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Original Study

New analytical solutions and modulation instability analysis for the nonlinear (1+1)-dimensional Phi-four model

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Abstract

This paper deals with the nonlinear (1+1)-dimensional Phi-four equation in the sense of the Katugampola operator, which can be used to model a variety of real-world applications. To solve this equation, we propose a generalized double auxiliary equation method that yields several new exact solutions. We also use linear stability analysis to discuss the instability modulation analysis for stationary solutions. Other partial differential equations can have their exact solutions found using the proposed methodology.

Keywords: Generalized double auxiliary equation method, Katugampola derivative, instability modulation analysis, exact solutions.
AMS 2020 codes: 35C07; 35Dxx; 34Axx.

1 Introduction

Partial differential equations have recently received a lot of attention because of their capacity to simulate various complex phenomena in the real world. These phenomena can occur in a wide range of contexts, such as signal processing control, complex biological systems, particle vibration, tsunamis, human diseases, physics, engineering, finance and so on.

In this paper, we deal with the nonlinear (1+1)-dimensional Phi-four equation, an important partial differential equation used for decades to model the kink and anti-kink solitary wave interactions that occur in biological structures and nuclear particle physics. The nonlinear (1+1)-dimensional Phi-four equation [1] is given as follows:

$$D_{tt}^{2\alpha}\Theta - aD_{xx}^{2\alpha}\Theta - \Theta + \Theta^3 = 0, \tag{1}$$

where $a > 0, 0 < \alpha \leq 1$ and D_t^{α} is the Katugampola derivative of order α, which was first introduced in [2]. For a given function $\Psi : (0, +\infty) \longrightarrow R, \alpha \in (0, 1]$, the Katugampola derivative is defined as follows:

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$$D^{\alpha}(\Psi)(t)=\lim_{\varepsilon\to 0}\frac{\Psi\left(te^{\varepsilon t^{-\alpha}}\right)-\Psi(t)}{\varepsilon}. \tag{2}$$

The chain rule, which is the most important formula of the Katugampola derivative, is used:

$$D_t^{\alpha}\Psi(\phi(t))=\Psi'_{\phi}[\phi(t)]D_t^{\alpha}(\phi(t)).$$

Various approaches have been used to find exact solutions to nonlinear partial equations. Some of these approaches are as follows: the Khater II and Novel Kudryashov schemes [3], the modified extended tanh-function method [4], the $(\frac{G'}{G},\frac{1}{G})$-expansion technique [5], the homogeneous balance method [6], the q-homotopy analysis transform method [7], the hyperbolic function expansion method [8], the Jacobi elliptic function method [9], the F-expansion method [10], the He's homotopy perturbation method [11] and the bifurcation theory method of dynamical systems [12], the Weierstrass elliptic function method [13], the advanced $exp(-\varphi(\varepsilon))$ expansion approach with the conformable derivative [14], the generalized Kudryashov method [15], the improved Bernoulli sub equation function method [16], the modified exponential function method [17], the generalized exponential rational function [18] and various other methods [19–23].

The generalized double auxiliary equation method is a recent method proposed in [24] to solve the space-time fractional nonlinear equations, yielding promising new solutions. In addition, some studies investigate modulation instability analysis to describe the stability of stationary solutions [25, 26]. As a result, the first primary goal of the paper is to find new solutions to the nonlinear (1+1)-dimensional Phi-four equation using the generalized double auxiliary equation method. The second goal is to investigate the instability modulation for stationary solutions using linear stability analysis.

The paper is organized as follows. Section 2 describes the generalized double auxiliary equation method. In Section 3, the method is used to obtain exact solutions to the nonlinear (1+1)-dimensional Phi-four equation. Section 4 discusses instability modulation analysis for the stationary solutions. In Section 5, we conclude and discuss some perspectives of this research.

2 The generalized double auxiliary equation method

To describe the steps of the generalized double auxiliary equation method [24], the following general nonlinear equation with independent variables, $u(x,y,z,\cdots,t)$ is considered:

$$P(u,u_x,u_y,u_z,\cdots,D_x^{\alpha}u,D_y^{\alpha}u,D_z^{\alpha}u,D_t^{\alpha}u,\cdots)=0, \tag{3}$$

in which $u=u(x,y,z,\cdots,t)$, P is a polynomial in u and $D_x^{\alpha}u, D_y^{\alpha}u$, $D_z^{\alpha}u$ and $D_t^{\alpha}u$ are the Katugampola derivatives of u with respect to x, y, z and t, respectively.

- Step 1. Use the wave transformation

$$U(x,y,z,\cdots,t)=U(\xi),\xi=\frac{kx^{\alpha}}{\alpha}+\frac{ly^{\alpha}}{\alpha}+\frac{hz^{\alpha}}{\alpha}+\cdots-\frac{vt^{\alpha}}{\alpha}, \tag{4}$$

where k, l, h and v are four constants, to convert equation (3) to a nonlinear ordinary differential equation with new variable ξ

$$H\left(U,U',U'',U''',\cdots\right)=0, \tag{5}$$

where U' is the differentiation with respect to ξ.

- Step 2. Suppose the exact solution to equation (5) can be expressed as

$$U(\xi) = \frac{\sum_{i=0}^{N} a_i \left(\frac{h(\xi)}{g(\xi)}\right)^i}{\sum_{j=0}^{M} b_j \left(\frac{h(\xi)}{g(\xi)}\right)^j}, \tag{6}$$

where $a_0, \cdots, a_N, b_0, \cdots, b_M$ are constants to be determined later, $a_N^2 + b_M^2 \neq 0$ and $\left(\frac{h(\xi)}{g(\xi)}\right)$ is the solution of

$$\left(\frac{h(\xi)}{g(\xi)}\right)' = A\left(\frac{h(\xi)}{g(\xi)}\right)^2 + B\left(\frac{h(\xi)}{g(\xi)}\right) + C, \tag{7}$$

where $g(\xi) = \exp(\xi)(h(\xi))'$. As a result, the solutions to equation (7) are as follows:

1. Family 1: $\Delta = B^2 - 4AC > 0$ and $C \neq 0$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = \frac{-2C[1 - \tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}{B - B\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1) - \sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\xi) - \tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}, \quad \rho_1 \in \mathbb{R}. \tag{8}$$

2. Family 2: $\Delta = B^2 - 4AC < 0$ and $C \neq 0$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = \frac{-2C[1 + \tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}{B + B\tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1) + \sqrt{-\Delta}[\tan(\frac{\sqrt{-\Delta}}{2}\xi) - \tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}, \quad \rho_1 \in \mathbb{R}. \tag{9}$$

3. Family 3: $\Delta = B^2 - 4AC = 0$, $B > 0$ and $AC > 0$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = -\frac{C(\xi - \rho_1)}{\sqrt{AC}(\xi - \rho_1) - 1}, \quad \rho_1 \in \mathbb{R}. \tag{10}$$

4. Family 4: $A \neq 0$, $B \neq 0$ and $C = 0$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = -\frac{B\exp(B\xi)}{A\exp B\xi + B\rho_1}, \quad \rho_1 \in \mathbb{R}. \tag{11}$$

5. Family 5: $A \neq 0$, $B = 0$ and $C = 0$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = -\frac{1}{A\xi + \rho_1}, \quad \rho_1 \in \mathbb{R}. \tag{12}$$

6. Family 6: $A = 0$, $B = 0$ and $C \neq 0$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = C(\xi - \rho_1), \quad \rho_1 \in \mathbb{R}. \tag{13}$$

- Step 3: Balance both the highest order nonlinear and the highest order linear terms in equation (5) to get N and M.

- Step 4: Calculate all required derivatives $U', U'', \cdots$ by replacing equation (6) and equation (7) into equation (5). A polynomial of $\left(\frac{h(\xi)}{g(\xi)}\right)$ is obtained. The coefficients of the same power of $\left(\frac{h(\xi)}{g(\xi)}\right)$ are set to zero to get a system of algebraic equations, which is solved to determine $a_i\,(i = 0, \cdots, N)$, $b_j\,(j = 0, \cdots, M)$, A, B, C, $k, l, h, \cdots, v$. The solutions of equation (5) are obtained. Therefore, by using the solutions to equation (7), we can determine the exact solutions to equation (3).

3 Exact solutions to the nonlinear (1+1)-dimensional Phi-four equation

This section uses the method described in the previous section to obtain exact solutions by employing the traveling wave transformation given by equation (14), where k and l are user-defined constants representing wave numbers and frequency, respectively.

$$\xi = k(\frac{x^{\alpha}}{\alpha} - c\frac{t^{\alpha}}{\alpha}). \tag{14}$$

When equation (14) is substituted into equation (1), the result is

$$k^2(c^2 - a)\Theta'' - \Theta + \Theta^3 = 0. \tag{15}$$

Choosing $M = 1$ and applying the homogeneous balance method in equation (15) between Θ'' and Θ^3 yields $N = 2$. As a result, the solution to equation (15) is given by the generalized double auxiliary equation method as follows:

$$\Theta(\xi) = \frac{a_0 + a_1\left(\frac{h(\xi)}{g(\xi)}\right) + a_2\left(\frac{h(\xi)}{g(\xi)}\right)^2}{b_0 + b_1\left(\frac{h(\xi)}{g(\xi)}\right)}. \tag{16}$$

We get the following system of algebraic equations by replacing equation (16) in equation (15) and then equating to zero the equal coefficients of different powers of $\left(\frac{h(\xi)}{g(\xi)}\right)$:

$\left(\frac{h(\xi)}{g(\xi)}\right)^0 : BCc^2k^2a_0b_0b_1 + BCc^2k^2a_1b_0^2 + 2C^2c^2k^2a_0b_1^2 - 2C^2c^2k^2a_1b_0b_1 + 2C^2c^2k^2a_2b_0^2 + BCak^2a_0b_0b_1 - BCak^2a_1b_0^2 - 2C^2ak^2a_0b_1^2 + 2C^2ak^2a_1b_0b_1 - 2C^2ak^2a_2b_0^2 + a_0^3 - a_0b_0^2 = 0,$

$\left(\frac{h(\xi)}{g(\xi)}\right)^1 : -2ACc^2k^2a_0b_0b_1 + 2ACc^2k^2a_1b_0^2 - B^2c^2k^2a_0b_0b_1 + B^2c^2k^2a_1b_0^2 + 3BCc^2k^2a_0b_1^2 - 3BCc^2k^2a_1b_0b_1 + 6BCc^2k^2a_2b_0^2 + 2ACak^2a_0b_0b_1 - 2ACak^2a_1b_0^2 + B^2ak^2a_0b_0b_1 - B^2ak^2a_1b_0^2 - 3BCak^2a_0b_1^2 + 3BCak^2a_1b_0b_1 - 6BCak^2a_2b_0^2 + 3a_0^2a_1 - 2a_0b_0b_1 - a_1b_0^2 = 0,$

$\left(\frac{h(\xi)}{g(\xi)}\right)^2 : -3ABc^2k^2a_0b_0b_1 + 3ABc^2k^2a_1b_0^2 + 2ACc^2k^2a_0b_1^2 - 2ACc^2k^2a_1b_0b_1 + 8ACc^2k^2a_2b_0^2 + B^2c^2k^2a_0b_1^2 - B^2c^2k^2a_1b_0b_1 + 4B^2c^2k^2a_2b_0^2 + 3BCc^2k^2a_2b_0b_1 + 3ABak^2a_0b_0b_1 - 3ABak^2a_1b_0^2 - 2ACak^2a_0b_1^2 + 2ACak^2a_1b_0b_1 - 8ACak^2a_2b_0^2 - B^2ak^2a_0b_1^2 + B^2ak^2a_1b_0b_1 - 4B^2ak^2a_2b_0^2 - 3BCak^2a_2b_0b_1 + 3a_0^2a_2 + 3a_0a_1^2 - a_0b_1^2 - 2a_1b_0b_1 - a_2b_0^2 = 0.$

$\left(\frac{h(\xi)}{g(\xi)}\right)^3 : -2A^2c^2k^2a_0b_0b_1 + 2A^2c^2k^2a_1b_0^2 + ABc^2k^2a_0b_1^2 - ABc^2k^2a_1b_0b_1 + 10ABc^2k^2a_2b_0^2 + 6ACc^2k^2a_2b_0b_1 + 3B^2c^2k^2a_2b_0b_1 + BCc^2k^2a_2b_1^2 + 2A^2ak^2a_0b_0b_1 - 2A^2ak^2a_1b_0^2 - ABak^2a_0b_1^2 + ABak^2a_1b_0b_1 - 10ABak^2a_2b_0^2 - 6ACak^2a_2b_0b_1 - 3B^2ak^2a_2b_0b_1 - BCak^2a_2b_1^2 + 6a_0a_1a_2 + a_1^3 - a_1b_1^2 - 2a_2b_0b_1 = 0,$

$\left(\frac{h(\xi)}{g(\xi)}\right)^4 : 6A^2c^2k^2a_2b_0^2 + 9ABc^2k^2a_2b_0b_1 + 2ACc^2k^2a_2b_1^2 + B^2c^2k^2a_2b_1^2 - 6A^2ak^2a_2b_0^2 - 9ABak^2a_2b_0b_1 - 2ACak^2a_2b_1^2 - B^2ak^2a_2b_1^2 + 3a_0a_2^2 + 3a_1^2a_2 - a_2b_1^2 = 0,$

$\left(\frac{h(\xi)}{g(\xi)}\right)^5 : 6A^2c^2k^2a_2b_0b_1 + 3ABc^2k^2a_2b_1^2 - 6A^2ak^2a_2b_0b_1 - 3ABak^2a_2b_1^2 + 3a_1a_2^2 = 0,$

$\left(\frac{h(\xi)}{g(\xi)}\right)^6 : 2A^2c^2k^2a_2b_1^2 - 2A^2ak^2a_2b_1^2 + a_2^3 = 0,$

(17)

The solution to the above system yields three sets of solutions:

Set 1:

$$a_0 = \frac{\sqrt{2}Cb_1}{\sqrt{4AC-B^2}}, \quad a_1 = \frac{\sqrt{2}Bb_1}{\sqrt{4AC-B^2}}, \quad a_2 = \frac{\sqrt{2}Ab_1}{\sqrt{4AC-B^2}}, \quad b_0 = \frac{Bb_1}{2A}, b_1 = b_1, \quad k = k,$$
$$c = \frac{\sqrt{(4AC-B^2)(4ACak^2-B^2ak^2-1)}}{k(4AC-B^2)}. \tag{18}$$

Substituting equation (18) into equation (16) results in:

$$\Theta_1(\xi) = \frac{\frac{\sqrt{2}Cb_1}{\sqrt{4AC-B^2}} + \frac{\sqrt{2}Bb_1}{\sqrt{4AC-B^2}}\left(\frac{h(\xi)}{g(\xi)}\right) + \frac{\sqrt{2}Ab_1}{\sqrt{4AC-B^2}}\left(\frac{h(\xi)}{g(\xi)}\right)^2}{\frac{Bb_1}{2A} + b_1\left(\frac{h(\xi)}{g(\xi)}\right)}, \tag{19}$$

where $\xi = k(\frac{x^\alpha}{\alpha} - \frac{\sqrt{(4AC-B^2)(4ACak^2-B^2ak^2-1)}}{k(4AC-B^2)}\frac{t^\alpha}{\alpha})$.

Case 1.1 When $\Delta = B^2 - 4AC < 0$ and $C \neq 0$

$$\Theta_{1.1}(\xi) = \frac{\left\{\begin{array}{l}\sigma + \varsigma\left(\frac{-2C[1+\tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}{B+B\tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)+\sqrt{-\Delta}[\tan(\frac{\sqrt{-\Delta}}{2}\xi)-\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}\right) \\ \quad +\tau\left(\frac{-2C[1+\tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}{B+B\tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)+\sqrt{-\Delta}[\tan(\frac{\sqrt{-\Delta}}{2}\xi)-\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}\right)^2\end{array}\right.}{\frac{b_1B}{2A} + b_1\left(\frac{-2C[1+\tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}{B+B\tan(\frac{\sqrt{-\Delta}}{2}\xi)\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)+\sqrt{-\Delta}[\tan(\frac{\sqrt{-\Delta}}{2}\xi)-\tan(\frac{\sqrt{-\Delta}}{2}\rho_1)]}\right)}, \tag{20}$$

where $\xi = k(\frac{x^\alpha}{\alpha} - \frac{\sqrt{(4AC-B^2)(4ACak^2-B^2ak^2-1)}}{k(4AC-B^2)}\frac{t^\alpha}{\alpha}), \sigma = \frac{\sqrt{2}Cb_1}{\sqrt{4AC-B^2}}, \varsigma = \frac{\sqrt{2}Bb_1}{\sqrt{4AC-B^2}}, \tau = \frac{\sqrt{2}Ab_1}{\sqrt{4AC-B^2}}$, and $\rho_1 \neq 0$ reel constant.

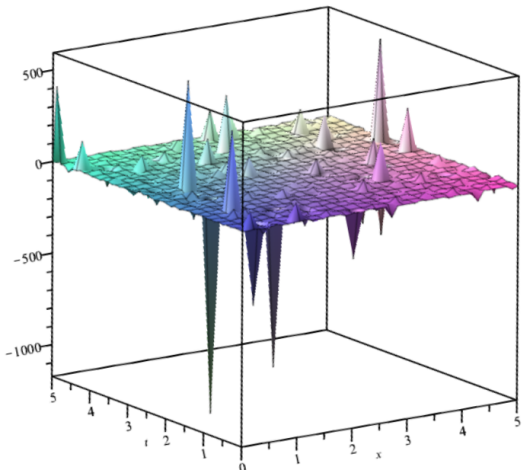


Fig. 1 Plot of $\Theta_{1.1}(\xi)$ for $A = 2, B = 1, C = 4, a = 4, k = 2, \rho_1 = 1$ and $\alpha = \frac{1}{2}$.

Set 2:

$$a_0 = \frac{b_1(B^2-2AC)}{2A\sqrt{B^2-4AC}}, \quad a_1 = \frac{Bb_1}{\sqrt{B^2-4AC}}, \quad a_2 = \frac{Ab_1}{\sqrt{B^2-4AC}}, \quad b_0 = \frac{Bb_1}{2A}, b_1 = b_1, \quad k = k,$$
$$c = \frac{\sqrt{2(4AC-B^2)(8ACak^2-2B^2ak^2+1)}}{2k(4AC-B^2)}. \tag{21}$$

Substituting equation (21) into equation (16) yields:

$$\Theta_2(\xi) = \frac{\frac{b_1\left(B^2-2AC\right)}{2A\sqrt{B^2-4AC}} + \frac{Bb_1}{\sqrt{B^2-4AC}}\left(\frac{h(\xi)}{g(\xi)}\right) + \frac{Ab_1}{\sqrt{B^2-4AC}}\left(\frac{h(\xi)}{g(\xi)}\right)^2}{\frac{Bb_1}{2A} + b_1\left(\frac{h(\xi)}{g(\xi)}\right)}, \tag{22}$$

where $\xi = k(\frac{x^\alpha}{\alpha} - \frac{\sqrt{2(4AC-B^2)(8ACak^2-2B^2ak^2+1)}}{2k(4AC-B^2)}\frac{t^\alpha}{\alpha})$.

Case 2.1 When $\Delta = B^2 - 4AC > 0$ and $C \neq 0$

$$\Theta_{2.1}(\xi) = \frac{\left\{\begin{array}{l}\tau + \sigma\left(\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\xi)-\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}\right) \\ +\omega\left(\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\xi)-\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}\right)^2\end{array}\right.}{\frac{1}{2}\frac{b_1B}{A} + b_1\left(\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\xi)-\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}\right)}, \tag{23}$$

where $\tau = \frac{b_1(B^2-2AC)}{2A\sqrt{B^2-4AC}}, \sigma = \frac{Bb_1}{\sqrt{B^2-4AC}}, \omega = \frac{Ab_1}{\sqrt{B^2-4AC}}, \xi = k(\frac{x^\alpha}{\alpha} - \frac{\sqrt{2(4AC-B^2)(8ACak^2-2B^2ak^2+1)}}{2k(4AC-B^2)}\frac{t^\alpha}{\alpha})$ and $\rho_1 \neq 0$ reel constant.

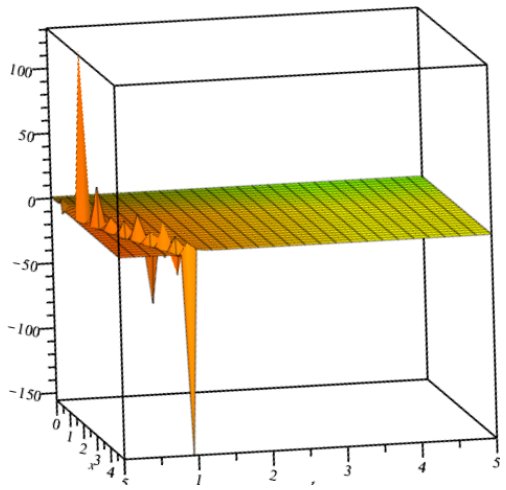


Fig. 2 Plot of $\Theta_{2.1}(\xi)$ for $A = 1, B = 4, C = 1, a = 2, k = 1, \rho_1 = 1$ and $\alpha = \frac{1}{2}$.

Case 2.2 When $A \neq 0$, $B \neq 0$ and $C = 0$

$$\Theta_{2.2}(\xi) = \frac{\frac{b_1B^2}{2A\sqrt{B^2}} + \frac{Bb_1}{\sqrt{B^2}}\left(-\frac{B\exp(B\xi)}{A\exp B\xi + B\rho_1}\right) + \frac{Ab_1}{\sqrt{B^2}}\left(-\frac{B\exp(B\xi)}{A\exp B\xi + B\rho_1}\right)^2}{\frac{Bb_1}{2A} + b_1\left(-\frac{B\exp(B\xi)}{A\exp B\xi + B\rho_1}\right)}, \tag{24}$$

where $\xi = k(\frac{x^\alpha}{\alpha} + \frac{\sqrt{2B^2(2B^2ak^2-1)}}{2kB^2}\frac{t^\alpha}{\alpha})$, and $\rho_1 \neq 0$ reel constant.

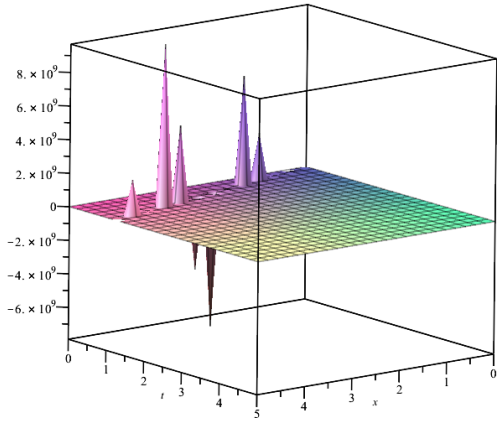


Fig. 3 Plot of $\Theta_{2.2}(\xi)$ for $A=1, B=1, C=0, k=1, \rho_1=1, c=2$ and $\alpha=\frac{1}{2}$.

Set 3:

$$a_0=\frac{Bb_0}{\sqrt{B^2-4AC}},\quad a_1=\frac{2Ab_0+Bb_1}{\sqrt{B^2-4AC}},\quad a_2=\frac{2Ab_1}{\sqrt{B^2-4AC}},\quad b_0=b_0, b_1=b_1,\quad k=k,$$
$$c=\frac{\sqrt{(4AC-B^2)(4ACak^2-B^2ak^2+2)}}{k(4AC-B^2)}. \tag{25}$$

By substituting equation (25) into equation (16), we get:

$$\Theta_3(\xi)=\frac{\frac{Bb_0}{\sqrt{B^2-4AC}}+\frac{2Ab_0+Bb_1}{\sqrt{B^2-4AC}}\left(\frac{h(\xi)}{g(\xi)}\right)+\frac{2Ab_1}{\sqrt{B^2-4AC}}\left(\frac{h(\xi)}{g(\xi)}\right)^2}{b_0+b_1\left(\frac{h(\xi)}{g(\xi)}\right)}, \tag{26}$$

where $\xi=k(\frac{x^\alpha}{\alpha}-\frac{\sqrt{(4AC-B^2)(4ACak^2-B^2ak^2+2)}}{k(4AC-B^2)}\frac{t^\alpha}{\alpha})$. Based on Set 2 and the results of equation (7), we will demonstrate how to solve equation (1):

Case 3.1 When $\Delta=B^2-4AC>0$ and $C\neq 0$

$$\Theta_{3.1}(\xi)=\frac{\left\{\begin{array}{l}\varepsilon+\nu\left(\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\xi)-\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}\right)\\ +\varsigma\left(\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\xi)-\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}\right)^2\end{array}\right.}{b_0+b_1\left(\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\xi)\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\xi)-\tanh(\frac{\sqrt{\Delta}}{2}\rho_1)]}\right)}, \tag{27}$$

where $\varepsilon=\frac{Bb_0}{\sqrt{B^2-4AC}}, \nu=\frac{2Ab_0+Bb_1}{\sqrt{B^2-4AC}}, \varsigma=\frac{2Ab_1}{\sqrt{B^2-4AC}}, \xi=k(\frac{x^\alpha}{\alpha}-\frac{\sqrt{(4AC-B^2)(4ACak^2-B^2ak^2+2)}}{k(4AC-B^2)}\frac{t^\alpha}{\alpha})$.

Case 3.2 When $A\neq 0$, $B\neq 0$ and $C=0$

$$\Theta_{3.2}(\xi)=\frac{\frac{Bb_0}{\sqrt{B^2}}+\frac{2Ab_0+Bb_1}{\sqrt{B^2}}\left(-\frac{B\exp(B\xi)}{A\exp B\xi+B\rho_1}\right)+\frac{2Ab_1}{\sqrt{B^2}}\left(-\frac{B\exp(B\xi)}{A\exp B\xi+B\rho_1}\right)^2}{b_0+b_1\left(-\frac{B\exp(B\xi)}{A\exp B\xi+B\rho_1}\right)},$$
$$\xi=k(\frac{x^\alpha}{\alpha}+\frac{\sqrt{B^2(B^2ak^2-2)}}{kB^2}\frac{t^\alpha}{\alpha}). \tag{28}$$

The multidimensional figures such as Figures (1,2,3,4,5) present to visually observe and comprehend the wave behavior of the obtained solutions. Maple 2022 has been used to basic computational calculations.

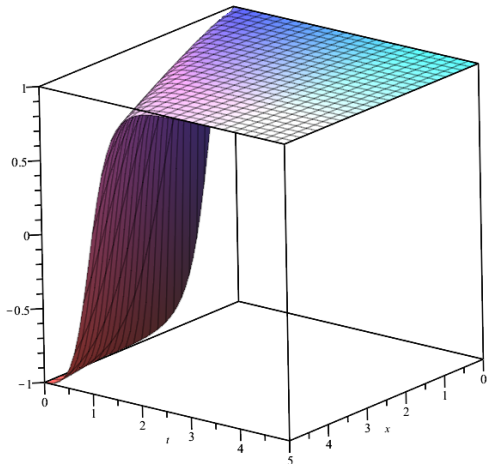


Fig. 4 Plot of $\Theta_{3.1}(\xi)$ for $A=1, B=4, C=1, a=2, k=1, \rho_1=1$ and $\alpha=\frac{1}{2}$.

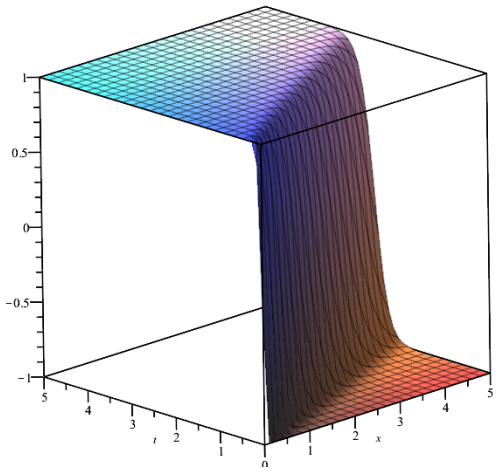


Fig. 5 Plot of $\Theta_{3.2}(\xi)$ for $A=11, B=4, C=0, a=2, c=1, k=2, a=2, \rho_1=2, b_0=b_1=1$ and $\alpha=\frac{1}{2}$.

4 Modulation instability analysis

In this section, we use linear stability analysis to perform instability modulation analysis (MI) on the stationary solutions of equation (1). The MI could be the exponential growth of small disturbances in the amplitude or phase of an optical wave. Assume that the stationary solutions to equation (1) are as follows:

$$\Theta(x,t)=(q+\mu\phi(x,t))\,e^{it\psi}, \tag{29}$$

where q and μ represent real trouble wave-numbers, $\phi(x,t)$ is a real valued function and ψ is the complex frequency. We obtain the following equation by substituting equation (29) with its necessary derivatives into equation (1):

$$\begin{aligned}&-e^{it\psi}q+e^{3it\psi}q^3-e^{it\psi}q\psi^2-e^{it\psi}\mu\phi(x,t)+3e^{3it\psi}q^2\mu\phi(x,t)\\&-e^{it\psi}\mu\psi^2\phi(x,t)+3e^{3it\psi}q\mu^2\phi(x,t)^2+e^{3it\psi}\mu^3\phi(x,t)^3\\&+2ie^{it\psi}\mu\psi\phi_t(x,t)+e^{it\psi}\mu\phi_{tt}(x,t)-e^{it\psi}\alpha\mu\phi_{xx}(x,t)=0,\end{aligned} \tag{30}$$

with $e^{it\psi}\to 1$ assuming that solutions of equation (30) are as follows:

$$\phi(x,t)=\gamma e^{i(\beta x+\lambda t)}+\delta e^{-i(\beta x+\lambda t)}. \tag{31}$$

Pushing equation (31) with the desired derivatives into equation (30) after some mathematical simplification yields

$$\gamma+\alpha\beta^2\gamma+\gamma^3-\delta+\alpha\beta^2\delta+3\gamma^2\delta+3\gamma\delta^2+\delta^3-\gamma\lambda^2-\delta\lambda^2=0. \tag{32}$$

The coefficient matrix of γ and τ can be calculated using equation (32).

$$\begin{pmatrix} 1+\alpha\beta^2+\gamma^2-\lambda^2 & 3\gamma^2 \\ 3\delta^2 & -1+\alpha\beta^2+\delta^2-\lambda^2 \end{pmatrix} * \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{33}$$

If the determinant equals zero, the coefficient matrix in equation (33) has a nontrivial solution. Extending the determinant yields the following result:

$$-1+\alpha^2\beta^4+\delta^2-\delta^2\lambda^2+\lambda^4+\alpha\beta^2\left(\gamma^2+\delta^2-2\lambda^2\right)-\gamma^2\left(1+8\delta^2+\lambda^2\right)=0. \tag{34}$$

Equation (34) has the following solutions for β:

$$\beta=-\frac{\sqrt{-\frac{\gamma^2+\delta^2+\sqrt{\gamma^4+\left(-2+\delta^2\right)^2+\gamma^2\left(4+34\delta^2\right)}-2\lambda^2}{\alpha}}}{\sqrt{2}},$$
$$\beta=\frac{\sqrt{-\frac{\gamma^2+\delta^2+\sqrt{\gamma^4+\left(-2+\delta^2\right)^2+\gamma^2\left(4+34\delta^2\right)}-2\lambda^2}{\alpha}}}{\sqrt{2}}.$$

and

$$\beta=-\frac{\sqrt{-\frac{\gamma^2+\delta^2-\sqrt{\gamma^4+\left(-2+\delta^2\right)^2+\gamma^2\left(4+34\delta^2\right)}-2\lambda^2}{\alpha}}}{\sqrt{2}},$$
$$\beta=\frac{\sqrt{-\frac{\gamma^2+\delta^2-\sqrt{\gamma^4+\left(-2+\delta^2\right)^2+\gamma^2\left(4+34\delta^2\right)}-2\lambda^2}{\alpha}}}{\sqrt{2}}.$$

The modulation instability of equation (1) is observed if the following inequality is true.

$$-\frac{\gamma^2+\delta^2+\sqrt{\gamma^4+(-2+\delta^2)^2+\gamma^2(4+34\delta^2)}-2\lambda^2}{\alpha}<0,$$

or

$$-\frac{\gamma^2+\delta^2-\sqrt{\gamma^4+(-2+\delta^2)^2+\gamma^2(4+34\delta^2)}-2\lambda^2}{\alpha}<0.$$

Now, in order to investigate the instability modulation gain spectrum, the following result is needed:

$$g(\lambda)=2Im(\beta)=\mp\sqrt{2}\sqrt{-\frac{\gamma^2+\delta^2-\sqrt{\gamma^4+(-2+\delta^2)^2+\gamma^2(4+34\delta^2)}-2\lambda^2}{\alpha}}.$$

It must be taken into account that non-zero constants γ,δ,α and λ, which show that MI gain is significantly artificial and that represents dispersion and diffraction of equation (29) as in the following two cases depending on the value of $g(\lambda)$:

Case 1 This case corresponds to $g(\lambda)=2Im(\beta)=-\sqrt{2}\sqrt{-\frac{\gamma^2+\delta^2-\sqrt{\gamma^4+(-2+\delta^2)^2+\gamma^2(4+34\delta^2)}-2\lambda^2}{\alpha}}$ and has four subcases depending on the constant values α, δ and γ:

Case 1.1 When $\alpha=1$, $\delta=2$ and $\gamma=1$, it results in

$$g_{1.1}(\lambda)=-\sqrt{2}\sqrt{-5-\sqrt{145}+2\lambda^2}.$$

Case 1.2 Once it is selected as $\alpha = \frac{4}{5}$, $\delta = 3$ and $\gamma = -1$, it give rise to

$$g_{1.2}(\lambda) = -\sqrt{5}\sqrt{-5-3\sqrt{10}+\lambda^2}.$$

Case 1.3 If $\alpha = \frac{3}{4}$, $\delta = \frac{1}{3}$ and $\gamma = \frac{-1}{2}$, it produces

$$g_{1.3}(\lambda) = -\frac{2}{3}\sqrt{-\frac{49}{3}+12\lambda^2}.$$

Case 1.4 Taking as $\alpha = \frac{5}{3}$, $\delta = \frac{1}{5}$ and $\gamma = \frac{-1}{3}$, one reads

$$g_{1.4}(\lambda) = -\frac{2\sqrt{-17-\sqrt{56314}+225\lambda^2}}{5\sqrt{15}}.$$

The graphs for these subcases are presented in Figure 6 where $-100 < \lambda < 100$.

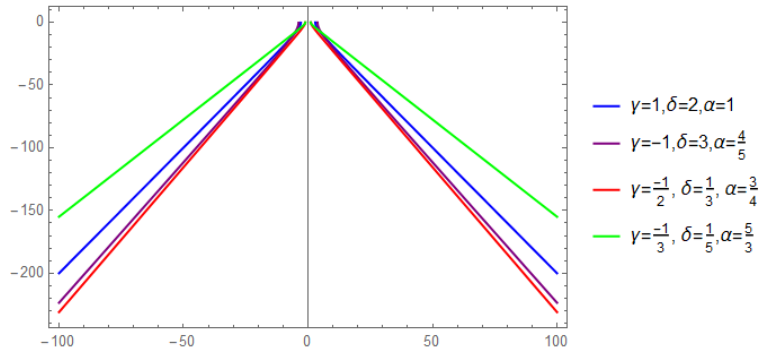


Fig. 6 The instability modulation gain spectra in the normal-GVD regime for Case 1.

Case 2 This case corresponds to $g(\lambda) = 2Im(\beta) = \sqrt{2}\sqrt{-\frac{\gamma^2+\delta^2+\sqrt{\gamma^4+(-2+\delta^2)^2+\gamma^2(4+34\delta^2)}-2\lambda^2}{\alpha}}$ and has four subcases depending on the constant values α, δ and γ as follows

Case 2.1 Considering $\alpha = \frac{5}{4}$, $\delta = \frac{4}{3}$ and $\gamma = \frac{2}{5}$, one reads

$$g_{2.1}(\lambda) = \frac{4}{15}\sqrt{-\frac{218}{5}-\frac{\sqrt{131449}}{5}+45\lambda^2}.$$

Case 2.2 Taking $\alpha = \frac{4}{5}$, $\delta = \frac{3}{2}$ and $\gamma = \frac{2}{3}$, it give rise to

$$g_{2.2}(\lambda) = \frac{1}{6}\sqrt{\frac{5}{2}}\sqrt{-97-\sqrt{46705}+72\lambda^2}.$$

Case 2.3 When $\alpha = \frac{3}{4}$, $\delta = \frac{5}{3}$ and $\gamma = \frac{3}{2}$, it results in

$$g_{2.3}(\lambda) = \frac{1}{3}\sqrt{\frac{2}{3}}\sqrt{-181-\sqrt{294409}+72\lambda^2}.$$

Case 2.4 Choosing $\alpha = \frac{2}{3}$, $\delta = \frac{4}{5}$ and $\gamma = \frac{4}{3}$, one reads

$$g_{2.4}(\lambda) = \frac{1}{5}\sqrt{\frac{2}{3}}\sqrt{-272-\sqrt{643009}+225\lambda^2}.$$

The graphs for these subcases are plotted by Figure 7 for $-100 < \lambda < 100$.

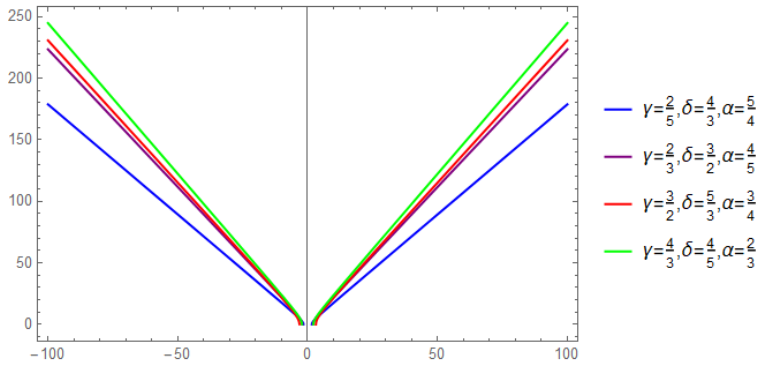


Fig. 7 The instability modulation gain spectra in the normal-GVD regime for Case 2.

5 Conclusion

We investigated in this paper the nonlinear (1+1)-dimensional Phi-four equation in the sense of the Katugampola operator. We obtained several traveling wave solutions using a generalized double auxiliary equation method, which has never been used before. We also discussed the instability modulation analysis for stationary solutions using linear stability analysis. It is worth noting that the (1+1)-dimensional Phi-four equation has been solved in the literature using various derivative operators such as the Beta-fractional derivative operator, the conformable order derivative, the Caputo fractional derivative and the classic derivative. No comparison can be made because we used a Katugampola derivative that had never been used before in the literature. As a result, it will be interesting to investigate the behavior of the generalized double auxiliary equation method using other types of derivative operators and on other complex partial differential equations.

6 Declarations

6.1 Conflict of interest:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

6.2 Funding:

No funding was received to assist with the preparation of this manuscript.

6.3 Authors Contribution:

B.G.-Supervision, Methodology, Software, Formal Analysis, Writing-Original Draft, Writing-Review Editing; A.C.-Validation; A.M.-Formal Analysis, Writing-Original Draft; L.A.-Conceptualization; Y.M.-Writing-Review Editing. All authors read and approved the final submitted version of this manuscript.

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6.5 Data availability statement:

All data that support the findings of this study are included within the article (and any supplementary files).

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