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Original Study

New generalized Mellin transform and applications to partial and fractional differential equations

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Abstract

In this paper, we introduce a generalized Mellin transformation in a general form that encompasses the generalized Mellin transformations found in the literature. Then, we give the fundamental properties of this new integral transformation and apply it to some elementary functions. Furthermore, we obtain the solutions of partial and fractional differential equations by means of this new integral transformation. Finally, we examine the relations between generalized Mellin transformations in the literature and the new generalized Mellin transformation and present a table showing the new integral transformations of functions commonly used in mathematics, physics and engineering applications.

Keywords: Mellin transform, partial differential equations, fractional differential equations, fractional derivatives and integrals.
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1 Introduction

Integral transformation is a powerful tool used in many fields of science and engineering to simplify difficult mathematical problems. It is the process of transforming a real or complex valued function f into a new function F using a mathematical operation. Applying integral transformation to a problem, we can often obtain simple algebraic operators containing F, which makes it easier to solve the problem. The solution to the problem is then completed taking the inverse integral transform.

There are many types of integral transformations used in applied mathematics, physics and engineering. Some of the most commonly used are Fourier, Fourier sine, Fourier cosine, Laplace, Mellin, Hilbert, Stieltjes and Hankel. Each of these transformations has unique properties and applications. Also for studies on these transformations, (Watugala [1], Khan and Khan [2], Elzaki [3], Upadhyaya [4], Jafari [5], Luchko et al. [6], Jumarie [7], Romero et al. [8], Mahor et al. [9] and Kumar [10]).

In this paper, we will focus on the Mellin transform, which is a type of integral transformation that is particularly useful in the study of special functions, differential equations and fractional calculus. We give the Mellin transform, which has many applications in fields such as number theory, combinatorics and probability theory below.

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The Mellin transform in Debnath and Bhatta [11] can be expressed as follows:

$$\mathfrak{M}\{f(x)\} = F(\alpha) = \int_0^\infty x^{\alpha-1} f(x)dx, \quad (\alpha \in \mathbb{C}),$$

where $f(x)$ is a real or complex valued function defined in the range $(0,\infty)$.

Scientists have introduced various generalizations of the Mellin transformation in the literature, (Aziz and Rehman [12], Duraković et al. [13], Erdoğan et al. [14], Fitouhi et al. [15], Saha and Singh [16], González Gaxiola and Santiago [17] and Yang et al. [18]). Also in recent developments in fractional calculus theory, (Belmor et al. [19], Goufo et al. [20], Veeresha et al. [21], Yao et al. [22] and Veeresha et al. [23]).

The purpose of this paper is to introduce a new generalized Mellin transformation that encompasses previously defined generalized Mellin transformations found in literature. Additionally, we will demonstrate how this new integral transformation can be used to solve partial and fractional differential equation problems.

The remainder of this paper is organized as follows: In Section 2, we give the preliminaries information needed throughout the paper. In Section 3, we introduce the new generalized Mellin transformation and discuss some of its properties. In Section 4, we derive the new generalized Mellin transformation of some elementary functions. In Section 5, we provide solutions to partial and fractional differential equations using the new generalized Mellin transform. In Section 6, we explore the relationships between generalized Mellin transforms in literature and the new generalized Mellin transform. Finally, we present a table of the new generalized Mellin transform.

2 Preliminaries

In this section, we give relevant material which will be used throughout the paper.

The gamma function in Andrews et al. [24] for $\Re(x) > 0$ is defined by

$$\Gamma(x) = \int_0^\infty \omega^{x-1} \exp(-\omega)\, d\omega.$$

The beta function in Andrews et al. [24] for $\Re(x) > 0$ and $\Re(y) > 0$ is given by

$$B(x,y) = \int_0^1 \omega^{x-1}(1-\omega)^{y-1} d\omega.$$

The Riemann zeta function in Andrews et al. [24] for $\Re(x) > 1$ is defined by the series

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

The Laplace transform in Debnath and Bhatta [11] for $\Re(s) > 0$ is given by

$$\mathfrak{L}\big[f(\omega)\big](s) = \int_0^\infty \exp(-s\omega)\, f(\omega) d\omega.$$

The Mellin convolutions in Debnath and Bhatta [11] of two functions f and g are defined by

$$f(x) * g(x) = \int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi}, \tag{1}$$

$$f(x) \circ g(x) = \int_0^\infty f(x\xi) g(\xi) d\xi. \tag{2}$$

The Riemann-Liouville fractional integral in Kilbas et al. [25] is given by

$$\left(I_{0+}^{\varepsilon} f\right)(x) = \frac{1}{\Gamma(\varepsilon)} \int_0^x (x-\tau)^{\varepsilon-1} f(\tau) d\tau, \quad (x>0;\ \Re(\varepsilon)>0). \tag{3}$$

The Riemann-Liouville and Caputo fractional derivatives in Kilbas et al. [25] respectively are defined by

$$\left(D_{0+}^{\varepsilon} f\right)(x) = \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{dx^m} \int_0^x (x-\tau)^{m-\varepsilon-1} f(\tau) d\tau, \tag{4}$$

$$\left({}^{c}D_{0+}^{\varepsilon} f\right)(x) = \frac{1}{\Gamma(m-\varepsilon)} \int_0^x (x-\tau)^{m-\varepsilon-1} f^{(m)}(\tau) d\tau, \tag{5}$$

where $x>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$.

3 Generalized Mellin transform and its properties

In this section, we define new generalizations of the Mellin and inverse Mellin transforms. We also provide properties of these new generalizations of the Mellin transform.

Definition 1. Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then, the generalized Mellin and inverse Mellin transforms respectively are defined as

$${}_p\mathfrak{M}_q\Big[f(x)\Big](\alpha) = \widehat{f}\big(q(\alpha)\big) := p(\alpha)\int_0^{\infty} x^{q(\alpha)-1} f(x) dx, \tag{6}$$

and

$${}_p\mathfrak{M}_q^{-1}\Big[\widehat{f}\big(q(\alpha)\big)\Big](x) = f(x) := \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{x^{-q(\alpha)} q'(\alpha)\widehat{f}\big(q(\alpha)\big)}{p(\alpha)} d\alpha. \tag{7}$$

Note that, the integration contour in the generalized inverse Mellin transform (7) is Bromwich contour.

Remark 1. The generalized Mellin transform (6) exists if the function f is piecewise continuous in every closed interval $[a,b]\subset(0,\infty)$ and $\gamma_1<\Re(q(\alpha))=\gamma<\gamma_2$, the following inequalities must be satisfied:

$$|p(\alpha)|\int_0^1 |f(x)|x^{\gamma_1-1}dx<\infty \quad \text{and} \quad |p(\alpha)|\int_1^{\infty} |f(x)|x^{\gamma_2-1}dx<\infty.$$

We present the basic properties of the generalized Mellin transform in the following theorem.

Theorem 1. *Let $f,g:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$.*

Linearity property: *Then, for $\lambda_1,\lambda_2\in\mathbb{R}$,*

$${}_p\mathfrak{M}_q\Big[\lambda_1 f(x)+\lambda_2 g(x)\Big](\alpha) = \lambda_1\ {}_p\mathfrak{M}_q\Big[f(x)\Big](\alpha) + \lambda_2\ {}_p\mathfrak{M}_q\Big[g(x)\Big](\alpha).$$

Scaling property: *Then, for $a>0$,*

$${}_p\mathfrak{M}_q\Big[f(ax)\Big](\alpha) = \frac{\widehat{f}\big(q(\alpha)\big)}{a^{q(\alpha)}}.$$

Shifting property: *Then,*

$${}_p\mathfrak{M}_q\Big[x^a f(x)\Big](\alpha) = \widehat{f}\big(q(\alpha)+a\big). \tag{8}$$

Proof. Using the generalized Mellin transform (6), we have

$$\begin{aligned}
{}_p\mathfrak{M}_q\Big[\lambda_1 f(x)+\lambda_2 g(x)\Big](\alpha) &= p(\alpha)\int_0^\infty x^{q(\alpha)-1}\left(\lambda_1 f(x)+\lambda_2 g(x)\right)dx,\\
&= \lambda_1\ p(\alpha)\int_0^\infty x^{q(\alpha)-1} f(x)dx+\lambda_2\ p(\alpha)\int_0^\infty x^{q(\alpha)-1} g(x)dx,\\
&= \lambda_1\ {}_p\mathfrak{M}_q\Big[f(x)\Big](\alpha)+\lambda_2\ {}_p\mathfrak{M}_q\Big[g(x)\Big](\alpha),
\end{aligned}$$

for linearity property;

$$\begin{aligned}
{}_p\mathfrak{M}_q\Big[f(ax)\Big](\alpha) &= p(\alpha)\int_0^\infty x^{q(\alpha)-1} f(ax)dx, \quad (\text{Taking } ax=t),\\
&= \frac{p(\alpha)}{a^{q(\alpha)}}\int_0^\infty t^{q(\alpha)-1} f(t)dt,\\
&= \frac{\widehat{f}\big(q(\alpha)\big)}{a^{q(\alpha)}},
\end{aligned}$$

for scaling property; and

$$\begin{aligned}
{}_p\mathfrak{M}_q\Big[x^a f(x)\Big](\alpha) &= p(\alpha)\int_0^\infty x^{q(\alpha)+a-1} f(x)dx,\\
&= \widehat{f}\big(q(\alpha)+a\big),
\end{aligned}$$

for shifting property.

We provide generalized Mellin transformations for several functions in the following theorems.

Theorem 2. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[f\left(x^a\right)\Big](\alpha) = \frac{1}{a}\widehat{f}\left(\frac{q(\alpha)}{a}\right).$$

Proof. Using the generalized Mellin transform (6), we have

$$\begin{aligned}
{}_p\mathfrak{M}_q\Big[f\left(x^a\right)\Big](\alpha) &= p(\alpha)\int_0^\infty x^{q(\alpha)-1} f\left(x^a\right)dx, \quad (\text{Taking } x^a=t),\\
&= \frac{p(\alpha)}{a}\int_0^\infty t^{\frac{q(\alpha)}{a}-1} f(t)dt,\\
&= \frac{1}{a}\widehat{f}\left(\frac{q(\alpha)}{a}\right).
\end{aligned}$$

Theorem 3. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[f^{(n)}(x)\Big](\alpha) = (-1)^n\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-n\big)}\widehat{f}\big(q(\alpha)-n\big),$$

where $\Re\big(q(\alpha)\big)>0$ and $\Re\big(q(\alpha)-n\big)>0$, $n\in\mathbb{N}$.

Proof. We use induction to prove the following equation.

For $n=1$, we have

$$\begin{aligned}
{}_p\mathfrak{M}_q\Big[f'(x)\Big](\alpha) &= p(\alpha)\int_0^\infty x^{q(\alpha)-1}f'(x)dx,\\
&= p(\alpha)\lim_{A\to\infty}\Big[x^{q(\alpha)-1}f(x)\Big]_0^A-(q(\alpha)-1)p(\alpha)\int_0^\infty x^{q(\alpha)-2}f(x)dx,\\
&= -(q(\alpha)-1)p(\alpha)\int_0^\infty x^{q(\alpha)-2}f(x)dx,\\
&= -\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-1\big)}\widehat{f}\big(q(\alpha)-1\big),
\end{aligned}$$

where we assume $x^{q(\alpha)-1}f(x)=0$ as $x\to 0$ and $x\to\infty$.

For $n=k$, assume the following equation is true:

$${}_p\mathfrak{M}_q\Big[f^{(k)}(x)\Big](\alpha)=(-1)^k\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-k\big)}\widehat{f}\big(q(\alpha)-k\big).$$

For $n=k+1$, we get

$$\begin{aligned}
{}_p\mathfrak{M}_q\Big[f^{(k+1)}(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\left[\frac{d}{dx}f^{(k)}(x)\right](\alpha),\\
&= (-1)^k(q(\alpha)-1)(q(\alpha)-2)\dots(q(\alpha)-k)p(\alpha)\int_0^\infty x^{q(\alpha)-(k+1)}f'(x)dx,\\
&= (-1)^{k+1}(q(\alpha)-1)(q(\alpha)-2)\dots(q(\alpha)-(k+1))\widehat{f}\big(q(\alpha)-(k+1)\big),\\
&= (-1)^{k+1}\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-(k+1)\big)}\widehat{f}\big(q(\alpha)-(k+1)\big),
\end{aligned}$$

in which we assume that the first terms from the partial integration approach zero as $x\to 0$ and $x\to\infty$.

Theorem 4. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[x^nf^{(n)}(x)\Big](\alpha)=(-1)^n\frac{\Gamma\big(q(\alpha)+n\big)}{\Gamma\big(q(\alpha)\big)}\widehat{f}\big(q(\alpha)\big), \tag{9}$$

where $\Re\big(q(\alpha)\big)>0$ and $\Re\big(q(\alpha)+n\big)>0$, $n\in\mathbb{N}$.

Proof. We use induction again.

For $n=1$, we have

$$\begin{aligned}
{}_p\mathfrak{M}_q\Big[xf'(x)\Big](\alpha) &= p(\alpha)\int_0^\infty x^{q(\alpha)}f'(x)dx,\\
&= p(\alpha)\lim_{A\to\infty}\Big[x^{q(\alpha)}f(x)\Big]_0^A-q(\alpha)p(\alpha)\int_0^\infty x^{q(\alpha)-1}f(x)dx,\\
&= -q(\alpha)p(\alpha)\int_0^\infty x^{q(\alpha)-1}f(x)dx,\\
&= -\frac{\Gamma\big(q(\alpha)+1\big)}{\Gamma\big(q(\alpha)\big)}\widehat{f}\big(q(\alpha)\big),
\end{aligned}$$

in which we assume $x^{q(\alpha)}f(x)=0$ as $x\to 0$ and $x\to\infty$.

For $n=k$, let the equation be true:

$$ {}_p\mathfrak{M}_q\Big[x^k f^{(k)}(x)\Big](\alpha)=(-1)^k\frac{\Gamma\big(q(\alpha)+k\big)}{\Gamma\big(q(\alpha)\big)}\widehat{f}\big(q(\alpha)\big). $$

For $n=k+1$, we get

$$ \begin{aligned} {}_p\mathfrak{M}_q\Big[x^{k+1}f^{(k+1)}(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\Big[x\frac{d}{dx}x^k f^{(k)}(x)\Big](\alpha)-{}_p\mathfrak{M}_q\Big[kx^k f^{(k)}(x)\Big](\alpha),\\ &=-q(\alpha)(-1)^k q(\alpha)(q(\alpha)+1)(q(\alpha)+2)\dots(q(\alpha)+k-1)\widehat{f}\big(q(\alpha)\big)\\ &\quad -k(-1)^k q(\alpha)(q(\alpha)+1)(q(\alpha)+2)\dots(q(\alpha)+k-1)\widehat{f}\big(q(\alpha)\big),\\ &=(-1)^{k+1}q(\alpha)(q(\alpha)+1)(q(\alpha)+2)\dots(q(\alpha)+k)\widehat{f}\big(q(\alpha)\big),\\ &=(-1)^{k+1}\frac{\Gamma\big(q(\alpha)+k+1\big)}{\Gamma\big(q(\alpha)\big)}\widehat{f}\big(q(\alpha)\big), \end{aligned} $$

where we assume that the first terms from the partial integration approach zero as $x\to 0$ and $x\to\infty$.

Theorem 5. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\Big[\Big(x\frac{d}{dx}\Big)^n f(x)\Big](\alpha)=(-1)^n\big(q(\alpha)\big)^n\widehat{f}\big(q(\alpha)\big), $$

where $n\in\mathbb{N}$.

Proof. We use induction.

For $n=1$, we have

$$ \begin{aligned} {}_p\mathfrak{M}_q\Big[\Big(x\frac{d}{dx}\Big)f(x)\Big](\alpha) &= p(\alpha)\int_0^\infty x^{q(\alpha)-1}\Big(x\frac{d}{dx}\Big)f(x)dx,\\ &=-q(\alpha)\widehat{f}\big(q(\alpha)\big), \end{aligned} $$

in which we assume $x^{q(\alpha)}f(x)=0$ as $x\to 0$ and $x\to\infty$.

For $n=k$, let the equation be true:

$$ {}_p\mathfrak{M}_q\Big[\Big(x\frac{d}{dx}\Big)^k f(x)\Big](\alpha)=(-1)^k\big(q(\alpha)\big)^k\widehat{f}\big(q(\alpha)\big). $$

For $n=k+1$, we get

$$ \begin{aligned} {}_p\mathfrak{M}_q\Big[\Big(x\frac{d}{dx}\Big)^{k+1}f(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\Big[\Big(x\frac{d}{dx}\Big)\Big(x\frac{d}{dx}\Big)^k f(x)\Big](\alpha),\\ &=(-1)^k\big(q(\alpha)\big)^k p(\alpha)\int_0^\infty x^{q(\alpha)-1}\Big(x\frac{d}{dx}\Big)f(x)dx,\\ &=(-1)^{k+1}\big(q(\alpha)\big)^{k+1}\widehat{f}\big(q(\alpha)\big), \end{aligned} $$

where we assume that the first terms from the partial integration approach zero as $x\to 0$ and $x\to\infty$.

Theorem 6. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\Big[I_n f(x)\Big](\alpha)=(-1)^n\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)+n\big)}\widehat{f}\big(q(\alpha)+n\big), $$

where $n\in\mathbb{N}$ and $I_n f(x)=\underbrace{\int_0^x\int_0^x\dots\int_0^x}_{n\text{ times}}f(t)dt.$

Proof. Setting $F(x) = I_n f(x)$, we get $F^{(n)}(x) = f(x)$. We use induction.

For $n = 1$, we have

$$
\begin{aligned}
{}_p\mathfrak{M}_q\Big[F'(x) = f(x)\Big](\alpha) &= -\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-1\big)}\widehat{F}\big(q(\alpha)-1\big), \\
&= -\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-1\big)}\widehat{I_1 f}\big(q(\alpha)-1\big).
\end{aligned}
$$

Replacing $q(\alpha) \to q(\alpha)+1$, we get

$$
{}_p\mathfrak{M}_q\Big[I_1 f(x)\Big](\alpha) = -\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)+1\big)}\widehat{f}\big(q(\alpha)+1\big).
$$

For $n = k$, assume the following equation is true:

$$
{}_p\mathfrak{M}_q\Big[I_k f(x)\Big](\alpha) = (-1)^k\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)+k\big)}\widehat{f}\big(q(\alpha)+k\big). \tag{10}
$$

For $n = k+1$, we obtain

$$
{}_p\mathfrak{M}_q\Big[F^{(k+1)}(x) = f(x)\Big](\alpha) = {}_p\mathfrak{M}_q\left[\frac{d}{dx}F^{(k)}(x) = f(x)\right](\alpha).
$$

Considering Eq. (10), we have

$$
\widehat{f}\big(q(\alpha)\big) = (-1)^{k+1}\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-k-1\big)}\widehat{I_{k+1}f}\big(q(\alpha)-k-1\big).
$$

Substituting $q(\alpha) \to q(\alpha)+k+1$, we get

$$
{}_p\mathfrak{M}_q\Big[I_{k+1}f(x)\Big](\alpha) = (-1)^{k+1}\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)+k+1\big)}\widehat{f}\big(q(\alpha)+k+1\big).
$$

We give the convolution property of the generalized Mellin transform with the following theorem.

Theorem 7. *Let $f, g : (0,\infty) \to \mathbb{R}$, $p : \mathbb{C} \to \mathbb{C} - \{0\}$, $q : \mathbb{C} \to \mathbb{C}$. Then,*

$$
{}_p\mathfrak{M}_q\Big[f(x) * g(x)\Big](\alpha) = \frac{\widehat{f}\big(q(\alpha)\big)\widehat{g}\big(q(\alpha)\big)}{p(\alpha)}, \tag{11}
$$

$$
{}_p\mathfrak{M}_q\Big[f(x) \circ g(x)\Big](\alpha) = \frac{\widehat{f}\big(q(\alpha)\big)\widehat{g}\big(1-q(\alpha)\big)}{p(\alpha)}. \tag{12}
$$

Proof. Using the generalized Mellin transform (6) and Eq. (1), we have

$$
\begin{aligned}
{}_p\mathfrak{M}_q\Big[f(x) * g(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\left[\int_0^\infty f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right](\alpha), \\
&= p(\alpha)\int_0^\infty\int_0^\infty x^{q(\alpha)-1}f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}dx, \quad \left(\text{Taking } \frac{x}{\xi} = \eta\right), \\
&= p(\alpha)\int_0^\infty \xi^{q(\alpha)-1}f(\xi)d\xi\int_0^\infty \eta^{q(\alpha)-1}g(\eta)d\eta.
\end{aligned}
$$

Multiplying and dividing the right-hand side of the equation by $p(\alpha)$, we get

$$
\begin{aligned}
{}_p\mathfrak{M}_q\Big[f(x)*g(x)\Big](\alpha) &= p(\alpha)\int_0^\infty \xi^{q(\alpha)-1}f(\xi)d\xi\ \frac{p(\alpha)}{p(\alpha)}\int_0^\infty \eta^{q(\alpha)-1}g(\eta)d\eta,\\
&= \frac{\widehat{f}\big(q(\alpha)\big)\widehat{g}\big(q(\alpha)\big)}{p(\alpha)}.
\end{aligned}
$$

Similarly, using the generalized Mellin transform (6) and Eq. (2), we have

$$
\begin{aligned}
{}_p\mathfrak{M}_q\Big[f(x)\circ g(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\left[\int_0^\infty f(x\xi)g(\xi)d\xi\right](\alpha),\\
&= p(\alpha)\int_0^\infty\int_0^\infty x^{q(\alpha)-1}f(x\xi)g(\xi)d\xi dx, \quad \big(\text{Taking } x\xi=\eta\big),\\
&= p(\alpha)\int_0^\infty \eta^{q(\alpha)-1}f(\eta)d\eta\int_0^\infty \xi^{-q(\alpha)}g(\xi)d\xi.
\end{aligned}
$$

Multiplying and dividing the right-hand side of the equation by $p(\alpha)$, we get

$$
\begin{aligned}
{}_p\mathfrak{M}_q\Big[f(x)\circ g(x)\Big](\alpha) &= p(\alpha)\int_0^\infty \eta^{q(\alpha)-1}f(\eta)d\eta\ \frac{p(\alpha)}{p(\alpha)}\int_0^\infty \xi^{-q(\alpha)}g(\xi)d\xi,\\
&= \frac{\widehat{f}\big(q(\alpha)\big)\widehat{g}\big(1-q(\alpha)\big)}{p(\alpha)}.
\end{aligned}
$$

Corollary 8. *Applying the generalized inverse Mellin transform (7) to Eq. (12), we have*

$$
f(x)\circ g(x) = {}_p\mathfrak{M}_q^{-1}\left[\frac{\widehat{f}\big(q(\alpha)\big)\widehat{g}\big(1-q(\alpha)\big)}{p(\alpha)}\right](x).
$$

Using Eq. (2), we get

$$
\int_0^\infty f(x\xi)g(\xi)d\xi = {}_p\mathfrak{M}_q^{-1}\left[\frac{\widehat{f}\big(q(\alpha)\big)\widehat{g}\big(1-q(\alpha)\big)}{p(\alpha)}\right](x).
$$

Taking $f(x)=\exp(-x)$, we obtain

$$
\mathfrak{L}\Big[g(\xi)\Big](x) = {}_p\mathfrak{M}_q^{-1}\Big[\Gamma\big(q(\alpha)\big)\widehat{g}\big(1-q(\alpha)\big)\Big](x).
$$

Theorem 9. *Let $f,g:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$
{}_p\mathfrak{M}_q\left[x^\lambda\int_0^\infty \tau^\eta f(x\tau)g(\tau)d\tau\right](\alpha) = \frac{\widehat{f}\big(q(\alpha)+\lambda\big)\widehat{g}\big(\eta+1-q(\alpha)-\lambda\big)}{p(\alpha)}. \tag{13}
$$

Proof. From the generalized Mellin transform (6), we have

$$
\begin{aligned}
{}_p\mathfrak{M}_q\left[x^\lambda\int_0^\infty \tau^\eta f(x\tau)g(\tau)d\tau\right](\alpha) &= p(\alpha)\int_0^\infty\int_0^\infty x^{q(\alpha)-1}x^\lambda\tau^\eta f(x\tau)g(\tau)d\tau dx, \quad \big(\text{Taking } x\tau=t\big),\\
&= p(\alpha)\int_0^\infty t^{q(\alpha)+\lambda-1}f(t)dt\int_0^\infty \tau^{\eta-q(\alpha)-\lambda}g(\tau)d\tau.
\end{aligned}
$$

Multiplying and dividing the right-hand side of the equation by $p(\alpha)$, we get

$$
\begin{aligned}
{}_p\mathfrak{M}_q\left[x^{\lambda}\int_0^{\infty}\tau^{\eta}f(x\tau)g(\tau)d\tau\right](\alpha) &= p(\alpha)\int_0^{\infty}t^{q(\alpha)+\lambda-1}f(t)dt\,\frac{p(\alpha)}{p(\alpha)}\int_0^{\infty}\tau^{\eta-q(\alpha)-\lambda}g(\tau)d\tau,\\
&= \frac{\widehat{f}\big(q(\alpha)+\lambda\big)\widehat{g}\big(\eta+1-q(\alpha)-\lambda\big)}{p(\alpha)}.
\end{aligned}
$$

Theorem 10 (Parseval's type property)**.** *Let $f,g:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[f(x)g(x)\Big](\alpha) = \frac{1}{p(\alpha)}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\widehat{f}\big(q(s)\big)q'(s)\widehat{g}\big(q(\alpha)-q(s)\big)ds.$$

Proof. Using the generalized Mellin and inverse Mellin transforms, we have

$$
\begin{aligned}
{}_p\mathfrak{M}_q\Big[f(x)g(x)\Big](\alpha) &= p(\alpha)\int_0^{\infty}x^{q(\alpha)-1}f(x)g(x)dx,\\
&= \frac{p(\alpha)}{p(\alpha)}\frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma-i\infty}^{\gamma+i\infty}x^{q(\alpha)-1}x^{-q(s)}g(x)\widehat{f}\big(q(s)\big)q'(s)dsdx,\\
&= \frac{1}{p(\alpha)}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\widehat{f}\big(q(s)\big)q'(s)\widehat{g}\big(q(\alpha)-q(s)\big)ds. \qquad (14)
\end{aligned}
$$

Corollary 11. *Taking $p(\alpha)=q(\alpha)=1$ and $q(s)=s$ in Eq. (14), we get*

$${}_p\mathfrak{M}_q\Big[f(x)g(x)\Big](\alpha) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\widehat{f}(s)\widehat{g}(1-s)ds.$$

Theorem 12. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[\big(I_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) = \frac{\Gamma\big(1-q(\alpha)-\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)+\varepsilon\big),$$

where $\Re(\varepsilon)>0$ and $\Re(q(\alpha)+\varepsilon)<1$.

Proof. Using Eq. (3), we have

$$
\begin{aligned}
\big(I_{0+}^{\varepsilon}f\big)(x) &= \frac{1}{\Gamma(\varepsilon)}\int_0^{x}(x-\tau)^{\varepsilon-1}f(\tau)d\tau, \quad (\text{Taking } \tau=x\xi),\\
&= \frac{x^{\varepsilon}}{\Gamma(\varepsilon)}\int_0^{1}(1-\xi)^{\varepsilon-1}f(x\xi)d\xi,\\
&= \frac{x^{\varepsilon}}{\Gamma(\varepsilon)}\int_0^{\infty}f(x\xi)g(\xi)d\xi, \qquad (15)
\end{aligned}
$$

where the function g is as follows:

$$g(x) = \begin{cases}(1-x)^{\varepsilon-1} & \text{for } 0\le x<1,\\ 0 & \text{for } x\ge 1.\end{cases}$$

Applying the generalized Mellin transform (6) to the function g, we get

$$\begin{aligned}\widehat{g}\big(q(\alpha)\big) &= p(\alpha)\int_0^{\infty} x^{q(\alpha)-1}g(x)dx,\\ &= p(\alpha)\int_0^{1} x^{q(\alpha)-1}(1-x)^{\varepsilon-1}dx,\\ &= p(\alpha)B\big(q(\alpha),\varepsilon\big),\\ &= p(\alpha)\frac{\Gamma\big(q(\alpha)\big)\Gamma(\varepsilon)}{\Gamma\big(q(\alpha)+\varepsilon\big)}.\end{aligned} \tag{16}$$

Using Eqs. (13), (15) and (16), we obtain

$$\begin{aligned}{}_p\mathfrak{M}_q\Big[\big(I_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\left[\frac{x^{\varepsilon}}{\Gamma(\varepsilon)}\int_0^{\infty} f(x\xi)g(\xi)d\xi\right](\alpha),\\ &= \frac{\widehat{f}\big(q(\alpha)+\varepsilon\big)\widehat{g}\big(1-q(\alpha)-\varepsilon\big)}{p(\alpha)\Gamma(\varepsilon)},\\ &= \frac{\Gamma\big(1-q(\alpha)-\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)+\varepsilon\big).\end{aligned}$$

Theorem 13. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[\big(D_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) = \frac{\Gamma\big(1-q(\alpha)+\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)-\varepsilon\big),$$

where $\Re(\varepsilon)>0$ and $\Re(q(\alpha))<1$.

Proof. The relationship between Eqs. (3) and (4) is as follows:

$$\begin{aligned}\big(D_{0+}^{\varepsilon}f\big)(x) &= \frac{d^m}{dx^m}\frac{1}{\Gamma(m-\varepsilon)}\int_0^{x}(x-\tau)^{m-\varepsilon-1}f(\tau)d\tau,\\ &= \frac{d^m}{dx^m}\big(I_{0+}^{m-\varepsilon}f\big)(x),\end{aligned}$$

where

$$g(x) := \big(I_{0+}^{m-\varepsilon}f\big)(x). \tag{17}$$

Then, we have

$$\big(D_{0+}^{\varepsilon}f\big)(x) = g^{(m)}(x).$$

Applying the generalized Mellin transform (6) to both sides of the equation, we get

$$\begin{aligned}{}_p\mathfrak{M}_q\Big[\big(D_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\Big[g^{(m)}(x)\Big](\alpha),\\ &= (-1)^m\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-m\big)}\widehat{g}\big(q(\alpha)-m\big),\\ &= \frac{\Gamma\big(m+1-q(\alpha)\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{g}\big(q(\alpha)-m\big).\end{aligned} \tag{18}$$

Applying the generalized Mellin transform (6) to Eq. (17), we obtain

$$\begin{aligned}\widehat{g}\big(q(\alpha)\big) &= {}_p\mathfrak{M}_q\Big[\big(I_{0+}^{m-\varepsilon}f\big)(x)\Big](\alpha),\\ &= \frac{\Gamma\big(1-q(\alpha)-m+\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)+m-\varepsilon\big).\end{aligned} \tag{19}$$

Taking $q(\alpha)-m$ instead of $q(\alpha)$ in Eq. (19) and then using in Eq. (18), we have

$${}_p\mathfrak{M}_q\Big[\big(D_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) = \frac{\Gamma\big(1-q(\alpha)+\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)-\varepsilon\big).$$

Theorem 14. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[\big({}^cD_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) = \frac{\Gamma\big(1-q(\alpha)+\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)-\varepsilon\big), \tag{20}$$

where $\Re(\varepsilon)>0$ and $\Re(q(\alpha))<1$.

Proof. Let the following equation be used in Eq. (5):

$$g(x) := f^{(m)}(x). \tag{21}$$

Considering Eq. (21), the relationship between Eqs. (3) and (5) is as follows:

$$\begin{aligned}\big({}^cD_{0+}^{\varepsilon}f\big)(x) &= \frac{1}{\Gamma(m-\varepsilon)}\int_0^x (x-\tau)^{m-\varepsilon-1}g(\tau)d\tau,\\ &= \big(I_{0+}^{m-\varepsilon}g\big)(x).\end{aligned}$$

Applying the generalized Mellin transform (6) to both sides of the equation, we get

$$\begin{aligned}{}_p\mathfrak{M}_q\Big[\big({}^cD_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) &= {}_p\mathfrak{M}_q\Big[\big(I_{0+}^{m-\varepsilon}g\big)(x)\Big](\alpha),\\ &= \frac{\Gamma\big(1-q(\alpha)-m+\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{g}\big(q(\alpha)+m-\varepsilon\big).\end{aligned} \tag{22}$$

Applying the generalized Mellin transform (6) to Eq. (21), we obtain

$$\begin{aligned}\widehat{g}\big(q(\alpha)\big) &= {}_p\mathfrak{M}_q\Big[f^{(m)}(x)\Big](\alpha),\\ &= (-1)^m\frac{\Gamma\big(q(\alpha)\big)}{\Gamma\big(q(\alpha)-m\big)}\widehat{f}\big(q(\alpha)-m\big),\\ &= \frac{\Gamma\big(m+1-q(\alpha)\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)-m\big).\end{aligned} \tag{23}$$

Taking $q(\alpha)+m-\varepsilon$ instead of $q(\alpha)$ in Eq. (23) and then using in Eq. (22), we have

$${}_p\mathfrak{M}_q\Big[\big({}^cD_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) = \frac{\Gamma\big(1-q(\alpha)+\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)}\widehat{f}\big(q(\alpha)-\varepsilon\big).$$

Theorem 15. *Let $f:(0,\infty)\to\mathbb{R}$, $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$${}_p\mathfrak{M}_q\Big[x^{\varepsilon}\big({}^cD_{0+}^{\varepsilon}f\big)(x)\Big](\alpha) = \frac{\Gamma\big(1-q(\alpha)\big)}{\Gamma\big(1-q(\alpha)-\varepsilon\big)}\widehat{f}\big(q(\alpha)\big), \tag{24}$$

where $\Re(\varepsilon)>0$ and $\Re(q(\alpha)+\varepsilon)<1$.

Proof. The desired result is obtained by evaluating Eqs. (8) and (20) together.

4 Generalized Mellin transforms of some functions

In this section, we obtain the generalized Mellin transformations of some elementary functions.

Example 16. *Let $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\Big[\exp(-ax)\Big](\alpha)=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)}{a^{q(\alpha)}}, \tag{25}$$

where $a>0$ and $\Re(q(\alpha))>0$.

Using the generalized Mellin transform (6), we have

$$\begin{aligned} {}_p\mathfrak{M}_q\Big[\exp(-ax)\Big](\alpha)&=p(\alpha)\int_0^\infty x^{q(\alpha)-1}\exp(-ax)dx,\quad \big(\text{Taking } ax=t\big),\\ &=\frac{p(\alpha)}{a^{q(\alpha)}}\int_0^\infty t^{q(\alpha)-1}\exp(-t)dt,\\ &=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)}{a^{h(p)}}. \end{aligned}$$

Example 17. *Let $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\left[\frac{1}{\exp(x)-1}\right](\alpha)=p(\alpha)\Gamma\big(q(\alpha)\big)\zeta\big(q(\alpha)\big), \tag{26}$$

where $\Re(q(\alpha))>1$.

Using the formula $\sum_{n=1}^{\infty}\exp(-nx)=\frac{1}{\exp(x)-1}$ in the generalized Mellin transform (6), we have

$$\begin{aligned} {}_p\mathfrak{M}_q\left[\frac{1}{\exp(x)-1}\right](\alpha)&=p(\alpha)\sum_{n=1}^{\infty}\int_0^\infty x^{q(\alpha)-1}\exp(-nx)\,dx,\\ &=p(\alpha)\sum_{n=1}^{\infty}\frac{\Gamma\big(q(\alpha)\big)}{n^{q(\alpha)}},\\ &=p(\alpha)\Gamma\big(q(\alpha)\big)\zeta\big(q(\alpha)\big). \end{aligned}$$

Example 18. *Let $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\left[\frac{2}{\exp(2x)-1}\right](\alpha)=2^{1-q(\alpha)}p(\alpha)\Gamma\big(q(\alpha)\big)\zeta\big(q(\alpha)\big), \tag{27}$$

where $\Re(q(\alpha))>1$.

Using the generalized Mellin transform (6), we have

$$\begin{aligned} {}_p\mathfrak{M}_q\left[\frac{2}{\exp(2x)-1}\right](\alpha)&=2p(\alpha)\sum_{n=1}^{\infty}\int_0^\infty x^{q(\alpha)-1}\exp(-2nx)\,dx,\\ &=2^{1-q(\alpha)}p(\alpha)\sum_{n=1}^{\infty}\frac{\Gamma\big(q(\alpha)\big)}{n^{q(\alpha)}},\\ &=2^{1-q(\alpha)}p(\alpha)\Gamma\big(q(\alpha)\big)\zeta\big(q(\alpha)\big). \end{aligned}$$

Example 19. *Let $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\left[\frac{1}{\exp(x)+1}\right](\alpha)=\left(1-2^{1-q(\alpha)}\right)p(\alpha)\Gamma\big(q(\alpha)\big)\zeta\big(q(\alpha)\big), $$

where $\Re(q(\alpha))>1$.

Using the formula $\left(\frac{1}{\exp(x)-1}-\frac{1}{\exp(x)+1}\right)=\frac{2}{\exp(2x)-1}$ in the generalized Mellin transform (6), we have

$$ {}_p\mathfrak{M}_q\left[\frac{1}{\exp(x)+1}\right](\alpha)={}_p\mathfrak{M}_q\left[\frac{1}{\exp(x)-1}\right](\alpha)-{}_p\mathfrak{M}_q\left[\frac{2}{\exp(2x)-1}\right](\alpha). $$

Considering Eqs. (26) and (27), we get

$$ {}_p\mathfrak{M}_q\left[\frac{1}{\exp(x)+1}\right](\alpha)=\left(1-2^{1-q(\alpha)}\right)p(\alpha)\Gamma\big(q(\alpha)\big)\zeta\big(q(\alpha)\big). $$

Example 20. *Let $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\left[\frac{1}{1+x}\right](\alpha)=p(\alpha)B(q(\alpha),1-q(\alpha)), $$

where $0<\Re(q(\alpha))<1$.

Using the generalized Mellin transform (6), we have

$$ \begin{aligned} {}_p\mathfrak{M}_q\left[\frac{1}{1+x}\right](\alpha)&=p(\alpha)\int_0^\infty x^{q(\alpha)-1}\left(\frac{1}{1+x}\right)dx,\quad \left(\text{Taking } x=\frac{t}{1-t}\right),\\ &=p(\alpha)\int_0^1 t^{q(\alpha)-1}(1-t)^{-q(\alpha)}dt,\\ &=p(\alpha)B\big(q(\alpha),1-q(\alpha)\big). \end{aligned} $$

Example 21. *Let $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\left[\frac{1}{(1+x)^n}\right](\alpha)=p(\alpha)B\big(q(\alpha),n-q(\alpha)\big), $$

where $0<\Re(q(\alpha))<n$, $n>0$.

Using the generalized Mellin transform (6), we have

$$ \begin{aligned} {}_p\mathfrak{M}_q\left[\frac{1}{(1+x)^n}\right](\alpha)&=p(\alpha)\int_0^\infty x^{q(\alpha)-1}\frac{1}{(1+x)^n}dx,\quad \left(\text{Taking } x=\frac{t}{1-t}\right),\\ &=p(\alpha)\int_0^1 t^{q(\alpha)-1}(1-t)^{n-q(\alpha)-1}dt,\\ &=p(\alpha)B\big(q(\alpha),n-q(\alpha)\big). \end{aligned} $$

Example 22. *Let $p:\mathbb{C}\to\mathbb{C}-\{0\}$, $q:\mathbb{C}\to\mathbb{C}$. Then,*

$$ {}_p\mathfrak{M}_q\Big[\cos(nx)\Big](\alpha)=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)\cos\left(\frac{q(\alpha)\pi}{2}\right)}{n^{q(\alpha)}}, $$
$$ {}_p\mathfrak{M}_q\Big[\sin(nx)\Big](\alpha)=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)\sin\left(\frac{q(\alpha)\pi}{2}\right)}{n^{q(\alpha)}}, $$

where $\Re(q(\alpha))>0$.

In Eq. (25), taking $a=in$, we have

$$ \begin{aligned} {}_p\mathfrak{M}_q\Big[\exp(-inx)\Big](\alpha)&={}_p\mathfrak{M}_q\Big[\cos(nx)-i\sin(nx)\Big](\alpha),\\ &=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)}{(in)^{q(\alpha)}}. \end{aligned} $$

Using the formula $i^{-q(\alpha)}=\exp\left(-\frac{iq(\alpha)\pi}{2}\right)=\cos\left(\frac{q(\alpha)\pi}{2}\right)-i\sin\left(\frac{q(\alpha)\pi}{2}\right)$, we get

$$ {}_p\mathfrak{M}_q\Big[\cos(nx)-i\sin(nx)\Big](\alpha)=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)}{n^{q(\alpha)}}\left(\cos\left(\frac{q(\alpha)\pi}{2}\right)-i\sin\left(\frac{q(\alpha)\pi}{2}\right)\right). $$

Then, we obtain

$$ {}_p\mathfrak{M}_q\Big[\cos(nx)\Big](\alpha)=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)\cos\left(\frac{q(\alpha)\pi}{2}\right)}{n^{q(\alpha)}}, $$

and

$$ {}_p\mathfrak{M}_q\Big[\sin(nx)\Big](\alpha)=\frac{p(\alpha)\Gamma\big(q(\alpha)\big)\sin\left(\frac{q(\alpha)\pi}{2}\right)}{n^{q(\alpha)}}. $$

5 Applications to differential equations

In this section, we describe how to obtain solutions of partial and fractional differential equations using the generalized Mellin transform.

Open Problem 23. *We consider the following fractional differential equation*

$$ \sum_{k=0}^{m}A_k\,x^{\varepsilon+k}\left({}^cD_{0+}^{\varepsilon+k}y\right)(x)=f(x), $$

where $\Re(\varepsilon)>0$ and $A_k\in\mathbb{R}$ for $k=0,1,\ldots,m$.

Proof. Applying the generalized Mellin transform (6) to the given fractional differential equation, we have

$$ \sum_{k=0}^{m}A_k\;{}_p\mathfrak{M}_q\left[x^{\varepsilon+k}\left({}^cD_{0+}^{\varepsilon+k}y\right)(x)\right](\alpha)={}_p\mathfrak{M}_q\Big[f(x)\Big](\alpha). $$

Considering Eq. (24), we get

$$ \sum_{k=0}^{m}A_k\,\frac{\Gamma\big(1-q(\alpha)\big)}{\Gamma\big(1-q(\alpha)-\varepsilon-k\big)}\widehat{y}\big(q(\alpha)\big)=\widehat{f}\big(q(\alpha)\big), $$

where

$$\widehat{h}_{\varepsilon}\big(1-q(\alpha)\big) := \sum_{k=0}^{m} A_k \, \frac{\Gamma\big(1-q(\alpha)\big)}{\Gamma\big(1-q(\alpha)-\varepsilon-k\big)}.$$

From here, we obtain

$$\widehat{y}\big(q(\alpha)\big) = \frac{\widehat{f}\big(q(\alpha)\big)}{\widehat{h}_{\varepsilon}\big(1-q(\alpha)\big)},$$

where

$$\widehat{g}_{\varepsilon}\big(1-q(\alpha)\big) := \frac{1}{\widehat{h}_{\varepsilon}\big(1-q(\alpha)\big)},$$

then

$$\widehat{y}\big(q(\alpha)\big) = \widehat{f}\big(q(\alpha)\big)\widehat{g}_{\varepsilon}\big(1-q(\alpha)\big).$$

Multiplying and dividing the right-hand side of the equation by $p(\alpha)$ and using Eq. (12), we have

$$\begin{aligned}
\widehat{y}\big(q(\alpha)\big) &= p(\alpha)\frac{\widehat{f}\big(q(\alpha)\big)\widehat{g}_{\varepsilon}\big(1-q(\alpha)\big)}{p(\alpha)}, \\
&= p(\alpha)\ {}_{p}\mathfrak{M}_{q}\Big[f(x)\circ g_{\varepsilon}(x)\Big](\alpha), \\
&= p(\alpha)\ {}_{p}\mathfrak{M}_{q}\left[\int_{0}^{\infty} f(x\tau)g_{\varepsilon}(\tau)d\tau\right](\alpha).
\end{aligned}$$

Applying the generalized inverse Mellin transform (7) to both sides of the equation, we obtain

$$y(x) = p(\alpha)\int_{0}^{\infty} f(x\tau)g_{\varepsilon}(\tau)d\tau.$$

Open Problem 24. *We consider the following fractional differential equation*

$$x^{\varepsilon+1}\left({}^{c}D_{0+}^{\varepsilon+1}y\right)(x) + \lambda\ x^{\varepsilon}\left({}^{c}D_{0+}^{\varepsilon}y\right)(x) = f(x),$$

where $\lambda \in \mathbb{R}$ *and* $\Re(\varepsilon) > 0$.

Proof. Applying the generalized Mellin transform (6) to the given fractional differential equation, we have

$${}_{p}\mathfrak{M}_{q}\Big[x^{\varepsilon+1}\left({}^{c}D_{0+}^{\varepsilon+1}y\right)(x)\Big](\alpha) + \lambda\ {}_{p}\mathfrak{M}_{q}\Big[x^{\varepsilon}\left({}^{c}D_{0+}^{\varepsilon}y\right)(x)\Big](\alpha) = {}_{p}\mathfrak{M}_{q}\Big[f(x)\Big](\alpha).$$

Considering Eq. (24), we get

$$\frac{\Gamma\big(1-q(\alpha)\big)\big(1-q(\alpha)+\lambda-\varepsilon-1\big)}{\Gamma\big(1-q(\alpha)-\varepsilon\big)}\widehat{y}\big(q(\alpha)\big) = \widehat{f}\big(q(\alpha)\big).$$

From here, we obtain

$$\widehat{y}\big(q(\alpha)\big) = \frac{\Gamma\big(1-q(\alpha)-\varepsilon\big)}{\Gamma\big(1-q(\alpha)\big)\big(1-q(\alpha)+\lambda-\varepsilon-1\big)}\widehat{f}\big(q(\alpha)\big),$$

where

$$\widehat{g}_{\varepsilon,\lambda}\left(1-q(\alpha)\right) := \frac{\Gamma\left(1-q(\alpha)-\varepsilon\right)}{\Gamma\left(1-q(\alpha)\right)\left(1-q(\alpha)+\lambda-\varepsilon-1\right)}, \tag{28}$$

then

$$\widehat{y}\left(q(\alpha)\right) = \widehat{f}\left(q(\alpha)\right)\widehat{g}_{\varepsilon,\lambda}\left(1-q(\alpha)\right).$$

Multiplying and dividing the right-hand side of the equation by $p(\alpha)$ and using Eq. (12), we have

$$\begin{aligned}
\widehat{y}\left(q(\alpha)\right) &= p(\alpha)\frac{\widehat{f}\left(q(\alpha)\right)\widehat{g}_{\varepsilon,\lambda}\left(1-q(\alpha)\right)}{p(\alpha)}, \\
&= p(\alpha)\ {}_p\mathfrak{M}_q\Big[f(x)\circ g_{\varepsilon,\lambda}(x)\Big](\alpha), \\
&= p(\alpha)\ {}_p\mathfrak{M}_q\left[\int_0^\infty f(x\tau)g_{\varepsilon,\lambda}(\tau)d\tau\right](\alpha).
\end{aligned}$$

Applying the generalized inverse Mellin transform (7) to both sides of the equation, we obtain

$$y(x) = p(\alpha)\int_0^\infty f(x\tau)g_{\varepsilon,\lambda}(\tau)d\tau. \tag{29}$$

Here we can show that $g_{\varepsilon,\lambda}(x) = 0$ for $x > 1$ with the following steps. Writing Eq. (28) as follows, we have

$$\widehat{g}_{\varepsilon,\lambda}\left(q(\alpha)\right) := \frac{\Gamma\left(q(\alpha)-\varepsilon\right)}{\Gamma\left(q(\alpha)\right)\left(q(\alpha)+\lambda-\varepsilon-1\right)}, \tag{30}$$

where

$$\widehat{g}_1\left(q(\alpha)\right) := \frac{\Gamma\left(q(\alpha)-\varepsilon\right)}{\Gamma\left(q(\alpha)\right)}, \tag{31}$$

and

$$\widehat{g}_2\left(q(\alpha)\right) := \frac{1}{\left(q(\alpha)+\lambda-\varepsilon-1\right)}, \tag{32}$$

then

$$\widehat{g}_{\varepsilon,\lambda}\left(q(\alpha)\right) = \widehat{g}_1\left(q(\alpha)\right)\widehat{g}_2\left(q(\alpha)\right). \tag{33}$$

Applying the generalized inverse Mellin transform (7) to Eqs. (31) and (32), we obtain

$$g_1(x) = \begin{cases} \frac{x^{-\varepsilon}(1-x)^{\varepsilon-1}}{p(\alpha)\Gamma(\varepsilon)} & \text{for } 0 < x < 1, \\ 0 & \text{for } x > 1, \end{cases} \tag{34}$$

and

$$g_2(x) = \begin{cases} \frac{x^{-\varepsilon-1+\lambda}}{p(\alpha)} & \text{for } 0 < x < 1, \\ 0 & \text{for } x > 1. \end{cases} \tag{35}$$

Multiplying and dividing Eq. (33) by $p(\alpha)$ and using Eq. (11), we have

$$\begin{aligned}\widehat{g}_{\varepsilon,\lambda}\big(q(\alpha)\big) &= p(\alpha)\frac{\widehat{g}_1\big(q(\alpha)\big)\widehat{g}_2\big(q(\alpha)\big)}{p(\alpha)},\\ &= p(\alpha)\ {}_p\mathfrak{M}_q\Big[g_1(x)*g_2(x)\Big](\alpha),\\ &= p(\alpha)\ {}_p\mathfrak{M}_q\left[\int_0^\infty g_1\left(\frac{x}{\tau}\right)g_2(\tau)\frac{d\tau}{\tau}\right](\alpha).\end{aligned}$$

Applying the generalized inverse Mellin transform (7), we get

$$g_{\varepsilon,\lambda}(x) = p(\alpha)\int_0^\infty g_1\left(\frac{x}{\tau}\right)g_2(\tau)\frac{d\tau}{\tau}.$$

Thus, Eqs. (34) and (35) show that $g_{\varepsilon,\lambda}(x) = 0$ for $x > 1$. So Eq. (29) has the form

$$y(x) = p(\alpha)\int_0^1 f(x\tau)g_{\varepsilon,\lambda}(\tau)d\tau. \tag{36}$$

Applying the generalized inverse Mellin transform (7) to Eq. (30), we obtain

$$g_{\varepsilon,\lambda}(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{q'(\alpha)}{p(\alpha)}\frac{\Gamma\big(q(\alpha)-\varepsilon\big)x^{-q(\alpha)}}{\Gamma\big(q(\alpha)\big)\big(q(\alpha)+\lambda-1-\varepsilon\big)}d\alpha,$$

where $q(\alpha) = \varepsilon+1-\lambda$ and $q_k(\alpha) = \varepsilon-k$ for $k\in\mathbb{N}_0$ are the poles. Using the Cauchy rezidu theorem in Zill and Shanahan [26], we have

$$\begin{aligned}g_{\varepsilon,\lambda}(x) = \operatorname{Res}&\left[\frac{q'(\alpha)}{p(\alpha)}\frac{\Gamma\big(q(\alpha)-\varepsilon\big)x^{-q(\alpha)}}{\Gamma\big(q(\alpha)\big)\big(q(\alpha)+\lambda-1-\varepsilon\big)}, q(\alpha)=\varepsilon+1-\lambda\right]\\ &+\sum_{k=0}^{\infty}\operatorname{Res}\left[\frac{q'(\alpha)}{p(\alpha)}\frac{\Gamma\big(q(\alpha)-\varepsilon\big)x^{-q(\alpha)}}{\Gamma\big(q(\alpha)\big)\big(q(\alpha)+\lambda-1-\varepsilon\big)}, q(\alpha)=\varepsilon-k\right],\end{aligned}$$

then

$$g_{\varepsilon,\lambda}(x) = \frac{A(\varepsilon,\lambda)}{p(\alpha)}\frac{\Gamma(1-\lambda)}{\Gamma(\varepsilon+1-\lambda)}x^{\lambda-\varepsilon-1}+\sum_{k=0}^{\infty}\frac{B(\varepsilon,k)}{p(\alpha)}\frac{x^{k-\varepsilon}(-1)^k}{(\lambda-k-1)\Gamma(\varepsilon-k)k!}, \tag{37}$$

where

$$A(\varepsilon,\lambda) := \lim_{q(\alpha)\to\varepsilon+1-\lambda} q'(\alpha),$$

and

$$B(\varepsilon,k) := \lim_{q(\alpha)\to\varepsilon-k} q'(\alpha).$$

Thus, using Eq. (37) in Eq. (36), we obtain

$$y(x) = \int_0^1 f(x\tau)\left(\frac{A(\varepsilon,\lambda)\Gamma(1-\lambda)}{\Gamma(\varepsilon+1-\lambda)}\tau^{\lambda-\varepsilon-1}+\sum_{k=0}^{\infty}\frac{B(\varepsilon,k)\tau^{k-\varepsilon}(-1)^k}{(\lambda-k-1)\Gamma(\varepsilon-k)k!}\right)d\tau. \tag{38}$$

Corollary 25. *Taking $A(\varepsilon,\lambda)=B(\varepsilon,k)=1$ in Eq. (38), we have*

$$y(x)=\int_0^1 f(x\tau)\left(\frac{\Gamma(1-\lambda)}{\Gamma(\varepsilon+1-\lambda)}\tau^{\lambda-\varepsilon-1}+\sum_{k=0}^{\infty}\frac{\tau^{k-\varepsilon}(-1)^k}{(\lambda-k-1)\Gamma(\varepsilon-k)k!}\right)d\tau.$$

This result coincides with the solution of the classical Mellin transform, (Kilbas et al. [25]).

Open Problem 26. *We consider the following partial differential equation*

$$x^2u_{xx}+xu_x+u_{yy}=0,\quad 0\le x<\infty,\ 0<y<1,$$

with the initial conditions

$$u(x,0)=0\ \text{ and }\ u(x,1)=\begin{cases}\lambda & \text{for } 0\le x\le 1,\\ 0 & \text{for } x>1,\end{cases}$$

where λ is constant.

Proof. The generalized Mellin transform (6) of the function $u(x,y)$ with respect to the parameter x is as follows:

$$\widehat{u}\big(q(\alpha),y\big)=p(\alpha)\int_0^{\infty}x^{q(\alpha)-1}u(x,y)dx.$$

Applying the generalized Mellin transform (6) to the given partial differential equation, we have

$${}_p\mathfrak{M}_q\Big[x^2u_{xx}\Big](\alpha)+{}_p\mathfrak{M}_q\Big[xu_x\Big](\alpha)+{}_p\mathfrak{M}_q\Big[u_{yy}\Big](\alpha)=0.$$

Considering Eq. (9), we get

$$\frac{d^2\,\widehat{u}\big(q(\alpha),y\big)}{dy^2}+\big(q(\alpha)\big)^2\,\widehat{u}\big(q(\alpha),y\big)=0. \tag{39}$$

Applying the generalized Mellin transform (6) to the given initial conditions, we obtain

$$\widehat{u}\big(q(\alpha),0\big)=0\ \text{ and }\ \widehat{u}\big(q(\alpha),1\big)=\lambda p(\alpha)\int_0^1 x^{q(\alpha)-1}dx=\frac{\lambda p(\alpha)}{q(\alpha)}. \tag{40}$$

Using Eqs. (40) in Eq. (39), we have

$$\widehat{u}\big(q(\alpha),y\big)=\frac{\lambda p(\alpha)}{q(\alpha)\sin\big(q(\alpha)\big)}\sin\big(q(\alpha)y\big),\quad \big(0<\Re(q(\alpha))<1\big). \tag{41}$$

Applying the generalized inverse Mellin transform (7) to Eq. (41), we get

$$u(x,y)=\frac{\lambda}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{x^{-q(\alpha)}q'(\alpha)}{q(\alpha)}\frac{\sin\big(q(\alpha)y\big)}{\sin\big(q(\alpha)\big)}d\alpha. \tag{42}$$

Note that the expression $\widehat{u}\big(q(\alpha),y\big)$ is analytic in the range $0<\Re\big(q(\alpha)\big)=\gamma<\pi$. Eq. (42) has simple poles at $q(\alpha)=n\pi$ for $n\in\mathbb{N}$. These simple poles are located on a semicircular arc in the right half-plane. Therefore, for $x>1$, we can use the Cauchy residue theorem in Zill and Shanahan [26] to obtain:

$$\begin{aligned}u(x,y)&=2\pi i\sum_{n=1}^{\infty}\frac{\lambda}{2\pi i}\frac{x^{-n\pi}A(n)}{n\pi}\frac{\sin(n\pi y)}{\sin(n\pi)}\sin(n\pi),\\&=\frac{\lambda}{\pi}\sum_{n=1}^{\infty}\frac{A(n)}{n}(-1)^nx^{-n\pi}\sin(n\pi y),\end{aligned} \tag{43}$$

where

$$A(n):=\lim_{q(\alpha)\to n\pi}q'(\alpha).$$

Corollary 27. *Taking $A(n) = 1$ in Eq. (43), we have*

$$u(x,y) = \frac{\lambda}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n x^{-n\pi}\sin(n\pi y).$$

This result coincides with the solution of the classical Mellin transform, (Debnath and Bhatta [11]).

6 Concluding remarks

The new generalized Mellin transform has a general form that encompasses other generalized Mellin transforms found in the literature. In this section, we provide definitions for some of the recently defined generalized Mellin transforms and we also present their relationships with the new generalized Mellin transform.

Classic form of the Mellin transform in Debnath and Bhatta [11] is defined as:

$$\mathscr{M}\{f(x)\} = \int_0^{\infty} x^{\alpha-1} f(x)dx,$$

and the relationship with our generalization is:

$${}_1\mathfrak{M}_{\alpha}\left[f(x)\right](\alpha) = \mathscr{M}\{f(x)\}.$$

Erdoğan et al. [14], defined a generalization of Mellin transform as:

$$\mathscr{M}_n\{f(x);\alpha\} = \int_0^{\infty} x^{n\alpha-1} f(x)dx,$$

and the relationship with our generalization is:

$${}_1\mathfrak{M}_{n\alpha}\left[f(x)\right](\alpha) = \mathscr{M}_n\{f(x);\alpha\}.$$

Gonzalez-Gaxiola and Santiago [17], defined a generalization of Mellin transform as:

$$\mathscr{M}_{\alpha}(f(x),s_{\alpha}) = \int_0^{\infty} x^{s_{\alpha}-1} f(x)dx, \quad \left(s_{\alpha} = a + iw^{\frac{1}{\alpha}}\right),$$

and the relationship with our generalization is:

$${}_1\mathfrak{M}_{s_{\alpha}}\left[f(x)\right](\alpha) = \mathscr{M}_{\alpha}(f(x),s_{\alpha}).$$

Yang et al. [18], defined a generalization of Mellin transform as:

$$M_f^m(\alpha) = \alpha\int_0^{\infty} x^{\alpha-1} f(x)dx,$$

and the relationship with our generalization is:

$${}_{\alpha}\mathfrak{M}_{\alpha}\left[f(x)\right](\alpha) = M_f^m(\alpha).$$

This paper presented a new and generalized version of the Mellin transform that encompasses previously defined generalized Mellin transformations found in literature. This new integral transformation was shown to have many useful properties, including the ability to solve partial and fractional differential equations. The paper provided various examples of the new integral transformation applied to elementary functions and also explored the relationships between the new integral transformation and previously defined generalized Mellin transformations. Finally, we present Table 1, which shows the new generalized Mellin transform of functions commonly used in mathematics, physics and engineering applications.

In conclusion, the generalized Mellin transform has been shown to have a wide range of applications in the field of differential equations. Further research in this area is necessary to fully understand the potential of the generalized Mellin transform and its impact on modern mathematics.

Table 1 New Generalized Mellin Transform Table

$f(x)$	${}_p\mathfrak{M}_q\left[f(x)\right](\alpha)=\widehat{f}\left(q(\alpha)\right)=p(\alpha)\int_0^\infty x^{q(\alpha)-1}f(x)dx$	Conditions
$\exp(-ax)$	$\frac{p(\alpha)\Gamma(q(\alpha))}{a^{q(\alpha)}}$	$\Re(q(\alpha))>0$
$\frac{1}{\exp(x)-1}$	$p(\alpha)\Gamma\left(q(\alpha)\right)\zeta\left(q(\alpha)\right)$	$\Re(q(\alpha))>1$
$\frac{2}{\exp(2x)-1}$	$2^{1-q(\alpha)}p(\alpha)\Gamma\left(q(\alpha)\right)\zeta\left(q(\alpha)\right)$	$\Re(q(\alpha))>1$
$\frac{1}{\exp(x)+1}$	$\left(1-2^{1-q(\alpha)}\right)p(\alpha)\Gamma\left(q(\alpha)\right)\zeta\left(q(\alpha)\right)$	$\Re(q(\alpha))>1$
$\frac{1}{1+x}$	$\frac{p(\alpha)\Gamma(q(\alpha))\Gamma(1-q(\alpha))}{\Gamma(1)}$	$0<\Re(q(\alpha))<1$
$\frac{1}{(1+x)^n}$	$\frac{p(\alpha)\Gamma(q(\alpha))\Gamma(n-q(\alpha))}{\Gamma(n)}$	$0<\Re(q(\alpha))<n$
$\cos(kx)$	$\frac{p(\alpha)\Gamma(q(\alpha))}{k^{q(\alpha)}}\cos\left(\frac{q(\alpha)\pi}{2}\right)$	$\Re(q(\alpha))>0$
$\sin(kx)$	$\frac{p(\alpha)\Gamma(q(\alpha))}{k^{q(\alpha)}}\sin\left(\frac{q(\alpha)\pi}{2}\right)$	$\Re(q(\alpha))>0$
$f'(x)$	$-\frac{\Gamma\left(q(\alpha)\right)}{\Gamma\left(q(\alpha)-1\right)}\widehat{f}\left(q(\alpha)-1\right)$	$\Re(q(\alpha))>1$
$f''(x)$	$\frac{\Gamma\left(q(\alpha)\right)}{\Gamma\left(q(\alpha)-2\right)}\widehat{f}\left(q(\alpha)-2\right)$	$\Re(q(\alpha))>2$
$f^{(n)}(x)$	$(-1)^n\frac{\Gamma\left(q(\alpha)\right)}{\Gamma\left(q(\alpha)-n\right)}\widehat{f}\left(q(\alpha)-n\right)$	$\Re(q(\alpha))>n$
$xf'(x)$	$-\frac{\Gamma(q(\alpha)+1)}{\Gamma(q(\alpha))}\widehat{f}\left(q(\alpha)\right)$	$\Re(q(\alpha))>0$
$x^2f''(x)$	$\frac{\Gamma(q(\alpha)+2)}{\Gamma(q(\alpha))}\widehat{f}\left(q(\alpha)\right)$	$\Re(q(\alpha))>0$
$x^nf^{(n)}(x)$	$(-1)^n\frac{\Gamma(q(\alpha)+n)}{\Gamma(q(\alpha))}\widehat{f}\left(q(\alpha)\right)$	$\Re(q(\alpha))>0$
$\left(x\frac{d}{dx}\right)^2f(x)$	$(q(\alpha))^2\widehat{f}\left(q(\alpha)\right)$	
$\left(x\frac{d}{dx}\right)^nf(x)$	$(-1)^n(q(\alpha))^n\widehat{f}\left(q(\alpha)\right)$	
$I_nf(x)$	$(-1)^n\frac{\Gamma\left(q(\alpha)\right)}{\Gamma\left(q(\alpha)+n\right)}\widehat{f}\left(q(\alpha)+n\right)$	$\Re(q(\alpha))>0$
$\left(I_{0+}^{\varepsilon}f\right)(x)$	$\frac{\Gamma\left(1-q(\alpha)-\varepsilon\right)}{\Gamma\left(1-q(\alpha)\right)}\widehat{f}\left(q(\alpha)+\varepsilon\right)$	$\Re(q(\alpha)+\varepsilon)<1$
$\left(D_{0+}^{\varepsilon}f\right)(x)$	$\frac{\Gamma\left(1-q(\alpha)+\varepsilon\right)}{\Gamma\left(1-q(\alpha)\right)}\widehat{f}\left(q(\alpha)-\varepsilon\right)$	$\Re(q(\alpha))<1$
$\left({}^cD_{0+}^{\varepsilon}f\right)(x)$	$\frac{\Gamma\left(1-q(\alpha)+\varepsilon\right)}{\Gamma\left(1-q(\alpha)\right)}\widehat{f}\left(q(\alpha)-\varepsilon\right)$	$\Re(q(\alpha))<1$
$f(x)*g(x)$	$\frac{\widehat{f}\left(q(\alpha)\right)\widehat{g}\left(q(\alpha)\right)}{p(\alpha)}$	
$f(x)\circ g(x)$	$\frac{\widehat{f}\left(q(\alpha)\right)\widehat{g}\left(1-q(\alpha)\right)}{p(\alpha)}$	

7 Declarations

7.1 Conflict of interest:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

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All data that support the findings of this study are included within the article (and any supplementary files).

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