



Original Study

Dynamical analysis of a Tumor Growth model under the effect of fractal fractional Caputo-Fabrizio derivative

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Abstract

Fractal-fractional derivatives, which are still rather new, are frequently used to look into the complexities of an issue. Today, tumors are a prevalent and difficult-to-treat condition. The Caputo-Fabrizio-fractal-fractional derivative, which is a non-singular derivative, has been used to explore the tumor-growth model both quantitatively and numerically. By using fixed-point theorems, it has been demonstrated that the model underneath the Caputo-Fabrizio-fractal-fractional derivative exists and is unique. The Ulam-Hyres stability of the model was evaluated using non-linear analysis. Using Lagrangian-piecewise interpolation and the fundamentals of fractional calculus, we can develop an algorithm that will enable us to determine the numerical solutions for the new model. In order to show the method's dependability and effectiveness, numerical simulations are also included. Utilizing an exponential-decay kernel, we evaluated the dynamics of the Tumor Growth model to see if the non-singular fractal fractional operator offered better dynamics for the model under consideration.

Keywords: Tumor Growth model, fractal-fractional derivative, fixed-point theorem, exponential-decay kernel, Ulam-Hyres stability.

AMS 2020 codes: 49M37; 65Lxx; 65L20; 65Mxx.

1 Introduction

Fractional calculus is an appealing area of applied analysis having the purpose of simulating biological issues in science. The modeling of numerous diseases has benefited greatly from the use of fractional calculus [1–6]. Mathematical models are regarded as a useful tool for debating hypotheses, validating experiments and simulating the dynamics of complicated objects. It is a very significant and well-known piece of mathematical research to develop and numerically solve an infectious disease mathematical model in biological science. Three distinct kinds of operators are primarily used in fractional calculus: the Caputo-Fabrizio fractional derivative, the Riemann-Liouville and Caputo derivatives [7]. Simulation and modeling processes utilizing the above three different types of kernels are significantly impacted by fractional order differential and integral equations. As can be observed, the fractional order differential and integral equations model is more precise than the classical model [8, 9].

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A lump of abnormal cells that develops in the body is called a tumor. A tumor might not always indicate cancer. There are several benign tumors that are not cancerous, but malignant tumors are. Cancer is sometimes referred to as "malignant." Malignant tumors continue to be among the most prevalent diseases that determine a person's death, despite advances in science and medicine. This is a multiple-step illness that arises when aberrant cells undergo mutations in their DNA. Malignant tumors are caused by unchecked cell growth and the human body experiences cell mutations in large numbers everyday. While some of these altered cells pass away, others survive and develop into malignant cells. A tumor may form when the body's defenses against these cells, such as the immune system and other systems, fail. Malignant cells can be separated from healthy cells by their volume, structure, texture, amount, discrimination, functionality and capacity to spread to different tissues and viscera [1]. Ahmad Shabir et al. [3] presented a work in 2021 that claimed fixed point theorems had been used to demonstrate the validity of the tumor-immune interaction model under the fractal-fractional derivative. Kumar Sunil et al. [4] proposed a tumor-immune model in an arbitrary order. The Adams-Bashforth-Moulton method (ABM) and the Atangana-Toufik technique (AT) as two numerical schemes to capture the dynamical behavior of the suggested model was used. Shabir et al. [5] presented a piece of work in which they claimed fixed point theorems had been employed to show the tumor-immune interaction model's viability under the Caputo-Fabrizio and Caputo fractal-fractional derivatives. The model has been shown to be Ulam-Hyres stable.

When the research is explored, many cancer models that address various perspectives on tumors can be found. Among them, Watanabe et al. [10] describe a Tumor Growth model for the evolution in time of the tumor volume prior to and following a radio surgical surgery.

The model that is used in this investigation will be one that considers the growth of the tumor following a single instant dose $\mathcal{D}$ of radiation given to the tumor at time t .

We work with the Tumor Growth model provided by

$$\begin{aligned}\frac{du(t)}{dt} &= w(t)q(\mathcal{D})u(t) - h(\mathcal{D})u(t) \\ \frac{dv(t)}{dt} &= \mathfrak{h}(\mathcal{D})u(t) - \mu_d v(t) \\ \frac{dw(t)}{dt} &= -\omega w(0)w(t).\end{aligned}\tag{1}$$

The functions and parameter of the considered model are as follows.

- $\mathcal{D}$ stands for dose rate of radiation,
- $u(t)$ stands for the tumor's proliferating volume,
- $v(t)$ stands for the volume made up of non-diverging cells,
- $w(t)$ stands for the rate of Tumor Growth,
- $w(0)$ stands for the tumor's initial growth rate,
- μ_d stands for the rate of cell clearance,
- ω stands for the component that retards vascular growth,
- $q(\mathcal{D})$ is a function that stands for the cell proliferation probability,
- $\mathfrak{h}(\mathcal{D})$ is a function that stands for the non-diverging state at a rate.

Similarly, if the order of the fractal approaches 1, the idea of classical differentiation is expanded to include fractal differentiation equivalent to the classical derivative. The concept of fractal derivatives offers solutions to

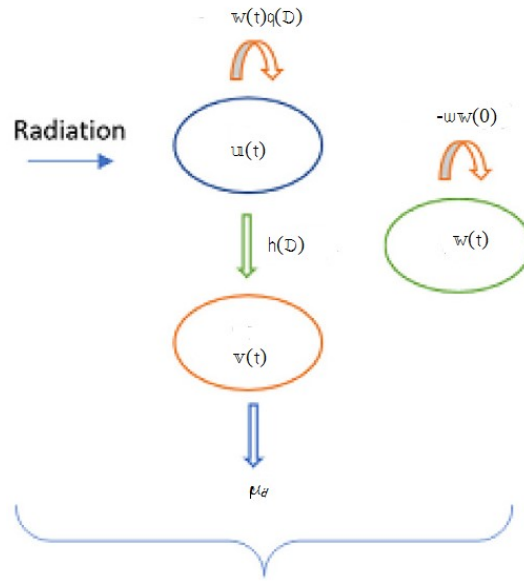


Fig. 1 Diagram for the Tumor Growth model.

a variety of issues in nature [11]. The behavior of each sub-class was simulated from a variety of viewpoints, such as changes in fractal-fractional dimension orders and the system's approximate solutions were derived using the Adams-Bashforth method. The work mentioned above provides our inspiration as we examine the model (1) as fractal-fractional in the Caputo-Fabrizio (CF) sense.

By using fresh mathematical techniques, in this study we change this model. As a result of the $\mathcal{FF}\mathcal{E}$ derivative success in simulating infectious diseases and being motivated by the aforementioned advantageous uses of particular fractional derivatives in the epidemic model, we expand the model (1). The model is generalized to be under $\mathcal{FF}\mathcal{E}$ fractional derivative of order σ .

$$\begin{aligned}\mathcal{FF}\mathcal{E}\mathcal{D}_{0,t}^{\sigma,\vartheta}u(t) &= w(t)q(\mathcal{D})u(t) - h(\mathcal{D})u(t) \\ \mathcal{FF}\mathcal{E}\mathcal{D}_{0,t}^{\sigma,\vartheta}v(t) &= h(\mathcal{D})u(t) - \mu_d v(t) \\ \mathcal{FF}\mathcal{E}\mathcal{D}_{0,t}^{\sigma,\vartheta}w(t) &= -\omega w(0)w(t).\end{aligned}\tag{2}$$

with the initial condition

$$u(0) = u_0, v(0) = v_0, w(0) = w_0.\tag{3}$$

The following is an outline of the manuscript: The introduction and motivation portion are located in section 1. Fractional calculus fundamental definitions are covered in section 2. The model's existence, uniqueness and HU-stability are presented in section 3 which also presents the numerical findings obtained using the fractional Lagrangian-piecewise interpolation approach. The numerical simulations section includes examples of numerical findings for various fractional orders in section 4. Discussion and numerical explanation is given in section 5. The article's conclusion is offered in section 6.

2 Preliminaries

The fractal-fractional calculus's fundamental ideas are discussed in this section. For simplicity of use, we write the fractal fractional, including the exponential decay kernel respectively, as $\mathcal{FF}\mathcal{E}$. Let $\mathfrak{U}(t)$ be a fractal-differentiable continuous function on (m, n) . The fractal-fractional derivatives are subsequently defined as:

Definition 1. [9] If $0 \leq \sigma, \vartheta \leq 1$ are given, then the fractal-fractional derivative of $\mathfrak{U}(t)$ with fractional order σ and fractal dimension ϑ in the Riemann-Liouville sense, including the exponential decay type kernel, is written as:

$${}^{\mathcal{F}\mathcal{F}\mathcal{E}}\mathfrak{D}_{0,t}^{\sigma,\vartheta}\mathfrak{U}(t) = \frac{\mathcal{M}(\sigma)}{1-\sigma} \frac{d}{dt^\vartheta} \int_0^t \exp\left[-\frac{\sigma}{1-\sigma}(t-\zeta)^{\alpha-\sigma-1}\right] \mathfrak{U}(\zeta) d\zeta,$$

where $\sigma > 0$, $\vartheta \leq \alpha \in \mathcal{N}$, and $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

Definition 2. [9] Fractal-fractional integral for $\mathfrak{U}(t)$ having a exponential decay type kernel is defined for fractional order σ and fractal dimension ϑ as:

$${}^{\mathcal{F}\mathcal{F}\mathcal{E}}\mathfrak{I}_{0,t}^{\sigma,\vartheta}\mathfrak{U}(t) = \frac{\vartheta(1-\sigma)t^{\vartheta-1}\mathfrak{U}(t)}{\mathcal{M}(\sigma)} + \frac{\vartheta\sigma}{\mathcal{M}(\sigma)} \int_0^t \zeta^{\sigma-1}\mathfrak{U}(\zeta) d\zeta.$$

3 Existence and stability theory

Using the fixed point theory, we demonstrate that the model under evaluation has at least one unique solution. Due to the integral being differentiable, we can formulate the suggested model as follows:

$$\begin{aligned} {}^{\mathcal{C}\mathcal{F}}\mathfrak{D}_t^\sigma u(t) &= \gamma t^{\gamma-1} \mathcal{U}(t, u, v, w), \\ {}^{\mathcal{C}\mathcal{F}}\mathfrak{D}_t^\sigma v(t) &= \gamma t^{\gamma-1} \mathcal{V}(t, u, v, w), \\ {}^{\mathcal{C}\mathcal{F}}\mathfrak{D}_t^\sigma w(t) &= \gamma t^{\gamma-1} \mathcal{W}(t, u, v, w), \end{aligned} \quad (4)$$

where

$$\begin{cases} \mathcal{U}(t, u, v, w) = w(t) \mathfrak{q}(\mathcal{D})u(t) - \mathfrak{h}(\mathcal{D})u(t), \\ \mathcal{V}(t, u, v, w) = \mathfrak{h}(\mathcal{D})u(t) - \mu_d v(t), \\ \mathcal{W}(t, u, v, w) = -\omega w(0) w(t). \end{cases}$$

Equation (4) can be written as:

$$\begin{cases} {}^{\mathcal{C}\mathcal{F}}\mathfrak{D}_t^\sigma \mathfrak{Y}(t) = \gamma t^{\gamma-1} \Omega(t, \mathfrak{Y}(t)), \\ \mathfrak{Y}(0) = \mathfrak{Y}_0, \end{cases}$$

where $\mathfrak{Y}(t)$ and $\Omega(t, \mathfrak{Y}(t))$ are defined above and using the Caputo-Fabrizio fractional integral [12], we obtain

$$\mathfrak{Y}(t) = \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma\vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(t)) d\xi, \quad (5)$$

where

$$\mathfrak{Y}(t) = \begin{cases} u(t) \\ v(t) \\ w(t) \end{cases}, \mathfrak{Y}(0) = \begin{cases} u(0) \\ v(0) \\ w(0) \end{cases}, \Omega(t, \mathfrak{Y}(t)) = \begin{cases} \mathcal{U}(t, u, v, w), \\ \mathcal{V}(t, u, v, w), \\ \mathcal{W}(t, u, v, w). \end{cases}$$

A Banach space $\mathcal{B} = \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ is defined for the existence theory, where $\mathcal{C} = [0, \mathcal{T}]$ under the norm, $\|\mathfrak{Y}\| = \max_{t \in [0, \mathcal{T}]} |u(t) + v(t) + w(t)|$.

Define an operator $\mathfrak{X} : \mathcal{B} \rightarrow \mathcal{B}$ as:

$$\mathfrak{X}(\mathfrak{Y})(t) = \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma\vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(t)) d\xi. \quad (6)$$

Now, let's assume that the nonlinear function $\Omega(t, \mathfrak{Y}(t))$ fulfills the required Lipschitz and growth conditions.

- For each $\mathfrak{Y} \in \mathcal{B}$, $\exists$ constants $\mathcal{P}_\Omega > 0$ and $\mathcal{Q}_\Omega$ such that

$$|\Omega(t, \mathfrak{Y}(t))| \leq \mathcal{P}_\Omega |\mathfrak{Y}(t)| + \mathcal{Q}_\Omega. \quad (7)$$

- For each $\mathfrak{Y}, \overline{\mathfrak{Y}} \in \mathcal{B}$, $\exists$ constants $\mathcal{O}_\Omega > 0$ such that

$$|\Omega(t, \mathfrak{Y}(t)) - \Omega(t, \overline{\mathfrak{Y}}(t))| \leq \mathcal{O}_\Omega |\mathfrak{Y}(t) - \overline{\mathfrak{Y}}(t)|. \quad (8)$$

Theorem 1. Suppose that the Lipschitz condition (7) holds. Let Ω be a continuous function from $[0, \mathcal{T}] \times \mathcal{B}$ to $\mathfrak{R}$. If so, at least one solution exists for the suggested model.

Proof. The operator $\mathfrak{X}$ described by (6) must first be proven to be absolutely continuous. Because Ω is continuous, likewise $\mathfrak{X}$ must be continuous.

Let $\mathcal{B}_\mathfrak{R} = \{\mathfrak{Y} \in \mathcal{B} : \|\mathfrak{Y}\| \leq \mathfrak{R}, \mathfrak{R} > 0\}$. Now for any $\mathfrak{Y} \in \mathcal{B}$, we have

$$\begin{aligned} |\mathfrak{X}(\mathfrak{Y})| &= \max_{t \in [0, \mathcal{T}]} \left| \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(\xi)) d\xi \right| \\ &\leq \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}\| + \mathcal{Q}_\Omega) + \max_{t \in [0, \mathcal{T}]} \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} |\Omega(\xi, \mathfrak{Y}(\xi))| d\xi \\ &\leq \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}\| + \mathcal{Q}_\Omega) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}\| + \mathcal{Q}_\Omega) \mathcal{T}^\vartheta \mathfrak{L}(\sigma, \vartheta) \\ &\leq \mathfrak{R}. \end{aligned}$$

As a result, the operator $\mathfrak{X}$, where $\mathfrak{L}(\sigma, \vartheta)$ denotes the beta function, is uniformly bounded. Let's use $t_1 < t_2 \leq \mathcal{T}$ to determine the equicontinuity of $\mathfrak{X}$.

$$\begin{aligned} |\mathfrak{X}(\mathfrak{Y})(t_2) - \mathfrak{X}(\mathfrak{Y})(t_1)| &= \left| \frac{\vartheta t_2^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t_2, \mathfrak{Y}(t_2)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^{t_2} \xi^{\vartheta-1} \Omega(\xi, \mathfrak{Y}(\xi)) d\xi - \frac{\vartheta t_1^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t_1, \mathfrak{Y}(t_1)) \right. \\ &\quad \left. - \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^{t_1} \xi^{\vartheta-1} \Omega(\xi, \mathfrak{Y}(\xi)) d\xi \right| \\ &\leq \frac{\vartheta t_2^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}\| + \mathcal{Q}_\Omega) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}\| + \mathcal{Q}_\Omega) t_2^\vartheta \mathfrak{L}(\sigma, \vartheta) - \frac{\vartheta t_1^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \\ &\quad \times (\mathcal{P}_\Omega \|\mathfrak{Y}\| + \mathcal{Q}_\Omega) - \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}\| + \mathcal{Q}_\Omega) t_1^\vartheta \mathfrak{L}(\sigma, \vartheta) \end{aligned}$$

when $t_1 \rightarrow t_2$ then $|\mathfrak{X}(\mathfrak{Y})(t_2) - \mathfrak{X}(\mathfrak{Y})(t_1)| \rightarrow 0$. Therefore, we can state that

$\|\mathfrak{X}(\mathfrak{Y})(t_2) - \mathfrak{X}(\mathfrak{Y})(t_1)\| \rightarrow 0$, as $t_1 \rightarrow t_2$.

That $\mathfrak{X}$ is equicontinuous can thus be seen from this. According to the "Arzela-Ascoli theorem", $\mathfrak{X}$ is absolutely continuous. Furthermore, according to "Schauder's fixed theorem," the model under consideration has at least one solution.

Theorem 2. Suppose that equation (8) holds. Then the equation (5) possesses at most one solution if

$$\mathfrak{p} = \left[\frac{\vartheta t^{\vartheta-1} (1-\sigma) \mathcal{O}_\Omega}{\mathcal{M}(\sigma)} + \frac{\sigma \mathcal{T}^\vartheta \mathcal{O}_\Omega}{\mathcal{M}(\sigma)} \right] < 1,$$

i.e. considers model has a unique solution.

Proof. The Picard operator $\mathfrak{X} : \mathcal{B} \rightarrow \mathcal{B}$ is defined as follows:

$$\mathfrak{X}(\mathfrak{Y})(t) = \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(\xi)) d\xi.$$

$$\begin{aligned}
\|\mathfrak{X}(\mathfrak{Y}(t)) - \mathfrak{Y}(0)\| &= \left\| \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(t)) d\xi - \mathfrak{Y}(0) \right\| \\
&= \left\| \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(t)) d\xi \right\| \\
&\leq \left\| \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) \right\| + \left\| \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(t)) d\xi \right\| \\
&\leq \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \|\Omega(t, \mathfrak{Y}(t))\| + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \|\Omega(t, \mathfrak{Y}(t))\| d\xi \\
&\leq \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}(t)\| + \mathcal{Q}_\Omega) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} (\mathcal{P}_\Omega \|\mathfrak{Y}(t)\| + \mathcal{Q}_\Omega) \\
&\leq \left(\frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \right) (\mathcal{P}_\Omega \|\mathfrak{Y}(t)\| + \mathcal{Q}_\Omega) < \infty.
\end{aligned}$$

Now, for $\mathfrak{Y}, \overline{\mathfrak{Y}} \in \mathcal{B}$, we have

$$\begin{aligned}
\|\mathfrak{X}(\mathfrak{Y}(t)) - \mathfrak{X}(\overline{\mathfrak{Y}}(t))\| &= \left\| \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(\xi, \mathfrak{Y}(\xi)) d\xi - \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \overline{\mathfrak{Y}}(t)) - \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(\xi, \overline{\mathfrak{Y}}(\xi)) d\xi \right\| \\
&= \left\| \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} (\Omega(t, \mathfrak{Y}(t)) - \Omega(t, \overline{\mathfrak{Y}}(t))) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} (\Omega(\xi, \mathfrak{Y}(\xi)) - \Omega(\xi, \overline{\mathfrak{Y}}(\xi))) d\xi \right\| \\
&\leq \frac{\vartheta t^{\vartheta-1}(1-\sigma) \mathcal{O}_\Omega}{\mathcal{M}(\sigma)} \|\mathfrak{Y}(t) - \overline{\mathfrak{Y}}(t)\| + \frac{\sigma \mathcal{T}^\vartheta \mathcal{O}_\Omega}{\mathcal{M}(\sigma)} \|\mathfrak{Y}(t) - \overline{\mathfrak{Y}}(t)\| \\
&\leq \left[\frac{\vartheta t^{\vartheta-1}(1-\sigma) \mathcal{O}_\Omega}{\mathcal{M}(\sigma)} + \frac{\sigma \mathcal{T}^\vartheta \mathcal{O}_\Omega}{\mathcal{M}(\sigma)} \right] \|\mathfrak{Y}(t) - \overline{\mathfrak{Y}}(t)\| \\
&\leq \mathfrak{p} \|\mathfrak{Y}(t) - \overline{\mathfrak{Y}}(t)\|.
\end{aligned}$$

Hence, $\mathfrak{X}$ is a contraction. According to the Banach principle of contraction, the suggested model has a unique solution.

3.1 UH-stability

In this section, we will show the UH-stability of the suggested model for the Caputo-Fabrizio operator.

Definition 3. The suggested model is UH-stable if $\exists \mathfrak{X}_{\sigma, \vartheta} \geq 0$ such that for any $\delta > 0$, and for every $\mathfrak{Y} \in (\mathfrak{C}[0, \mathcal{T}], \mathfrak{R})$ the following inequality is satisfied:

$$\left| {}^{\mathfrak{F}\mathfrak{F}\mathfrak{C}}\mathfrak{D}_t^{\sigma, \vartheta} \mathfrak{Y}(t) - \Omega(t, \mathfrak{Y}(t)) \right| \leq \delta, \quad t \in [0, \mathcal{T}],$$

and $\exists$ a unique solution $\rho \in (\mathfrak{C}[0, \mathcal{T}], \mathfrak{R})$ such that

$$|\mathfrak{Y}(t) - \rho(t)| \leq \mathfrak{X}_{\sigma, \vartheta} \delta, \quad t \in [0, \mathcal{T}].$$

Suppose a small perturbation $\varphi \in \mathfrak{C}[0, \mathcal{T}]$, such that $\varphi(0) = 0$. Let

1. $|\varphi(t)| \leq \delta$ for $\delta > 0$.
2. ${}^{\mathfrak{F}\mathfrak{F}\mathfrak{C}}\mathfrak{D}_t^{\sigma, \vartheta} \mathfrak{Y}(t) = \Omega(t, \mathfrak{Y}(t)) + \varphi(t)$.

Lemma 3. *The answer to the perturbed method*

$$\begin{cases} \mathfrak{D}_t^{\sigma, \vartheta} \mathfrak{Y}(t) = \Omega(t, \mathfrak{Y}(t)) + \varphi_{up}(t), \\ \mathfrak{Y}(0) = \mathfrak{Y}_0 \end{cases}$$

satisfies the relationship given below

$$\left| \mathfrak{X}(\mathfrak{Y}) - \left(\mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(t, \mathfrak{Y}(t)) d\xi \right) \right| \leq \sigma_{\sigma, \vartheta}^* \delta, \quad (9)$$

where $\sigma_{\sigma, \vartheta}^* \delta = \frac{\vartheta \mathcal{T}^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \mathcal{T}^{\sigma+\vartheta-1} \mathfrak{L}(\sigma, \vartheta)$.

Proof. Easy to provide proof. It can be established by using equation (6). The left side of equation (9) would need to be adjusted in order to provide the desired outcomes.

Lemma 4. When Lemma 3 and situation (8) are combined, the solution of suggested model is UH-stable if $\mathfrak{p} < 1$.

Proof. $\rho \in \mathcal{B}$ has to be a unique solution and $\mathfrak{Y} \in \mathcal{B}$ has to be any solution, because we have verified that the model has a unique solution.

$$\begin{aligned} |\mathfrak{Y}(t) - \rho(t)| &= \left| \mathfrak{Y}(t) - \left(\rho(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \rho(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(\xi, \rho(\xi)) d\xi \right) \right| \\ &\leq \left| \mathfrak{Y}(t) - \left(\mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(\xi, \mathfrak{Y}(\xi)) d\xi \right) \right| \\ &\quad + \left| \mathfrak{Y}(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(\xi, \mathfrak{Y}(\xi)) d\xi \right| \\ &\quad - \left| \rho(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \Omega(t, \mathfrak{Y}(t)) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \xi^{\vartheta-1} \Omega(\xi, \mathfrak{Y}(\xi)) d\xi \right| \\ &\leq \sigma_{\sigma, \vartheta}^* \delta \left(\frac{\vartheta \mathcal{T}^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \mathcal{T}^{\sigma+\vartheta-1} \mathfrak{L}(\sigma, \vartheta) \right) \mathcal{O}_{\Omega} |\mathfrak{Y}(t) - \rho(t)| \\ &\leq \sigma_{\sigma, \vartheta}^* \delta + \mathfrak{p} |\mathfrak{Y}(t) - \rho(t)|. \end{aligned}$$

As a result, one may write

$$\|\mathfrak{Y}(t) - \rho(t)\| \leq \sigma_{\sigma, \vartheta}^* \delta + \mathfrak{p} \|\mathfrak{Y} - \rho\|.$$

The above relationship can be expressed as being

$$\|\mathfrak{Y}(t) - \rho(t)\| \leq \mathfrak{X}_{\sigma, \vartheta} \delta,$$

where $\mathfrak{X}_{\sigma, \vartheta} \delta = \frac{\sigma_{\sigma, \vartheta}^*}{1-\mathfrak{p}}$. Hence, UH-stability characterizes the solution of suggested problem.

4 Numerical scheme of model

The numerical scheme for the suggested model underneath the Caputo-Fabrizio-fractal-fractional derivative will be established in this part using lagrangian piecewise interpolation. The numerical Adams-Bashforth method is the basis of the proposed approach. Applying the Caputo-Fabrizio integral to the equation (4), we have

$$\begin{aligned}\square(t) &= u(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{U}(t, u, v, w) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \chi^{\vartheta-1} \mathcal{U}(\chi, u, v, w) d\chi, \\ v(t) &= v(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{V}(t, u, v, w) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \chi^{\vartheta-1} \mathcal{V}(\chi, u, v, w) d\chi, \\ w(t) &= w(0) + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{W}(t, u, v, w) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \chi^{\vartheta-1} \mathcal{W}(\chi, u, v, w) d\chi.\end{aligned}\quad (10)$$

We now deduce the mathematical scheme at $t = t_{c+1}$,

$$\begin{aligned}u^{c+1} &= u^0 + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{U}(t_c, u^c, v^c, w^c) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \chi^{\vartheta-1} \mathcal{U}(\chi, u, v, w) d\chi, \\ v^{c+1} &= v^0 + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{V}(t_c, u^c, v^c, w^c) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \chi^{\vartheta-1} \mathcal{V}(\chi, u, v, w) d\chi, \\ w^{c+1} &= w^0 + \frac{\vartheta t^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{W}(t_c, u^c, v^c, w^c) + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_0^t \chi^{\vartheta-1} \mathcal{W}(\chi, u, v, w) d\chi.\end{aligned}\quad (11)$$

The approximate model for the above is then presented below:

$$\begin{aligned}u^{c+1} &= u^c + \frac{\vartheta t_c^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{U}(t_c, u^c, v^c, w^c) - \frac{\vartheta t_{c-1}^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{U}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}) \\ &\quad + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_{t_c}^{t_{c+1}} \chi^{\vartheta-1} \mathcal{U}(\chi, u, v, w) d\chi, \\ v^{c+1} &= v^c + \frac{\vartheta t_c^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{V}(t_c, u^c, v^c, w^c) - \frac{\vartheta t_{c-1}^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{V}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}) \\ &\quad + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_{t_c}^{t_{c+1}} \chi^{\vartheta-1} \mathcal{V}(\chi, u, v, w) d\chi, \\ w^{c+1} &= w^c + \frac{\vartheta t_c^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{W}(t_c, u^c, v^c, w^c) - \frac{\vartheta t_{c-1}^{\vartheta-1}(1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{W}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}) \\ &\quad + \frac{\sigma \vartheta}{\mathcal{M}(\sigma)} \int_{t_c}^{t_{c+1}} \chi^{\vartheta-1} \mathcal{W}(\chi, u, v, w) d\chi.\end{aligned}\quad (12)$$

Currently, we obtain the following results using Lagrangian polynomial piecewise interpolation and integration,

$$\begin{aligned}
 u^{c+1} &= u^c + \frac{\vartheta t_c^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{U}(t_c, u^c, v^c, w^c) - \frac{\vartheta t_{c-1}^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{U}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}) \\
 &\quad + \frac{\vartheta \sigma}{\mathcal{M}(\sigma)} \frac{3}{2} (\Delta t) t_c^{\vartheta-1} \mathcal{U}(t_c, u^c, v^c, w^c) - \frac{\vartheta \sigma}{\mathcal{M}(\sigma)} \frac{(\Delta t)}{2} t_{c-1}^{\vartheta-1} \mathcal{U}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}), \\
 v^{c+1} &= v^c + \frac{\vartheta t_c^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{V}(t_c, u^c, v^c, w^c) - \frac{\vartheta t_{c-1}^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{V}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}) \\
 &\quad + \frac{\vartheta \sigma}{\mathcal{M}(\sigma)} \frac{3}{2} (\Delta t) t_c^{\vartheta-1} \mathcal{V}(t_c, u^c, v^c, w^c) - \frac{\vartheta \sigma}{\mathcal{M}(\sigma)} \frac{(\Delta t)}{2} t_{c-1}^{\vartheta-1} \mathcal{V}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}), \\
 w^{c+1} &= w^c + \frac{\vartheta t_c^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{W}(t_c, u^c, v^c, w^c) - \frac{\vartheta t_{c-1}^{\vartheta-1} (1-\sigma)}{\mathcal{M}(\sigma)} \mathcal{W}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}) + \\
 &\quad \frac{\vartheta \sigma}{\mathcal{M}(\sigma)} \frac{3}{2} (\Delta t) t_c^{\vartheta-1} \mathcal{W}(t_c, u^c, v^c, w^c) - \frac{\vartheta \sigma}{\mathcal{M}(\sigma)} \frac{(\Delta t)}{2} t_{c-1}^{\vartheta-1} \mathcal{W}(t_{c-1}, u^{c-1}, v^{c-1}, w^{c-1}).
 \end{aligned} \tag{13}$$

The broad numerical output of the model under consideration is represented by equation (13).

5 Discussion and numerical simulations

The parametric values indicated in Table 1 below, along with the initial conditions $u(0) = 10, v(0) = 10, w(0) = 10$, are taken from [2]. Using Matlab, the numerical outcomes are displayed in Figures (2,3,4,5) at various fractional orders and fractal dimension values. Our goal is to examine how changes in fractional order and fractional dimension affect Tumor Growth rate, non-deviating cells and tumor proliferation. In addition, the connection involving Tumor Growth rate, non-deviating cells and tumor proliferation is examined through graphs. It should be noted that decreasing fractional order and fractional dimension increases the amount of non-deviating cells, Tumor Growth rate and tumor proliferation. Due to a wider variety of fractional order and fractal dimension values, the growth and decay processes vary in each figure. The non-local and non-singular kernel of the new operator results in outstanding model dynamics. The model's curves resemble integer-order curves when both the fractal dimensions and fractional orders are trending toward unity. We get to the conclusion that the integer-order model of malignant Tumor Growth is less broad than the fractal-fractional order model.

Table 1 Values of the parameters for simulation.

Parameters	Values
$w(0)$	2
μ_d	0.6
ω	0.8
$q(\mathcal{D})$	1
$h(\mathcal{D})$	0.7

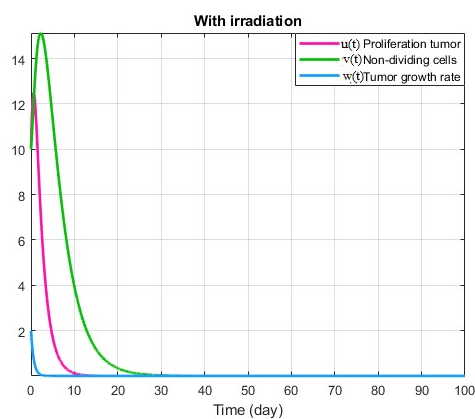


Fig. 2 Malignant Tumor Growth model numerical simulation for σ and $\vartheta = 1$.

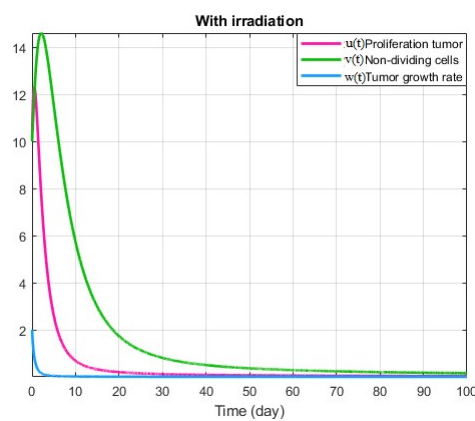


Fig. 3 Malignant Tumor Growth model numerical simulation for σ and $\vartheta = 0.9$.

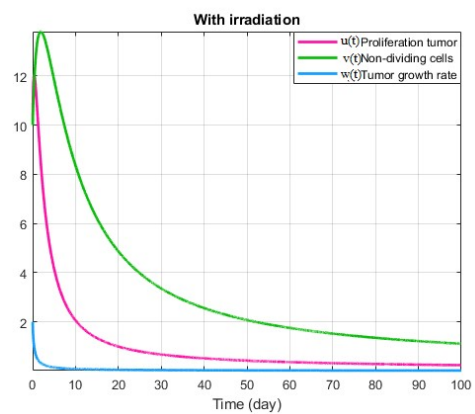


Fig. 4 Malignant Tumor Growth model numerical simulation for σ and $\vartheta = 0.72$.

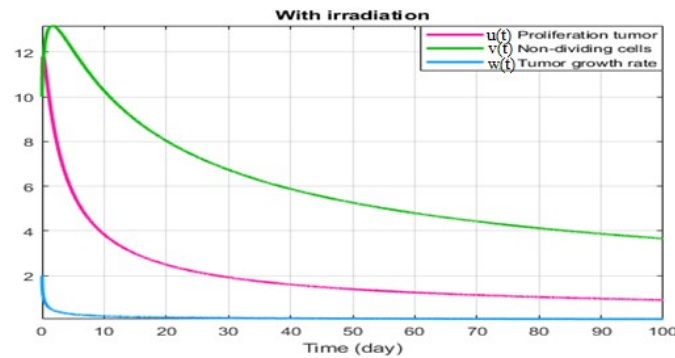


Fig. 5 Malignant Tumor Growth model numerical simulation for σ and $\vartheta = 0.54$.

6 Conclusion

In this paper, we used Caputo-Fabrizio-fractal-fractional derivative to provide a model for Tumor Growth. We looked into both the qualitative and quantitative aspects of the model using the creative operator. The existence and stability theories are covered in the qualitative portion, while the numerical solutions and simulations are covered in the quantitative section. We have shown the existence and uniqueness of the findings via the fixed-point Schauder and Banach theorems. The suggested model is Ulam-Hyres stable, as demonstrated by nonlinear analysis. For each of the model's several compartments, we created a numerical scheme and produced the corresponding numerical results. We analyzed the results using Matlab for various fractional order and fractal dimension values. We have found that when the Tumor Growth model is mathematically modeled, the innovative operator yields outstanding results. The graphs demonstrate that the dynamics of the tumor cells are affected by changing the fractal order. The figures show how fractional and fractal orders interact with one another. We will use this new operator to observe various physical processes [13–16].

7 Declarations

7.1 Conflict of interest:

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

7.2 Funding:

No funding was received for this work.

7.3 Authors Contribution:

R.S.-Conceptualization, Methodology, Software. J.M.-Formal Analysis, Writing-Original Draft, Writing-Review and Editing. V.K.G.-Conceptualization, Validation, Formal Analysis, Writing-Original Draft. All authors read and approved the final submitted version of this manuscript.

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7.5 Data availability statement:

All data that support the findings of this study are included within the article (and any supplementary files).

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