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Original Study

On n-times differentiable strongly s-convex functions

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Abstract

In this study, some new inequalities for n-times differentiable strongly s-convex functions are introduced. These inequalities are obtained via the perturbed trapezoid inequality. We obtain a better bound for the mentioned inequalities with the strongly s-convex functions. n^{th} derivatives of absolute values of the handled functions are strongly s-convex. Finally, the theorems presented for strongly s-convex functions are reduced to the ones given for s-convex functions when the constant from strongly s-convexity vanishes.

Keywords: s-convex function, strongly s-convex function, perturbed trapezoid inequalities.
AMS 2020 codes: 39B62; 26D07.

1 Introduction

Convex function classes, which are also inequalities, contribute to the many theories of inequality. It is well known that convex sets and functions have played a significant role in the developments of pure and applied sciences. They continue to add new and innovative applications to science. Convexity has also a great importance in the fields of mathematics as well as chemistry, biology, physics, health, industry, fine arts, human anatomy,etc. The general properties and various applications of convex functions widely appear in the literature [1–4].

Recently, the inequalities obtained for different types of convex functions generate a growing interest among many popular inequalities. n-times differentiable s-convex functions are an important class of mathematical functions used in optimization theory. Firstly, a function is said to be strongly s-convex if for any two points in its domain, the functions value at the midpoint of the line segment connecting these two points is less than or equal to the weighted average of the function's values at the two endpoints, with the weights depending on the value of s. When a function is also n-times differentiable, it means that its first n derivatives exist and are continuous. This additional property allows for more advanced mathematical techniques to be applied to analyze the function's behavior and optimize it.

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In 1960s, the Russian mathematician Aleksandrov introduced the concept of s-convexity for functions defined on Euclidean spaces. The author showed that s-convex functions have some useful properties, such as the mean value inequality, which plays an important role in the study of partial differential equations [5]. In order to estimate the mean value of continuous s-convex functions, the well-known Hermit-Hadamard inequality is frequently used. Dragomir and Fitzpatrick [6] introduced the variant of Hadamards inequality which holds for s-convex functions in second sense. This inequality is called s-Hadamard-type inequality for s-convex function in second sense. Hudzik and Maligrada [7] discussed s-convexity in second sense which is stronger than the first sense. Today, strongly convex functions are widely used in convex optimization and machine learning, and their development continues to be an active area of research.

The various extensions and generalizations for classical convexity have been much discussed in recent years. Some applications of the strongly convex functions have a great significance in optimization theory, variational inequalities and regarding areas [8–11]. Firstly, Polyak [12] presented the strongly convex functions. Erdem et al. [13] obtained some new results regarding the Hermite-Hadamard inequalities for the strongly s-convex functions in the second sense. Hua et al. [14] suggested a concept of strongly s-convex and presented some inequalities of Simpson type for strongly s-convex functions. Dönmez Demir and Şanal [15] considered some perturbed trapezoid inequalities for n-times differentiable convex, s-convex and s-convex. In this study, the authors introduce some new inequalities for s-times differentiable strongly s-convex functions. Then, the trapezoid inequality given for the numerical integration is defined as

$$\left| \int_{\alpha}^{\beta} \upsilon(x)\,dx - \frac{1}{2}(\beta-\alpha)(\upsilon(\alpha)+\upsilon(\beta)) \right| \leq \frac{1}{12} M_2 (\beta-\alpha)^3, \tag{1}$$

where $\upsilon : [\alpha,\beta] \to \mathbb{R}$ is supposed to be twice differentiable and bounded on (α,β) by $M_2 = \sup_{x\in(\alpha,\beta)} |\upsilon''(x)| < +\infty$ [16]. Let $\upsilon : [0,\infty) \to \mathbb{R}$. If the inequality

$$\upsilon(\tau x + (1-\tau)y) \leq \tau^s \upsilon(x) + (1-\tau)^s \upsilon(y), \tag{2}$$

is held for all $\alpha,\beta \in [0,\infty)$, $\tau \in [0,1]$ and for some fixed $s \in (0,1]$, then the function υ is called as s-convex in the second sense. Generally, K_s^2 describes the class of s-convex functions in the second sense. For $s = 1$, s-convexity is reduced to ordinary convexity of functions defined on $[0,\infty)$ [6].

n-times differentiable strongly s-convex functions are widely used, especially in optimization problems. These functions represent the extended class of convex optimization problems and have some properties: Strongly s-convex functions provide fast convergence, especially for large-sized problems. In some optimization algorithms, the number of iterations of the algorithm can remain constant while the amount of error remains constant, via the properties of strongly s-convex functions. The properties of strong s-convex functions make it possible to solve the optimization problem by guaranteeing the number of iterations. Thanks to these properties, n-times differentiable strongly s-convex functions are used in many fields, especially in large-scale data analysis, machine learning, image processing, signal processing and control systems. For example, n-time differentiable strongly s-convex functions are used in many application areas such as low-rank matrix completion, local-linear-fit regression, data analysis and model selection, portfolio optimization, and neural network training [17–19].

In our previous studies, we first started by extending the second order differentiable convex functions to n-order differentiable functions and obtaining some trapezoidal inequalities [20]. Similarly, n-order extensions were obtained for log-convex, tgs-convex, s-convex functions [15]. Subsequently, studies covering some numerical integration applications of the mentioned functions are presented.

This study consists of five parts. In the first two chapters, the introduction and preliminary are introduced. The theorem and proofs are presented in the third section. Finally, the conclusions and declarations are given in the last two sections.

2 Preliminaries

Definition 1. [14] A function $\upsilon : [\alpha,\beta] \to [0,\infty)$ is said to be strongly s-convex for $c > 0$ and $s \in [0,1]$ if

$$\upsilon(\tau x + (1-\tau)y) \leq \tau^s \upsilon(x) + (1-\tau)^s \upsilon(y) - c\tau(1-\tau)(x-y)^2, \tag{3}$$

holds for all $x, y \in [\alpha,\beta]$ and $\tau \in [0,1]$.

Lemma 1. *[20] Let $\upsilon : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping with n derivatives on I°, $\alpha,\beta \in I^\circ$ with $\alpha < \beta$ where n is even number. If $\upsilon^{(n)} \in [\alpha,\beta]$, then one obtains*

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)\,dx - \frac{\upsilon(\alpha)+\upsilon(\beta)}{2} + \cdots \\
&- \frac{(\beta-\alpha)^{(n-4)}\left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha) + \upsilon^{(n-4)}(\beta)\right] \\
&+ \frac{(\beta-\alpha)^{(n-3)}\left[n.(n-1).a_n + \cdots + 4.3a_4 + 3.2a_3 + 4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta) - \upsilon^{(n-3)}(\alpha)\right] \\
&- \frac{(\beta-\alpha)^{(n-2)}\left[n.a_n + \cdots + 2.a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha) + \upsilon^{(n-2)}(\beta)\right] \\
&+ \frac{(\beta-\alpha)^{(n-1)}\left[a_n + \cdots + a_1 + 2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta) - \upsilon^{(n-1)}(\alpha)\right] \\
&= \frac{(\beta-\alpha)^n}{2.n!.a_n}\int_0^1 (a_n\tau^n + \cdots + a_1\tau + a_0)\left[\upsilon^{(n)}(\tau\alpha + (1-\tau)\beta) + \upsilon^{(n)}(\tau\beta + (1-\tau)\alpha)\right]d\tau.
\end{aligned} \tag{4}$$

3 Main results

Theorem 2. *Let $s \in (0,1]$ and $\upsilon : [\alpha,\beta] \to [0,\infty)$ be a mapping with n derivatives on I° where $\alpha,\beta \in I^\circ$ with $\alpha < \beta$ such that n is even number. If $\left|\upsilon^{(n)}\right|$ is strongly s-convex on $[\alpha,\beta]$ with modulus $c > 0$, then the inequality in the following is held*

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)\,dx - \frac{\upsilon(\alpha)+\upsilon(\beta)}{2} + \cdots\right. \\
&- \frac{(\beta-\alpha)^{(n-4)}\left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha) + \upsilon^{(n-4)}(\beta)\right] \\
&+ \frac{(\beta-\alpha)^{(n-3)}\left[n.(n-1).a_n + \cdots + 4.3a_4 + 3.2a_3 + 4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta) - \upsilon^{(n-3)}(\alpha)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\beta-\alpha)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right] \\
&+ \frac{(\beta-\alpha)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\Bigg| \\
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\left[\left|\upsilon^{(n)}(\alpha)\right|+\left|\upsilon^{(n)}(\beta)\right|\right]\left[\sum_{i=0}^{n}\frac{|a_i|}{i+s+1}+\sum_{i=0}^{n}\frac{|a_i|\Gamma(i+1)\Gamma(s+1)}{i+s+2}\right]-c(\alpha-\beta)^2\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]\right\}.
\end{aligned}
\tag{5}
$$

Proof. From Lemma (1) and Definition (1), it is concluded that

$$
\begin{aligned}
&\Bigg|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)\,dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots \\
&- \frac{(\beta-\alpha)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right] \\
&+ \frac{(\beta-\alpha)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right] \\
&- \frac{(\beta-\alpha)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right] \\
&+ \frac{(\beta-\alpha)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\Bigg| \\
&= \left|\frac{(\beta-\alpha)^n}{2.n!.a_n}\left\{\int_0^1 (a_n\tau^n+\cdots+a_1\tau+a_0)\left[\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)+\upsilon^{(n)}(\tau\beta+(1-\tau)\alpha)\right]d\tau\right\}\right| \\
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\int_0^1 |a_n\tau^n+\cdots+a_1\tau+a_0|\left[\left|\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)\right|+\left|\upsilon^{(n)}(\tau\beta+(1-\tau)\alpha)\right|\right]d\tau\right\} \\
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\int_0^1 |a_n\tau^n+\cdots+a_1\tau+a_0|\left[\tau^s\left|\upsilon^{(n)}(\alpha)\right|+(1-\tau)^s\left|\upsilon^{(n)}(\beta)\right|\right.\right. \\
&\left.\left.+\tau^s\left|\upsilon^{(n)}(\beta)\right|+(1-\tau)^s\left|\upsilon^{(n)}(\alpha)\right|-c\tau(1-\tau)(\alpha-\beta)^2\right]d\tau\right\} \\
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\left[\left|\upsilon^{(n)}(\alpha)\right|+\left|\upsilon^{(n)}(\beta)\right|\right]\int_0^1 \left[|a_n|\tau^n+\cdots+|a_1|\tau+|a_0|\right]\left[\tau^s+(1-\tau)^s\right]d\tau\right. \\
&\left.-c(\alpha-\beta)^2\left\{\int_0^1\left[|a_n|\tau^n+\cdots+|a_1|\tau+|a_0|\right]\left[\tau(1-\tau)\right]d\tau\right\}\right\} \\
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\left[\left|\upsilon^{(n)}(\alpha)\right|+\left|\upsilon^{(n)}(\beta)\right|\right]\left[\sum_{i=0}^{n}\frac{|a_i|}{i+s+1}+\sum_{i=0}^{n}\frac{|a_i|\Gamma(i+1)\Gamma(s+1)}{i+s+2}\right]-c(\alpha-\beta)^2\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]\right\}.
\end{aligned}
\tag{6}
$$

Corollary 3. *Under conditions of Theorem (2), considering $c=0$ leads to Theorem 2.1 for s-convex functions included in Ref. ([15]) .*

Theorem 4. *Let $s \in (0,1]$ and $\upsilon : [\alpha,\beta] \to [0,\infty)$ be a mapping with n derivatives on I°, $\alpha,\beta \in I^\circ$ such that $\beta > \alpha$, $1 < p$ and $\frac{1}{p}+\frac{1}{q}=1$ where n is even number. If the mapping $\left|\upsilon^{(n)}\right|^q$ is strongly s-convex on $[\alpha,\beta]$, with modulus $c_1, c_2 > 0$, then one obtains*

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)\,dx - \frac{\upsilon(\alpha)+\upsilon(\beta)}{2} + \cdots \right.\\
&-\frac{(\beta-\alpha)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]\\
&+\frac{(\beta-\alpha)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]\\
&-\frac{(\beta-\alpha)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]\\
&\left.+\frac{(\beta-\alpha)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right|\\
&\le \frac{(\beta-\alpha)^n}{n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\\
&\times\left\{\left[\frac{\left|\upsilon^{(n)}(\alpha)\right|^q}{s+1}+\frac{\left|\upsilon^{(n)}(\beta)\right|^q\Gamma(s+1)}{\Gamma(s+2)}-\frac{c_1(\alpha-\beta)^2}{6}\right]^{\frac{1}{q}}+\left[\frac{\left|\upsilon^{(n)}(\beta)\right|^q}{s+1}+\frac{\left|\upsilon^{(n)}(\alpha)\right|^q\Gamma(s+1)}{\Gamma(s+2)}-\frac{c_2(\alpha-\beta)^2}{6}\right]^{\frac{1}{q}}\right\}
\end{aligned} \tag{7}$$

Proof. Using Lemma (1), Definition (1) and Hölder's integral inequality [21, 22] and Minkowski's integral inequality [23] gives

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)\,dx - \frac{\upsilon(\alpha)+\upsilon(\beta)}{2} + \cdots \right.\\
&-\frac{(\beta-\alpha)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]\\
&+\frac{(\beta-\alpha)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]\\
&-\frac{(\beta-\alpha)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]\\
&\left.+\frac{(\beta-\alpha)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right|\\
&=\left|\frac{(\beta-\alpha)^n}{2.n!.a_n}\int_0^1(a_n\tau^n+\cdots+a_1\tau+a_0)\left[\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)+\upsilon^{(n)}(\tau\beta+(1-\tau)\alpha)\right]d\tau\right|\\
&\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\int_0^1|a_n\tau^n+\cdots+a_1\tau+a_0|\left|\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)\right|d\tau\right.\\
&\left.+\int_0^1|a_n\tau^n+\cdots+a_1\tau+a_0|\left|\upsilon^{(n)}(\tau\beta+(1-\tau)\alpha)\right|d\tau\right]\\
&\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\left(\int_0^1|a_n\tau^n+\cdots+a_1\tau+a_0|^p\,d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)\right|^q d\tau\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 \left|a_n\tau^n+\cdots+a_1\tau+a_0\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \left|\upsilon^{n)}\left(\tau\alpha+(1-\tau)\beta\right)\right|^q d\tau\right)^{\frac{1}{q}}\Bigg] \\
&\leq \frac{(\beta-\alpha)^n}{n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\left\{\left[\frac{\left|\upsilon^{(n)}(\alpha)\right|^q}{s+1}+\frac{\left|\upsilon^{(n)}(\beta)\right|^q\Gamma(s+1)}{\Gamma(s+2)}-\frac{c_1(\alpha-\beta)^2}{6}\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{\left|\upsilon^{(n)}(\beta)\right|^q}{s+1}+\frac{\left|\upsilon^{(n)}(\alpha)\right|^q\Gamma(s+1)}{\Gamma(s+2)}-\frac{c_2(\alpha-\beta)^2}{6}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{8}
$$

such that $1/p+1/q=1$. Considering the strongly s-convexity of $\left|\upsilon^{(n)}\right|^q$, then we find

$$
\begin{aligned}
&\int_0^1 \left|\upsilon^{(n)}\left(\tau\alpha+(1-\tau)\beta\right)\right|^q d\tau \leq \int_0^1 \left[\tau^s\left|\upsilon^{(n)}(\alpha)\right|^q+(1-\tau)^s\left|\upsilon^{(n)}(\beta)\right|^q-c_1\tau(1-\tau)(\alpha-\beta)^2\right]d\tau \\
&\leq \frac{\left|\upsilon^{(n)}(\alpha)\right|^q}{s+1}+\frac{\left|\upsilon^{(n)}(\beta)\right|^q\Gamma(s+1)}{\Gamma(s+2)}-\frac{c_1(\alpha-\beta)^2}{6}
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\int_0^1 \left|\upsilon^{(n)}\left(\tau\beta+(1-\tau)\alpha\right)\right|^q d\tau \leq \int_0^1 \left[\tau^s\left|\upsilon^{(n)}(\beta)\right|^q+(1-\tau)^s\left|\upsilon^{(n)}(\alpha)\right|^q-c_1\tau(1-\tau)(\alpha-\beta)^2\right]d\tau \\
&\leq \frac{\left|\upsilon^{(n)}(\beta)\right|^q}{s+1}+\frac{\left|\upsilon^{(n)}(\alpha)\right|^q\Gamma(s+1)}{\Gamma(s+2)}-\frac{c_1(\alpha-\beta)^2}{6}
\end{aligned}
\tag{10}
$$

Using the Minkowski inequality, we have

$$
\begin{aligned}
&\left(\int_0^1 \left|a_n\tau^n+\cdots+a_1\tau+a_0\right|^p d\tau\right)^{\frac{1}{p}} \leq \left(\int_0^1 |a_n|^p\tau^{np}d\tau\right)^{\frac{1}{p}}+\cdots+\left(\int_0^1 |a_1|^p\tau^p d\tau\right)^{\frac{1}{p}}+\left(\int_0^1 |a_0|^p d\tau\right)^{\frac{1}{p}} \\
&\leq \frac{|a_n|}{(np+1)^{1/p}}+\cdots+\frac{|a_1|}{(p+1)^{1/p}}+|a_0| \\
&\leq \sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{1/p}}
\end{aligned}
\tag{11}
$$

Substituting (9), (10) and (11) to (8), then (7) is obtained.

Corollary 5. *If $c_1=c_2=0$, then Theorem (4) is reduced Theorem 2.2 ([15]) given for s-convex functions.*

Theorem 6. *Let $s\in(0,1]$ and $\upsilon:[\alpha,\beta]\to[0,\infty)$ be the function having n^{th} derivatives on I°, $\alpha,\beta\in I^\circ$ with $\alpha<\beta$ and $p>1$ where $1/p+1/q=1$ such that n is even number. If the mapping $\left|\upsilon^{(n)}\right|^p$ is strongly s-convex*

on $[\alpha,\beta]$ *with modulus* $c_1,c_2>0$, *then the inequality in the following is held*

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)\,dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots\right.\\
&-\frac{(\beta-\alpha)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]\\
&+\frac{(\beta-\alpha)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]\\
&-\frac{(\beta-\alpha)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]\\
&\left.+\frac{(\beta-\alpha)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right| \qquad (12)\\
&\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\sum_{i=0}^n\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\\
&\times\left\{\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+s+1}\right)\left|\upsilon^{(n)}(\alpha)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)\left|\upsilon^{(n)}(\beta)\right|^p-c_1(\beta-\alpha)^2\left[\sum_{i=0}^n\frac{|a_i|}{(i+2)(i+3)}\right]\right]^{\frac{1}{p}}\right.\\
&\left.+\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+s+1}\right)\left|\upsilon^{(n)}(\beta)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)\left|\upsilon^{(n)}(\alpha)\right|^p-c_2(\beta-\alpha)^2\left[\sum_{i=0}^n\frac{|a_i|}{(i+2)(i+3)}\right]\right]^{\frac{1}{p}}\right\}.
\end{aligned}$$

Proof. Using Lemma (1), definition (1) and power mean integral inequality [24], one obtains

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)\,dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots\right.\\
&-\frac{(\beta-\alpha)^{(n-4)}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]\\
&+\frac{(\beta-\alpha)^{(n-3)}\left[n.(n-1).a_n+\cdots+4.3a_4+3.2a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]\\
&-\frac{(\beta-\alpha)^{(n-2)}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]\\
&\left.+\frac{(\beta-\alpha)^{(n-1)}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right|\\
&=\left|\frac{(\beta-\alpha)^n}{2.n!.a_n}\int_0^1(a_n\tau^n+\cdots+a_1\tau+a_0)\left[\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)+\upsilon^{(n)}(\tau\beta+(1-\tau)\alpha)\right]d\tau\right|\\
&\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\int_0^1|a_n\tau^n+\cdots+a_1\tau+a_0|\left|\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)\right|d\tau\right.\\
&\left.+\int_0^1|a_n\tau^n+\cdots+a_1\tau+a_0|\left|\upsilon^{(n)}(\tau\beta+(1-\tau)\alpha)\right|d\tau\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left(\int_0^1 |a_n\tau^n+\cdots+a_1\tau+a_0|\,d\tau\right)^{1-\frac{1}{p}}\\
&\times\left\{\left(\int_0^1 |a_n\tau^n+\cdots+a_1\tau+a_0|\left|\upsilon^{(n)}(\tau\alpha+(1-\tau)\beta)\right|^p d\tau\right)^{\frac{1}{p}}\right.\\
&\left.+\left(\int_0^1 |a_n\tau^n+\cdots+a_1\tau+a_0|\left|\upsilon^{(n)}(\tau\beta+(1-\tau)\alpha)\right|^p d\tau\right)^{\frac{1}{p}}\right\}\\
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left(|a_n|\int_0^1\tau^n d\tau+\cdots+|a_1|\int_0^1\tau d\tau+\int_0^1|a_0|\,d\tau\right)^{1-\frac{1}{p}}\\
&\times\left\{\left(\int_0^1 |a_n\tau^n+\cdots+a_1\tau+a_0|\left(\tau^s\left|\upsilon^{(n)}(\alpha)\right|^p+(1-\tau)^s\left|\upsilon^{(n)}(\beta)\right|^p-c_1\tau(1-\tau)(\beta-\alpha)^2\right)d\tau\right)^{\frac{1}{p}}\right.\\
&\left.+\left(\int_0^1 |a_n\tau^n+\cdots+a_1\tau+a_0|\left(\tau^s\left|\upsilon^{(n)}(\beta)\right|^p+(1-\tau)^s\left|\upsilon^{(n)}(\alpha)\right|^p-c_2\tau(1-\tau)(\beta-\alpha)^2\right)d\tau\right)^{\frac{1}{p}}\right\}\\
&\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\sum_{i=0}^n\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\\
&\times\left\{\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+s+1}\right)\left|\upsilon^{(n)}(\alpha)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)\left|\upsilon^{(n)}(\beta)\right|^p-c_1(\beta-\alpha)^2\left[\sum_{i=0}^n\frac{|a_i|}{(i+2)(i+3)}\right]\right]^{\frac{1}{p}}\right.\\
&\left.+\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+s+1}\right)\left|\upsilon^{(n)}(\beta)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|\Gamma(s+1)\Gamma(i+1)}{\Gamma(i+s+2)}\right)\left|\upsilon^{(n)}(\alpha)\right|^p-c_2(\beta-\alpha)^2\left[\sum_{i=0}^n\frac{|a_i|}{(i+2)(i+3)}\right]\right]^{\frac{1}{p}}\right\}.
\end{aligned}
\tag{13}
$$

Corollary 7. *According to conditions of Theorem (6), taking $c_1=c_2=0$ leads to Theorem 2.3 ([15]) .*

4 Conclusion

The convex function classes and their associated inequalities play a crucial role in the theories of inequality. Convex sets and functions have made significant contributions to the advancements in pure and applied sciences, and they continue to provide novel and innovative applications in various fields.

The concept of strongly s-convex functions, which are n-times differentiable, has emerged as an important class of mathematical functions in optimization theory. Strongly s-convex functions satisfy a property where the functions value at the midpoint of a line segment connecting two points in its domain is less than or equal to the weighted average of the functions values at the two endpoints, with the weights determined by the value of s. The n-times differentiability of these functions allows for advanced mathematical techniques to be applied in analyzing their behavior and optimizing them.

The main difference between n-times differentiable strongly s-convex functions and s-convex functions is the strength of their convexity properties. Considering the inequalities of these functions leads to an additional term in the inequality provided by the strongly s-convex functions. This additional term contains the second power of the distance between x and y, which leads to a strongly convexity property. The strongly s-convexity

property implies the *s*-convexity property, but the converse is not necessarily true. The strongly convexity property of strongly s-convex functions often leads to faster convergence of optimization algorithms and may also provide additional guarantees on the behavior of the function.

5 Declarations

5.1 Conflict of interest:

The authors declare that they have no conflict of interest.

5.2 Authors' contributions:

D.D.D.-Methodology, Supervision, Resources, Methodology, Validation, Conceptualization, Formal analysis; G.Ş.-Data Curation, Writing-Original Draft, Investigation, Visualization. All authors read and approved the final submitted version of this manuscript.

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5.5 Data availability statement:

All data that support the findings of this study are included within the article.

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