General Mathematics Vol. 31, No. 2 (2023), 39–50

DOI: 10.2478/gm-2023-0011

sciendo

On analytic functions related with q-generalized bounded Mocanu variation [1]

Khalida Inayat Noor, Muhammad Aslam Noor

Abstract

Let $\mathcal{A}$ be the class of normalized functions which are analytic in the open unit disc. Jakson q-derivative represented by convolution operator

$$D_q f(z) = \frac{1}{z}\{f(z) \star \frac{z}{(1-qz)(1z)}\}, \quad f \in \mathcal{A}, \quad q \in (0,1)$$

is used to introduce a unified class $M_q(\alpha, \beta, \gamma), \quad \alpha \geq 0, \quad \beta, \gamma \in [0,1)$ and its various mapping properties are studied. For $q \to 1^{-1}, \quad \beta = \gamma = 0$ and $\alpha = 1$, this class reduces to the class K of close-to-convex univalent functions. A number of interesting results such as q-Bernardi integral operator and inclusion relations are included as part of this study. Applications are also pointed out as consequences of the main results.

2010 Mathematics Subject Classification: 30C45; 30C20; 30A20.
Key words and phrases: Analytic functions, q-calculus, Close-to-convex, q-derivative, Mocanu variation.

1 Introduction

Let $\mathcal{A}$ be the class of functions f given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

[1] *Received 15 April, 2023*
Accepted for publication (in revised form) 12 November, 2023

which are analytic in the open unit disc $E = \{z \in \mathbb{C} : \quad |z| < 1\}$. By $S, K, S^\star$ and C denote the subclasses of $\mathcal{A}$ which are respectively univalent, close-to-convex, starlike and convex in E. Let P be the class of Caratheodry function p given by

$$f(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2}$$

with $Re\{p(z)\} > 0$ for $z \in E$.
The convolution (Hadanard product) of two functions f and g, $f(z)$ given by (1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

is defined by

$$(f \star g)(z) = f(z) \star g(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (f \star g)(z), \quad z \in E.$$

The concept of q-calculus is the study of calculus without limit. In this paper, we shall use q-derivative, introduced in [4], convolution and differential subordination, see [7], in our investigation. Here we recall some basic definition of q-calculus.

(i). Let $f \in \mathcal{A}$. The q-derivative $D_q : \mathcal{A} \to \mathcal{A}$ is defined as:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}; \quad z \neq 0, \quad 0 < q < 1,$$

and

$$D_q f(0) = f'(0).$$

(ii). $D_q(z^n) = [n]_q z^{n-1}, \quad [n]_q = \frac{1-q^n}{1-q}$.

We note that $D_q f(z) \to f'(z)$ and $[n]_q \to n$ as $q \to 1^{-1}$.

(iii). q-integral of $f \in \mathcal{A}$ is defined in [5] as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(q^n z),$$

provided that series converges.

(iv). For $f \in \mathcal{A}$, q-logorithmic differentiation is given in [18] as

$$D_q\{\log f(z)\} = \frac{\log q}{q-1} \frac{D_q f(z)}{f(z)}.$$

Let the function $h_q \in \mathcal{A}$ be defined for $q \in (0,1)$ as

$$h_q(z) = \frac{1}{1-q}\log\frac{1-qz}{1-z} = \sum_{n=1}^{\infty}\left(\frac{1-q^n}{1-q}\right)\frac{z^n}{n} = \sum_{n=1}^{\infty}[n]_q\frac{z^n}{n}. \tag{3}$$

It can easily checked that

$$Re\left\{1+\frac{zh_q''(z)}{zh_q'(z)}\right\} > 0, \quad z \in E$$

which implies $h_q \in C$.

By Alexander relation between the classes $S^\star$ and C, it follows the function $zh_q'(z)$ is starlike univalent in E and is given by

$$zh_q'(z) = \frac{z}{(1-qz)(1-z)} = \sum_{n=1}^{\infty}[n]_q z^n, \quad z \in E. \tag{4}$$

In fact, with some calculations, it can be shown that $H_q \in S^\star(\sigma_1)$ and is called starlike of order $\sigma_1 = \frac{1-q}{2(1+q)}$, that is

$$Re\{\frac{zH_q'(z)}{H_q(z)}\} > \sigma_1, \quad \text{see [15].}$$

Also $H_q(z) = zh_q'(z) \to k(z) = \frac{z}{(1-z)^2}$, where $k(z)$ is well known Koebe function. From Definition of q-derivative and relation (4). we have

$$zD_qf(z) = f(z) \star zh_q'(z). \tag{5}$$

If $f \in C$, then it follows from (5) that $zD_qf(z)$ is convolution of $f(z)$ with a function belonging to $S^\star(\sigma_1)$, since it is known [17] that

$$C \star S^\star(\sigma_1) \subset S^\star(\sigma_1), \quad \sigma_1 = \frac{1-q}{2(1+q)}.$$

We note that

$$\sigma_1 = \frac{1-q}{2(1+q)}, \quad q \in (0,1). \tag{6}$$

Throughout this paper, we shall assume that $q \in (0,1)$, $f, g \in \mathcal{A}$, σ_1 is given by (6) and $F = f \star h_q$, $G = g \star h_q$ and $h_q \in C$ is defined in (3). Also, $z \in E$, $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$.

Definition 1 *Let $f \in \mathcal{A}$ and, with $F = f \star h_q$; $\frac{F(z)F'(z)}{z} \neq 0$. Let*

$$J_q(\alpha, \beta, \gamma, f(z)) = \frac{1-\alpha}{1-\beta}\{\frac{zF'(z)}{F(z)} - \beta\} + \frac{\alpha}{1-\alpha}\{\frac{(zF'(z))'}{F'(z)} - \gamma\}.$$

Then F is said to belong to the class $M_q(\alpha, \beta, \gamma)$, if and only if,

$$\int_{\theta_1}^{\theta_2} Re\big\{J_q(\alpha, \beta, \gamma, f(z))\big\}d\theta > -\pi, \quad \alpha \geq 0, \quad 0 \leq \beta, \gamma < 1, 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta}.$$

We note the following special cases.

(i). For $f \in M_q(1, 0, 0)$, we have

$$\int_{\theta_1}^{\theta_2} Re\big\{\frac{(zF'(z))'}{F'(z)}\big\}d\theta = \int_{\theta_1}^{\theta_2} Re\big\{\frac{(D_q(zD_qf(z))}{D_qf(z)}\big\}d\theta > -\pi.$$

This implies $F \in K$ and is close-to-convex univalent in E, see [6] and $f \in K_q$. We call K_q as the class of q-close-to-convex functions.

(ii). $M_q(\alpha, 0, 0) = M_q(\alpha)$. When $q \to 1^{-}$, the class $M_q(\alpha) \to M_\alpha^\star$, which consists of functions f with bounded Mocanu variation, see [8]

(iii). For $f \in M_q(0, 0, \gamma) = R_q^\star$, we have

$$\int_{\theta_1}^{\theta_2} Re\big\{\frac{zF'(z)}{F(z)}\big\}d\theta > -\pi, \quad F = f \star h_q.$$

For ore details, we refer to [11, 12, 13]

Lemma 1 *For p analytic in E with $p(0) = 1$ and for $a \geq 1, \quad c \geq 0, \quad 0 \leq \theta_1 < \theta_2 \leq 2\pi, \quad \}z = re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} Re\left\{p(z) + \frac{azp'(z)}{ac + p(z)}\right\}d\theta > -\pi$$

implies

$$\int_{\theta_1}^{\theta_2} Re\{p(z)\}d\theta > -\pi.$$

2 Main Results

Theorem 1 *Let $F = f \star h_q$. Then $f \in Mq(\alpha, \beta, \gamma), \quad \alpha > 0$, if and only if, there exists $G = g \star h_q, \quad g \in M_q(1, 0, 0)$ such that*

$$F(z) = \left[m \int_0^z t^{m-1} \big(D_q g(z)\big)^{\frac{1-\gamma}{\alpha}} dt \right]^{\frac{1}{m}} = z + \dots \tag{7}$$

where

$$m = 1 + \frac{(1-\alpha)(1-\gamma)}{\alpha(1-\beta)}. \tag{8}$$

Proof. From (7), with some computations, we have

$$\frac{(zF'(z))'}{F'(z)} + (m-1)\frac{zF'(z)}{F(z)} = \left\{ (m-1) + \frac{(1-\gamma)}{\alpha}(1 + \frac{zG''(z)}{G'(z)}) - \frac{1-\gamma}{\alpha} \right\}. \tag{9}$$

Using (8) in (9), we get after some simple computations

$$\begin{aligned} J_q(\alpha, \beta, \gamma, f(z)) &= \frac{1-\alpha}{1-\beta}\big(\frac{zF'(z)}{F(z)} - \beta\big) + \frac{\alpha}{1-\gamma}\big(\frac{(zF'(z))'}{F'(z)} - \gamma\big) \\ &= \frac{(zG'(z))'}{G'(z)} \end{aligned} \tag{10}$$

That implies

$$\int_{\theta_1}^{\theta_2} Re\big\{J_q(\alpha, \beta, \gamma, f(z))\big\} d\theta = \int_{\theta_1}^{\theta_2} Re\{\frac{(zG'(z))'}{G'(z)}\} d\theta.$$

Now the result follows from Definition 1 and the given hypothesis.

Corollary 1 *From Theorem 1, we have $G = g \star h_q \in M_q(1, 0, 0)$ and*

$$\int_{\theta_1}^{\theta_2} Re\{\frac{(zG'(z))'}{G'(z)}\} d\theta > -\pi$$

implies G is close-to-convex univalent function, see [6]. It is well known [3] that $G \in K$ is convex in

$$|z| < r_c = 2 - \sqrt{3}. \tag{11}$$

Therefore, from (10) and (11), we have

$$Re\{J_q(\alpha, \beta, \gamma, f(z))\} > 0 \quad for \quad |z| < r_c = 2 - \sqrt{3}.$$

That is $J_q(\alpha, \beta, \gamma, f(z)) \in P$ in $|z| < r_c$. The radius r_c is sharp.

Corollary 2 *Let*

$$F(z)=\left[\frac{1}{1-\alpha}\int_0^z t^{\frac{\alpha}{1-\alpha}}(D_qg(z))^{\frac{2-\alpha}{2(1-\alpha}}dt\right]^{(1-\alpha)}, \quad \alpha\neq 1. \tag{12}$$

(i). We note that $zD_qg(z)=zG'(z)$, $F=f\star h_q$ *and*

$$F\in M_q(\frac{\alpha}{2},0,0)\Leftrightarrow G\in M_q(1,0,0), \quad 0<\alpha\leq 2.$$

Also, for

$$\alpha>2, \quad f\in M_q(\frac{\alpha}{2},0,0)\Rightarrow F\in K(\eta_1),$$

where $K(\eta_1)$ *is the class of close-to-convex of higher order , discussed in [2, 9, 10]. We recall that* $F\in K(\eta_1)$ *implies*

$$\int_{\theta_1}^{\theta_2} Re\{\frac{(zF'(z))'}{F'(z)}\}d\theta>\eta_1\pi, \quad \eta_1=\frac{2\alpha-1}{\alpha}>1.$$

The function $F(z)$ *need not be of finitely-valent in* E, *[2].*

(ii). The class $M_q(\alpha,0,0)$, $\alpha>1$ *forms a subset of linear-invariant family of order* $(3-\frac{1}{\alpha})$, *[10, 16].*

Corollary 3 *From (12), we easily derive that*

$$F\in M_q(1-\beta_1,\beta_1,\beta_1) \quad \Leftrightarrow \quad M_q(1,0,0), \quad \beta_1=\frac{\beta}{2}.$$

Corollary 4 *Let* V_k, $k\geq 2$ *be the class of functions with bounded boundary rotation which satisfies the condition*

$$\int_0^{2\pi}\big|Re\frac{zG'(z))'}{G'(z)}\big|d\theta\leq k\pi.$$

For $2\leq k\leq 4$, V_k *consists of univalent functions, see [2].*

Now taking $G\in V_k$ *in (10), we have*

$$\int_0^{2\pi}\big|ReJ_q(\alpha,\beta,\alpha,f(z))\big|d\theta\leq k\pi.$$

As a special case, we can deduce that, for $\alpha=1$, $\beta=0$, $\gamma=0$, $F\in V_k, k\geq 2$.

Corollary 5 *Let*

$$G(z)=z\big\{\frac{F(z)}{z}\big\}^{\frac{1-\alpha}{1-\beta}}\big\{F'(z)\big\}^{\frac{\alpha}{1-\gamma}}, \tag{13}$$

where $G=g\star h_q$, $F=f\star h_q$. *Then* $f\in M_q(\alpha,\beta,\gamma)$ *implies* $g\in M_q(0,0,\gamma)$.

Proof. Proof is straight forward, when , with simple computations, we obtain from (13),

$$\begin{aligned}\frac{zG'(z)}{G(z)} &= 1+(\frac{1-\alpha}{1-\beta})[\frac{zF'(z)}{F(z)}-1]+(\frac{\alpha}{1-\gamma})[\frac{zF''(z)}{F'(z)}+1]-\frac{\alpha}{1-\gamma}\\ &= (\frac{1-\alpha}{1-\beta})[\frac{zF'(z)}{F(z)}-\beta]+(\frac{\alpha}{1-\gamma})[\frac{zF''(z)}{F'(z)}+1-\gamma]-\frac{\alpha}{1-\gamma}\\ &\quad +\{1+\frac{\beta(1-\alpha)}{1-\beta}-\frac{1-\alpha}{1-\beta}+\frac{\alpha\gamma}{1-\gamma}-\frac{\alpha}{1-\gamma}\}\\ &= (\frac{1-\alpha}{1-\beta})[\frac{zF'(z)}{F(z)}-\beta]+(\frac{\alpha}{1-\gamma})[\frac{zF''(z)}{F'(z)}+1-\gamma]\\ &= J_q(\alpha,\beta,\gamma,f(z)).\end{aligned}$$

Theorem 2 *Let, for* $\alpha \geq (1-\gamma)$, $f \in M_q(\alpha,\beta,\gamma)$. *Then*

$$\int_{\theta_1}^{\theta_2} Re\{\frac{zF'(z)}{F(z)}\}d\theta > -\pi,$$

where $F = f \star h_q$.

Proof. Let

$$\frac{zF'(z)}{F(z)} = p(z) = (1-\eta)p_1(z)+\eta. \tag{14}$$

From (14), we get

$$\frac{(zF'(z))'}{F'(z)} = p(z)+\frac{zp'(z)}{p(z)}. \tag{15}$$

Using (14) and (15), we have

$$J_q(\alpha,\beta,\gamma,f(z)) = \frac{1-\alpha}{1-\beta}p(z)-\frac{\beta(1-\alpha)}{1-\beta}+\frac{\alpha}{1-\gamma}p(z)+\frac{\alpha}{1-\gamma}[\frac{zp'(z)}{p(z)}-\gamma]. \tag{16}$$

In (14), choosing

$$\eta = \frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\gamma)+\alpha(1-\beta)}, \tag{17}$$

we have from (16),

$$J_q(\alpha,\beta,\gamma,f(z)) = p_1(z)+\frac{\frac{\alpha}{1-\gamma}zp_1'(z)}{p_1(z)+(\frac{\alpha}{1-\gamma})(\frac{\eta}{1-\eta})}. \tag{18}$$

Now, using Lemma 1, Definition 1 and from (14), (16), (17), (18), we obtain

$$\int_{\theta_1}^{\theta_2} Re\{\frac{zF'(z)}{F(z)}\}d\theta > -(1-\eta\pi) > -\pi.$$

This completes the proof.

Theorem 3 *For* $\alpha > \alpha_1 \geq 0, \quad M_q((\alpha,\beta,\gamma,f(z)) \subset M_q((\alpha_1,\beta,\gamma,f(z))$.

Proof. For $F = f \star h_q$, let $\quad f \in M_q(\alpha,\beta,\gamma,f(z))$. Then

$$\begin{aligned} J_q(\alpha,\beta,\gamma,f(z)) &= (\frac{(1-\alpha_1}{1-\beta})\{\frac{zF'(z)}{F(z)} - \beta\} + (\frac{\alpha_1}{1-\gamma})\{\frac{(zF'(z))'}{F'(z)} - \gamma\} \\ &= (1-\frac{\alpha_1}{\alpha})J_q(\alpha,\beta,\gamma,f(z)) + \frac{\alpha_1}{\alpha}J_q(\alpha,\beta,\gamma,f(z)) \\ &= (1-\frac{\alpha_1}{\alpha})(\frac{1}{1-\beta})\{\frac{zF'(z)}{F(z)} - \beta\} \\ &\quad + \frac{\alpha_1}{\alpha}\left[\frac{1-\alpha}{1-\beta}\{\frac{zF'(z)}{F(z)} - \beta\} + \frac{\alpha}{1-\gamma}\{\frac{(zF'(z))'}{F'(z)} - \gamma\}\right]. \end{aligned}$$

From Theorem 2, $\quad f \in M_q(\alpha,\beta,\gamma,f(z))$ and $\alpha_1 < \alpha$, we obtain

$$\int_{\theta_1}^{\theta_2} Re\{J_q(\alpha_1,\beta,\gamma,f(z))\}d\theta > -\pi.$$

This implies $\quad f \in M_q(\alpha_1,\beta,\gamma,f(z))$ and the proof is complete.

We shall study Bernardi integral operator [1, 11] for the class $\quad M_q(\alpha,\beta,\gamma)$.

Theorem 4 *Let*

$$F_{a,q}(z) = (F_a \star h_q)(z), \quad f_q(z) = (f \star h_q)(z), \quad a \geq 0.$$

For $f \in M_q(\alpha,\beta,\gamma), \quad \alpha \geq (1-\gamma)$, *define*

$$F_{a,q}(z) = \frac{(a+1)}{z^a}\int_0^z t^{a-1}f_q(t)dt. \tag{19}$$

Then $\quad F_{a,q} \in M_q(0,0,\gamma) \quad$ *in* $\quad E$.

Proof. Let p be analytic in E with $p(0) = 1$. Let

$$\frac{zF'_{a,q}(z)}{F_{a,q}} = p(z). \tag{20}$$

From (19), we get

$$(a+1)f_q(z) = aF_{a,q}(z) + zF'_{a,q}(z), \tag{21}$$

where

$$F'_{a,q}(z) = zD_qF_{a,q}(z).$$

Logarithmic differentiation of (21) gives us

$$\frac{zf_q'(z)}{f_q(z)} = \frac{azF_{a,q}'(z) + (zF_{a,q}')'}{aF_{a,q}(z) + zF_{a,q}'(z)}, \tag{22}$$

where

$$(zF_{a,q}'(z))' = D_q(zD_qF_{a,q}(z)).$$

Using (20) in (22), we obtain

$$\begin{aligned} \frac{zf_q'(z)}{f_q(z)} &= \frac{ap(z) + \{(p(z))^2 + zp'(z)\}}{a + p(z)} \\ &= p(z) + \frac{zp'(z)}{a + p(z)} = h(z) \end{aligned} \tag{23}$$

Logarithmic differentiation of (23) gives us

$$\frac{(zf_q'(z))'}{f_q'(z)} = h(z) + \frac{zh'(z)}{h(z)}. \tag{24}$$

From (23) and (24), we have

$$\begin{aligned} J_q(\alpha, \beta, \gamma, f_q(z)) &= (\frac{1-\alpha}{1-\beta})\{h(z) - \beta\} + \frac{\alpha}{1-\gamma}\{h(z) + \frac{zh'(z)}{h(z)} - \gamma\} \\ &= \left\{\left[\frac{(1-\alpha)(1-\gamma) + \alpha(1-\beta)}{(1-\alpha)(1-\gamma)}\right]h(z)\right. \\ &\quad \left. -\frac{\beta(1-\alpha)(1-\gamma) + \alpha\gamma(1-\beta)}{(1-\alpha)(1-\gamma} + \frac{\alpha}{1-\gamma}\frac{zh'(z)}{h(z)}\right\}. \end{aligned} \tag{25}$$

Choosing

$$\sigma = 1 - \frac{(1-\alpha)(1-\gamma)}{(1-\alpha)(1-\gamma) + \alpha(1-\beta)}$$

in (25), we have

$$J_q(\alpha, \beta, \gamma, f_q(z)) = h(z) + \frac{\frac{\alpha}{1-\gamma}zh'(z)}{h(z) + \frac{\alpha\sigma}{(1-\gamma)(1-\sigma)}}. \tag{26}$$

Thus, from (26) and Definition 1, it follows that

$$\int_{\theta_1}^{\theta_2} Re\left\{h(z) + \frac{\frac{\alpha}{1-\gamma}zh'(z)}{h(z) + \frac{\alpha\sigma}{(1-\gamma)(1-\sigma)}}\right\}d\theta > -\pi.$$

With $\frac{\alpha}{1-\gamma} \geq 1$, $\frac{\sigma}{1-\sigma} \geq 0$, we use Lemma 1, to have

$$\int_{\theta_1}^{\theta_2} Re\{h(z)\}d\theta > -\pi,$$

and using (23), it implies

$$\int_{\theta_1}^{\theta_2} Re\{p(z) + \frac{zp'(z)}{p(z)+a}\}d\theta > -\pi.$$

Hence, applying Lemma 1, it follows that

$$\int_{\theta_1}^{\theta_2} Re\{(pz)\}d\theta > -\pi.$$

That is $F_{a,q} \in M_q(0,0,\gamma)$.

3 Conclusion

The class $M_q(\alpha,\beta,\gamma)$ of anlatic functions has been introduced by using concepts of convolution and q-Koebe function. These functions are related to generalized bounded Mocanu variation and higher order close-to-convexity. Several mapping properties together with interesting consequences have been investigated.
Some growth rate problems with reference to coefficients, Hankel determinant and study of certain linear operators may be interest to researchers in this field. Also, with suitable choices of parameters involved, some generalizations may be explored.

Availability of data and materials: Not applicable.

Competing interests: The authors declare that they have no competing interests.

Authors contributions: All authors contributed equally. All authors read and approved the final manuscript.

References

[1] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., vol. 135, 1969, 429-446.

[2] A. W. Goodman, *On close-to-convex functions of higher order*, Annal. Univer. Sci. Budapest Eotvos Sect. Math., vol. 15, 1972, 17-30.

[3] A. W. Goodman, *Univalent Functions*, Polygonal Publishing House, Washington, New Jersey, USA, vol. I, II, 1983.

[4] F. H. Jackson, *On q-functions and certain differential operators*, Trans. R. Soc. Edinb., vol. 46, 1908, 253-281.

[5] F. H. Jackson, *On q-definite integrals*, Q. J. Pure Appl. Math., vol. 41, 1910, 193-203.

[6] W. Kaplan, *Close-to-convex Schlicht functions*, Mich. Math. J., vol. 1, 1952, 169-185.

[7] M. M. Miller, P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Marcel Dekker, Inc. New York, Basel, 2000.

[8] P. T. Mocanu, *Une propriete de la representation conforme*, Math(Cluj), vol. 11, 1969, 27-133.

[9] K. I. Noor, *Higher order close-to-convex functions*, Math. Japonica, vol. 37, 1993, 1-8.

[10] K. I. Noor, *On a generalization of close-to-convexity of complex order*, Tamkang J. Math., vol. 27, 1998, 73-87. .

[11] K. I. Noor, S. Riaz, M. A. Noor, *On q-Bernardi integral operator*, TWMS J. Pure Appl., vol. 8, 2017, 3-11.

[12] K. I. Noor, M. A. Noor, *Linear combination of generalized q-starlike functions*, Appl. Math. Inform. Sci., vol. 11, no. 3, 2017, 745-748.

[13] K. I. Noor, S. Riaz, *Generalized q-starlike functions*, Stud. Scient. Math. Hungarica, vol. 54, no. 4, 2017, 509-522.

[14] R. Parvatham, S. Radha, *On certian classes of analytic functions*, Annal. Polon. Math., vol. 49, 1988, 31-34.

[15] K. Piejeko, J. Sokol, *On convolution and q-calculus*, Bolet. dela Sociedad Matematica Mexcana, vol. 26, 2020, 349-359.

[16] Ch. Pommeranke, *Univalent Functions*, Vandenhocck and Ruprecht, Gottingen, 1975.

[17] S. Ruscheweyh, T. Sheil-Small, *Hadamard product of Schlicht functions and the Polya-Schoenberg conjecture*, Comment. Math. Helv., vol. 48, 1973, 119-135.

[18] H. E. O. Ucar, *Coefficient inequality for q-starlike functions*, Appl. Math. Comput., vol. 276, 2016, 122-126.

Khalida Inayat Noor
COMSATS University Islamabad
Department of Mathematics
Park Road, Islamabad, Pakistan
e-mail: khalidan@gmail.com

Muhammad Aslam Noor
COMSATS University Islamabad
Department of Mathematics
Park Road, Islamabad, Pakistan
e-mail: noormaslam@gmail.com