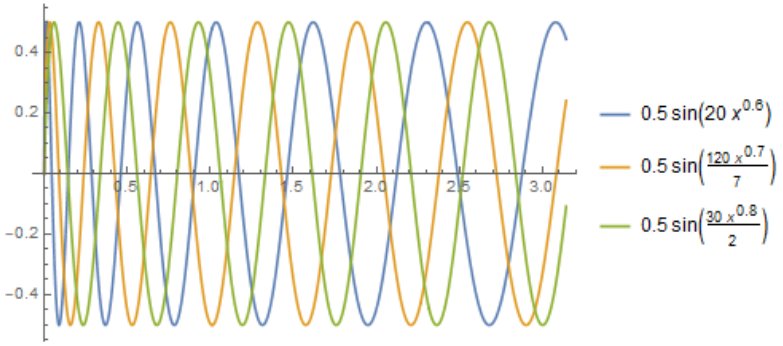


Fig. 2 Representation of solution for different alpha values

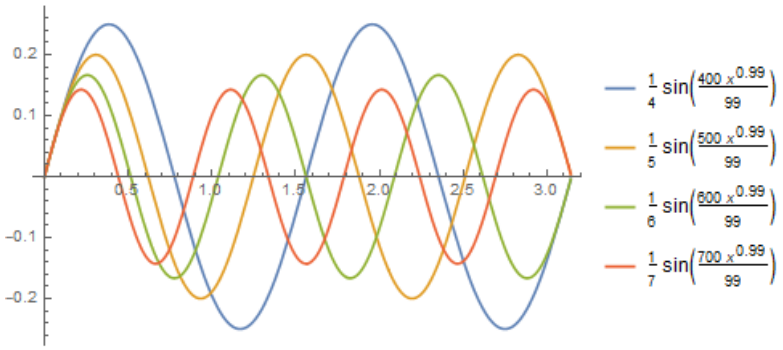


Fig. 3 Representation of solution for dissimilar s values

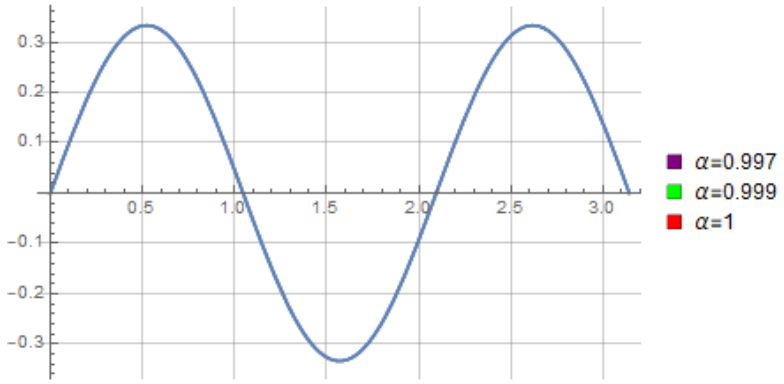


Fig. 4 Overlap with classical analysis

4 Conclusions

In this part of the study, we present some information about the novelty of a paper with different data of the M-Sturm-Liouville problem. In the light of the made observations, we point out that it is the general version of the classical Sturm-Liouville problem. For this aim, the graphs below sharply show us the representation movement of the solution obtained in the conclusions. We offer an elaborate for the dissimilar values of γ, α and s supported by graphs. In Figure 1, the representation of the solution is examined for various alpha values. It shows the representation of the solution by giving the values $s = 2, \gamma = 3, \alpha = 0.8$ and $\alpha = 1$ in equation (30). A remarkable image is obtained. In Figure 2, the representation of the solution in the gleam of $s = 2$, $\gamma = 3$, $\alpha = 0.6$, $\alpha = 0.7$ and $\alpha = 0.8$ values is plotted. In Figure 3, the representation of the solution in the light of $s = 4$, $s = 5$, $s = 6$, $s = 7$, $\alpha = 0.99$ and $\gamma = 1$ shows in detail of the values. In Figure 4, the representation of the solution in the shimmer of $s = 3$, $\gamma = 1$, $\alpha = 0.997$, α=0.999 and α=1 values is given notable. For the natural transform of the M-derivative, we produce the properties and relations of certain functions. As a result, this transformation we have obtained is the general version of the natural transform for the conformable derivative. By using this transformation, it is possible to find more precise values in the solution of differential equations. By modifying the Sturm-Liouville problem, which is very important in physics and engineering for the M-derivative, we have obtained the representation of the solution with the M-natural transform method. Finally, when the representation of the solution approaches α=1, it is observed that it coincides within the classical analysis. The results show that the new method is basic, efficient and accurate. M-natural transform will lead to open gates for the researchers concerning this area. Especially, representation of solution will be practicable for future studies.

5 Declarations

5.1 Conflict of interest:

According to the authors of this paper, there are no conflicts of interest to report regarding the article that is being presented.

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5.3 Authors contribution:

E.B.-Writing original manuscript, conceptualization, formal analysis, investigation, methodology, software. M.K.-Review and editing. The paper has been submitted with the knowledge and consent of all authors.

5.4 Acknowledgement:

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5.5 Data availability statement:

All data that support the findings of this study are included within the article.

References

[1] Podlubny I., Fractional Differential Equations, Mathematics in Science and Engineering, 198, 41-119, 1999.
[2] Miller K.S., Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, USA, 1993.
[3] Hilfer R., Applications of Fractional Calculus in Physics, World Scientific, Germany, 2000.
[4] Acay B., Inc M., Chu Y.M., Almohsen B., Modeling of pressure-volume controlled artificial respiration with local derivatives, Advances in Difference Equations, 46, 1-21, 2021.
[5] Pinar Z., On the explicit solutions of fractional Bagley-Torvik equation arises in engineering, An International Journal of Optimization and Control: Theory and Applications, 9(3), 52-58, 2019.
[6] Blohincev D.I., Foundations of Quantum Mechanics, GITTL, Moscow, Russia, 1949.
[7] Levitan B.M., Sargsjan I.S., Introduction to spectral theory: self-adjoint ordinary differential operators, American Mathematical Society, 39, 1-525, 1975.
[8] Derakhshan M.H., Ansari A., Fractional Sturm-Liouville problems for Weber fractional derivatives, International Journal of Computer Mathematics, 96(2), 217-237, 2019.
[9] Khalil R., Horani M.A., Yousef A., Sababheh M., A new definition of fractional derivative, Journal of Computational and Applied Mathematics, 264, 65-70, 2014.
[10] Sousa J.V.C., Oliveira E.C., A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties, International Journal of Analysis and Applications, 16(1), 83-96, 2018.
[11] Odabasi M., Pinar Z., Kocak H., Analytical solutions of some nonlinear fractional-order differential equations by different methods, Mathematical Methods in the Applied Sciences, 44(9), 7526-7537, 2021.
[12] Belgacem F.B.M., Silambarasan R., Theory of natural transform, Mathematics in Engineering Science and Aerospace, 3, 99-124, 2012.
[13] Belgacem F.B.M., Silambarasan R., Advances in the natural transform, AIP Conference Proceedings, 1493, 106-111, 2012.
[14] Khan Z.H., Khan W.A., N-transform-properties and applications, NUST Journal of Engineering Sciences, 1(1), 127-133, 2008.
[15] Bas E., Acay B., The direct spectral problem via local derivative including truncated Mittag-Leffler function, Applied Mathematics and Computation, 367, 124787, 2020.
[16] Acay B., Bas E., Abdeljawad T., Non-local fractional calculus from different viewpoint generated by truncated M-derivative, Journal of Computational and Applied Mathematics, 366, 112410, 2020.