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Some bounds for the elliptical integral of the first kind [1]

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Dedicated to Professor Ioan Raşa on the occasion of his 70^{th} birthday

Abstract

In this article we present a method to get upper bound for some integrals. The method is in connection with on a generalized trapezoidal quadrature formula. Applications for upper bounds of the elliptical integral of the first kind are presented. We compare these bounds with some previous results from the literature.

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1 Introduction

The elliptical integral of the first kind is defined by

$$K(r) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - r^2 \sin^2 t}} dt, \ r \in (0, 1). \tag{1}$$

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It has many applications in mathematics, physics and engineering. This attracted a lot of researcher to get bounds for the integral in (1). We present some of these bounds. We use the notations:

$$r' = \sqrt{1 - r^2}$$

and

$$\tanh^{-1}(r) = \frac{1}{2} \ln \frac{1+r}{1-r}.$$

The following inequalities were obtained by:

- Qiu, Vamanamurthy in [6] and Alzer in [1]

$$\left(1 + \left(\frac{\pi}{\ln 16} - 1\right) r'^2\right) \ln \frac{4}{r'} < K(r) < \left(1 + \frac{1}{4} r'^2\right) \ln \frac{4}{r'}; \tag{2}$$

- Anderson, Vamanamurthy, Vuorinen in [3] and Alzer, Qiu in [2]

$$\frac{\pi}{2} \left(\frac{\tanh^{-1} r}{r}\right)^{3/4} < K(r) < \frac{\pi}{2} \left(\frac{\tanh^{-1} r}{r}\right); \tag{3}$$

- Yang, Qian, Chu, Zhang in [8]

$$\frac{1 + (6p-7)r'}{p + (5p-6)r'} \frac{\pi}{2} \frac{\tanh^{-1} r}{r} < K(r) < \frac{1 + (6q-7)r'}{q + (5q-6)r'} \frac{\pi}{2} \frac{\tanh^{-1} r}{r} \tag{4}$$

if and only if $p \geq \pi/2$ and $1 < q \leq 89/69$.

In this paper we present a method to get upper bounds for the elliptical integral of the first kind. This method is related to a generalized trapezoidal quadrature formula. The bounds are in the simple form

$$K(r) \leq A + \frac{B}{r'},$$

where A,$B \in \mathbb{R}$. We show that the obtained bounds are smaller than the upper bounds in (2)-(4) for some values of r.

2 Generalized trapezoidal quadrature formula and related inequalities

Let $u : [a, b] \to [a, b]$ be a continuous differentiable and strictly increasing function having the properties: $u(a) = a$, $u(b) = b$. We assume that $\alpha + \beta u$, with $\alpha^2 + \beta^2 \neq 0$, has at most one root, counting the multiplicities.

We note by e_i, $i = 0, 1, ...$ the monomial functions, i.e. $e_i : [a, b] \to \mathbb{R}$, $e_i(x) = x^i$.

We consider the Bernstein type operator $B_u : C[a, b] \to C[a, b]$ given by

$$B_u(f)(x) = \frac{u(x) - b}{a - b} f(a) + \frac{u(x) - a}{b - a} f(b), \ x \in [a, b]$$

The operator B_u is linear and positive. It also preserves the constants and the function u.

Remark 1 *The operator B_u can be obtained from the operator defined in [4] for $n = 1$ and by extending it to the interval $[a, b]$. If $u = e_1$, then we get the first degree classical Bernstein polynomial.*

We remaind that a function $f \in C[a, b]$ is said to be convex with respect to the function u if

$$\begin{vmatrix} 1 & 1 & 1 \\ u(x_0) & u(x_1) & u(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0,\ a \leq x_0 < x_1 < x_2 \leq b$$

or equivalent to $f \circ u^{-1}$ is convex (in classical sense). As the function u is strictly increasing we get the following condition for the convexity of the function $f \in C^2[a, b]$ with respect to the function u:

$$f''(x)u'(x) - f'(x)u''(x) \geq 0, x \in [a, b].$$

We note that the convexity in the classical sense is obtained for $u = e_1$. For more about generalized convexity see [5].

From [9, Theorem 2] it follows that if the function f is convex with respect to the function u, then the following inequality holds:

$$f(x) \leq B_u(f)(x),\ x \in [a, b].$$

By integration on the interval $[a, b]$ we ge the upper bound

$$\int_a^b f(x)dx \leq A_u f(a) + B_u f(b), \tag{5}$$

where

$$A_u = \frac{1}{a-b}\int_a^b (u(x) - b)dx$$

and

$$B_u = \frac{1}{b-a}\int_a^b (u(x) - a)dx.$$

Definition 1 *The formula*

$$\int_a^b f(x)dx \approx A_u f(a) + B_u f(b)$$

will be called the generalized trapezoidal quadrature.

Remark 2 *For $u = e_1$ we get the classical trapezoidal formula (see for example [7])*

$$\int_a^b f(x)dx\backslash \approx \frac{b-a}{2}\left(f(a) + f(b)\right).$$

If $f \circ u^{-1} \in C^2[a,b]$, then the remainder (the error) can be represented in the integral form

$$R_u(f) = \frac{1}{2}\int_a^b \left(\int_a^b K(u(x),t)dx\right)\left(f\circ u^{-1}\right)''(t)dt,$$

where

$$K(x,t) = \begin{cases} -\frac{(x-a)(b-t)}{b-a}, & x \le t \\ -\frac{(t-a)(b-x)}{b-a}, & x \le t \end{cases}$$

3 Applications to the elliptical integral of the first kind

We take $[a,b] = \left[0, \frac{\pi}{2}\right]$ and $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{\sqrt{1 - r^2 \sin^2 x}},\ x \in \left[0, \frac{\pi}{2}\right]$$

for every $r \in (0,1)$. Let $u : \left[0, \frac{\pi}{2}\right] \to \left[0, \frac{\pi}{2}\right]$ be a function having the properties from the previous section.

From the inequality (5) we have: if the function f is convex with respect to the function u we get

$$K(r) \le A_u + \frac{B_u}{r'}, \tag{6}$$

where

$$A_u = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(u(x) - \frac{\pi}{2}\right)dx$$

and

$$B_u = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} u(x)dx.$$

Next, we present examples with different choices for the function u.

Example 1 *For the function*

$$u(x) = \frac{\pi}{2}\sin x,\ x \in \left[0, \frac{\pi}{2}\right]$$

we have

$$f''(x)u_1'(x) - f'(x)u_1''(x) = \frac{2\sqrt{2}\pi r^2 \cos^3 x(1 + 2r^2 \sin^2 x)}{(2 - r^2 + r^2\cos 2x)^{5/2}} \ge 0,\ x \in \left[0, \frac{\pi}{2}\right].$$

Thus, the function f is convex function with respect to the function u. Therefore, the inequality (6) holds, i.e.

$$K(r) \le \frac{\pi}{2} - 1 + \frac{1}{r'}. \tag{7}$$

We compare the inequality (7) with the upper bounds from (2), (3) and (4). In Figures 1.a, 1.b and 1.c respectively we give the plots of the functions g_i, $i = 1, 2, 3$ *on some subintervals from* $[0.01, 0.99]$, *where*

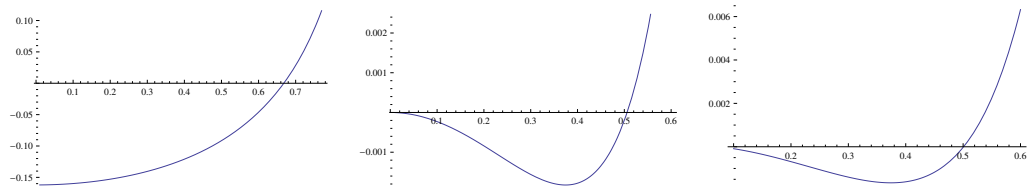
$$g_1(r) = \frac{\pi}{2} - 1 - \frac{1}{\sqrt{1-r^2}} - \left(\frac{5}{4} - \frac{1}{4}r^2\right) \ln \frac{4}{\sqrt{1-r^2}},$$

$$g_2(r) = \frac{\pi}{2} - 1 - \frac{1}{\sqrt{1-r^2}} - \frac{\pi}{2}\left(\frac{\tanh^{-1} r}{r}\right)$$

and

$$g_3(r) = \frac{\pi}{2} - 1 - \frac{1}{\sqrt{1-r^2}} - \frac{1+(6q-7)\sqrt{1-r^2}}{q+(5q-6)\sqrt{1-r^2}} \frac{\pi}{2} \frac{\tanh^{-1} r}{r}, \ q = 1.0001.$$

We observe that the bound from (7) is smaller than the upper bounds from (2), (3) and (4).



a b c

Figure 1: The functions g_i for $i = 1, 2, 3$.

Example 2 *Let* $\alpha > 0$. *We choose the function*

$$u(x) = \frac{\pi}{2(1+\alpha)} \left(\sin^2 x + \alpha \sin x\right), \ x \in \left[0, \frac{\pi}{2}\right].$$

The inequality

$$f''(x)u_1'(x) - f'(x)u_1''(x) = \frac{2\sqrt{2}\pi r^2 \cos^3 x(\alpha + 2\alpha r^2 \sin^2 x + 6r^2 \sin^3 x)}{(2 - r^2 + r^2 \cos 2x)^{5/2}} \geq 0$$

holds for every $x \in \left[0, \frac{\pi}{2}\right]$. *It follows that the function* f *is convex function with respect to the function* u. *We get the upper bound*

$$K(r) \leq \frac{\pi}{4} - \alpha + \left(\frac{\pi}{4} + \alpha\right) \frac{1}{r'}. \tag{8}$$

Taking limit $\alpha \to 0$ *in (8) we get*

$$K(r) \leq \frac{\pi}{4}\left(1 + \frac{1}{r'}\right).$$

Let the functions

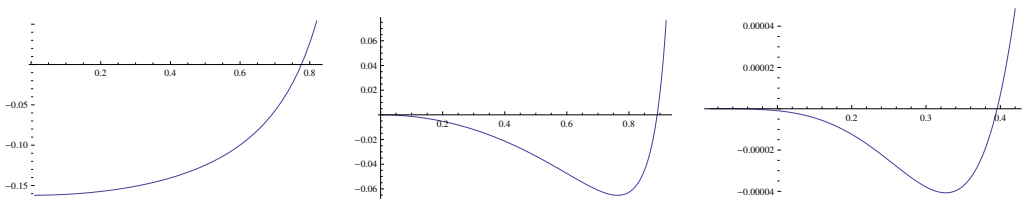
$$h_1(r) = \frac{\pi}{4}\left(1 + \frac{1}{\sqrt{1-r^2}}\right) - \left(\frac{5}{4} - \frac{1}{4}r^2\right)\ln\frac{4}{\sqrt{1-r^2}},$$

$$h_2(r) = \frac{\pi}{4}\left(1 + \frac{1}{\sqrt{1-r^2}}\right) - \frac{\pi}{2}\left(\frac{\tanh^{-1} r}{r}\right),$$

and

$$h_3(r) = \frac{\pi}{4}\left(1 + \frac{1}{\sqrt{1-r^2}}\right) - \frac{1 + (6q-7)\sqrt{1-r^2}}{q + (5q-6)\sqrt{1-r^2}}\frac{\pi}{2}\frac{\tanh^{-1} r}{r}, q = 1.09.$$

In Figures 2.a, 2.b and 2.c respectively we plotted the functions h_i, $i = 1, 2, 3$ *on some subintervals included in* $[0.01, 0.99]$. *We observe that for some values or* r *the inequality (8) improve the upper bounds in (2), (3) and (4).*



a b c

Figure 2: The functions g_i for $i = 1, 2, 3$.

Remark 3 *The approximation*

$$K(r) \approx \frac{\pi}{4}\left(1 + \frac{1}{r'}\right).$$

can also be obtained using the classical trapezoidal rule.

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