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Coincidence and Common Fixed Point Theorems for Hybrid Mappings[1]

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Abstract

In this paper, we prove a new coincidence and common fixed point theorem of hybrid mappings using the C-class function and T-weak commutativity. Finally, we give an example to illustrate our result.

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1 Introduction and Preliminaries

The study of common fixed points of mappings satisfying hybrid contraction conditions has witnessed a dramatic growth of interest in the last few decades. Apparently, it is absolutely crucial applications in disparate angles involving functional inclusions, optimization theory, fractal graphics, science and engineering and discrete dynamics. Because of that, authors conducted some fixed point results for mappings satisfying different contractive conditions (see [1], [2], [6], [8], [10], [12], [13], [14], [17], [20], [22], and [23]).

Our results unify, extend, and generalize various known comparable results in the literature.

We start with the following definitions and results which are frequently used in the main results.

Let (X, d) be a metric space. For $x \in X$ and $A \subset X$, we denote

$$D(x, A) = \inf\{d(x, y), y \in A\}.$$

Let $CB(X)$ be the set of all nonempty closed and bounded subsets of X.

The Hausdorff metric on $CB(X)$ is defined by

$$H(A, B) = max\left\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(A, b)\right\},$$

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for every $A, B \in CB(X)$.

It is well known that $(CB(X), H)$ is a metric space and if (X, d) is complete, then $(CB(X), H)$ is also complete.

Let $f : X \to X$ be a single-valued mapping and $S : X \to CB(X)$ be a multi-valued mapping.

(**i**) A point $x \in X$ is a fixed point of f (resp. S) if $fx = x$ (resp. $x \in Sx$).

(**ii**) A point $x \in X$ is a coincidence point of f and S if $fx \in Sx$.

We denote by $C(f, S)$ the set of all coincidence points of f and S.

(**iii**) A point $x \in X$ is a common fixed point of f and S if $x = fx \in Sx$.

We denote by $F(f, S)$ the set of all common fixed points of f and S.

Lemma 1 *[15] If $A, B \in CB(X)$ and $k > 1$, then for each $a \in A$, there exists $b \in B$ such that $d(a, b) \leq kH(A, B)$.*

Definition 1 *[7] f and S are said to be commuting in X if for all $x \in X$, $fSx \subset Sfx$.*

Definition 2 *[19] f and S are said to be weakly commuting on X if for all $x \in X$, $fSx \in CB(X)$ and*

$$H(fSx, Sfx) \leq D(fx, Sx).$$

Definition 3 *[16] f and S are said to be R-weakly commuting at $x \in X$, if $fSx \in CB(X)$ and there exists an $R > 0$ such that*

$$H(fSx, Sfx) \leq RD(fx, Sx). \tag{1}$$

Remark 1 *[9] Commuting implies weakly commuting, but the converse is not true in general*

We defined that f and S are said to be pointwise R-weakly commuting on X if for all $x \in X$, $fSx \in CB(X)$ and (1) holds for some $R > 0$.

Definition 4 *[21] 1) f and S are said to be (IS)-commuting at $x \in X$ if $fSx \subset Sfx$.*

2) A pointwise R-weakly commuting hybrid pair is not weakly compatible in general

3) IS-commutativity of f and S at a coincidence point is more general than their weak compatibility at the same point.

4) A pointwise R-weak commutativity at a coincidence point is equivalent to (IS) commutativity at this point.

Definition 5 *[10] 1) f is S-weakly commuting at $x \in X$ if $ffx \in Sfx$.*

2) For a hybrid pair (f, S), (IT) commuting at coincidence points implies that f is S-weakly commuting at these points.

Lemma 2 *[3] a) If f is S-weakly commuting at $x \in X$, then $fx \in C(f, S)$.*

b) If f is S-weakly commuting at $x \in X$ and $fx = ffx$, then fx is a common fixed point of f and S.

Lemma 3 *[18] Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_n, y_{n+1})$ is nonincreasing and*

$$\lim_{n \to +\infty} d(y_n, y_{n+1}) = 0.$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to +\infty$

$$d(x_{2n_k}, x_{2m_k}), d(x_{2n_k+1}, x_{2m_k}), d(x_{2n_k}, x_{2m_k-1}, d(x_{2n_k+1}, x_{2m_k-1}). \tag{2}$$

A. H. Ansari [4] introduced the concept of a C-class functions which covers a large class of contractive conditions.

Definition 6 *[4] A continuous function $F : [0,+\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s,t \in [0,+\infty)$; the following conditions hold*

*c*1) $F(s,t) \leq s$,

*c*2) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$.

An extra condition on F that $F(0,0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C-functions.

Example 1 *The following examples shows that the class C is nonempty:*

1. $F(s,t) = s - t$,
2. $F(s,t) = ms$, *for some* $m \in (0,1)$,
3. $F(s,t) = \frac{s}{(1+t)^r}$ *for some* $r \in (0,1)$,
4. $F(s,t) = \frac{\log(t+a^s)}{(1+t)}$, *for some* $a > 1$,

Let Φ_u denote the class of the functions $\varphi : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

a) φ is continuous,

b) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

Definition 7 *[11] A function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:*

i) ψ is non-decreasing and continuous,

ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 8 *[5] A tripled (ψ, φ, F) where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x, y \in [0,+\infty)$*

$$x \leq y \Rightarrow F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y)).$$

Example 2 *Let $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$*

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

2 Main Results

In the following theorem we obtain the coincidence and common fixed point for a hybrid pair of mappings via C-class function.

Theorem 1 *Let (X,d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$ satisfying*

$$T(X) \subset f(X) \tag{3}$$

$$\psi\left(cH(Tx,Ty)\right) \leq F\left(\psi\left(M\left(x,y\right)\right), \varphi\left(M\left(x,y\right)\right)\right), \tag{4}$$

where $c \geq 1$ and

$$M\left(x,y\right) = \max\left\{\left(d(fx,fy)\right), D(fx,Tx), D(fy,Ty), D(fx,Ty), D(Tx,fy)\right\}.$$

Suppose that $f(X)$ is complete. Then, there exists $p \in X$ such that $fp \in Tp$.

Further, if f is T-weakly commuting at the coincidence point, then z is a common fixed point of f and T.

Proof. First, assume that there exists $p \in X$ such that $D(fp, Tp) = 0$ which implies that $fp \in Tq$.

Now, assume that $D(fx, Ty) + D(Tx, fy) \neq 0$ for all $x, y \in X$.

Let $x_0 \in X$ be an arbitrary point. By (3) and Lemma 3, we define a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} \in Tx_{2n-1}, \quad y_{2n+1} = fx_{2n+1} \in Tx_{2n}.$$

$$d(y_{2n}, y_{2n+1}) \leq kH(Tx_{2n}, Tx_{2n-1}), \quad for\ n = 1, 2, ... \tag{5}$$

$$d(Tx_{2n-1}, y_{2n+1}) \leq kH(Tx_{2n}, Tx_{2n-1}), \quad for\ n = 1, 2, ... \tag{6}$$

Using (4), we have

$$\begin{aligned}
& \psi\left(cH(Tx_{2n}, Tx_{2n-1})\right) \\
\leq\ & F(\psi(\max\{d(y_{2n}, y_{2n-1}), D(y_{2n}, y_{2n+1}), D(y_{2n-1}, y_{2n}), 0, D(y_{2n+1}, y_{2n-1})\}), \\
& \varphi\left(\max\{d(y_{2n}, y_{2n-1}), D(y_{2n}, y_{2n+1}), D(y_{2n-1}, y_{2n}), 0, D(y_{2n+1}, y_{2n-1})\}\right)) \\
\leq\ & F(\psi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), 0, d(y_{2n+1}, y_{2n-1})\}), \\
& \varphi\left(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), 0, d(y_{2n+1}, y_{2n-1})\}\right)) \\
\leq\ & \psi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), 0, d(y_{2n+1}, y_{2n-1})\}).
\end{aligned}$$

By the properties of F and ψ, we have also

$$\begin{aligned}
\psi\left(cH(Tx_{2n}, Tx_{2n-1})\right) &\leq \psi\left(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), 0, d(y_{2n+1}, y_{2n-1})\}\right) \\
&\leq \psi(\{\max d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \\
&\quad 0, d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n+1})\}).
\end{aligned}$$

Using (5), we obtain

$$d(y_{2n}, y_{2n+1}) \leq d(y_{2n}, y_{2n-1}).$$

Then

$$d(y_n, y_{n+1}) \leq d(y_n, y_{n-1}).$$

Also, from (4), we get

$$\begin{aligned}
\psi\left(cH(Tx_{2n}, Tx_{2n+1})\right) \leq\ & F(\psi(\max\{fx_{2n}, fx_{2n+1}, D(fx_{2n}, Tx_{2n}), D(fx_{2n+1}, Tx_{2n+1}), \\
& D(fx_{2n}, Tx_{2n+1}), D(Tx_{2n}, fx_{2n+1})\}); \\
& \varphi(\max\{fx_{2n}, fx_{2n+1}, D(fx_{2n}, Tx_{2n}), D(fx_{2n+1}, Tx_{2n+1}), \\
& D(fx_{2n}, Tx_{2n+1}), D(Tx_{2n}, fx_{2n+1})\}).
\end{aligned}$$

So

$$\begin{aligned}
\psi\left(cH(Tx_{2n}, Tx_{2n+1})\right) \leq\ & F(\psi(\max\{d\left(y_{2n}, y_{2n+1}\right), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\
& d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0\}), \\
& \varphi(\max\{H(Tx_{2n}, Tx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\
& d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0\})).
\end{aligned} \tag{7}$$

By (6), we obtain

$$d(y_{2n+1}, y_{2n+2}) \leq d\left(y_{2n}, y_{2n+1}\right)$$

and hence

$$d(y_n, y_{n+1}) \leq d\left(y_n, y_{n-1}\right).$$

Then, the sequence $\{d(y_n, y_{n+1})\}$ is bounded below and non-increasing, hence there exist $r \geq 0$ such that

$$\lim_{n \to +\infty} d(y_n, y_{n+1}) = 0.$$

By taking $n \to +\infty$ in (7) and using continuity of φ and ψ, we deduce that

$$\psi(r) \leq F(\psi(r), \varphi(r)) \leq \psi(r).$$

So, $\varphi(r) = 0$ or $\psi(r) = 0$, It follows that $r = 0$.

Now, we prove that the sequence $\{y_{2n}\}$ is a Cauchy in the metric space (X, d). Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence in (X, d), then by lemma 3, there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ such that all sequences in (2) are tend to $\varepsilon > 0$, when $k \to +\infty$. Now, for $x = x_{2n(k)}$ and $y = x_{2m(k)+1}$ in equation (4), we get

$$\begin{aligned}
\psi\left(cH(Tx_{2n(k)}, Tx_{2m(k)+1})\right) &\leq F\left(\psi\left(M\left(x_{2n(k)}, x_{2m(k)+1}\right)\right), \varphi M\left(x_{2n(k)}, x_{2m(k)+1}\right)\right) \\
&\leq F(\psi(\max\{d(fx_{2n(k)}, fx_{2m(k)+1}), D(fx_{2n(k)}, Tx_{2n(k)}), \\
&\quad D(fx_{2m(k)+1}, Tx_{2m(k)+1}), D(fx_{2n(k)}, Tx_{2m(k)+1}), \\
&\quad D(Tx_{2n(k)}, fx_{2m(k)+1})\}), \\
&\quad \varphi(\max\{d(fx_{2n(k)}, fx_{2m(k)+1}), D(fx_{2n(k)}, Tx_{2n(k)}), \\
&\quad D(fx_{2m(k)+1}, Tx_{2m(k)+1}), \\
&\quad D(fx_{2n(k)}, Tx_{2m(k)+1}), D(Tx_{2n(k)}, fx_{2m(k)+1})\} \\
&\leq F(\psi(\max\{d(y_{2n(k)-1}, y_{2m(k)}), D(y_{2n(k)-1}, y_{2n(k)}), \\
&\quad D(y_{2m(k)}, y_{2m(k)+1}), \\
&\quad D(y_{2n(k)-1}, y_{2m(k)+1}), D(y_{2n(k)}, y_{2m(k)})\}) \\
&\quad \varphi(\{\max d(y_{2n(k)-1}, y_{2m(k)}), D(y_{2n(k)-1}, y_{2n(k)}), \\
&\quad D(y_{2m(k)}, y_{2m(k)+1}), \\
&\quad D(y_{2n(k)-1}, y_{2m(k)+1}), D(y_{2n(k)}, y_{2m(k)})\}). \qquad (8)
\end{aligned}$$

Therefore, taking $k \to +\infty$ in inequality (8), we get

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)) \leq \psi(\varepsilon).$$

So, $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, hence we get $\varepsilon = 0$ which contradicts being $\varepsilon > 0$. Thus $\{y_{2n}\}$ is a Cauchy sequence in (X, d), hence by (2), we deduce that the sequence $\{y_n\}$ is Cauchy sequence in X. As $f(X)$ is complete, it converges to $z \in f(X)$ and so there exists $p \in X$ such that $z = fp$.

From (4), we obtain

$$\begin{aligned}
\psi(cH(Tp, Tx_{2n-1})) &\leq F(\psi(\max\{d(fp, fx_{2n-1}), D(fp, Tp), D(fx_{2n-1}, Tx_{2n-1}), \\
&\quad D(fp, Tx_{2n-1}), D(Tp, fx_{2n-1})\}) \\
&\quad \varphi(\max\{d(fp, fx_{2n-1}), D(fp, Tp), D(fx_{2n-1}, Tx_{2n-1}), \\
&\quad D(fp, Tx_{2n-1}), D(Tp, fx_{2n-1})\}). \\
\psi(cD(Tp, y_{2n})) &\leq F((\psi \max\{d(fp, y_{2n-1}), D(fp, Tp), d(y_{2n-1}, y_{2n}), \\
&\quad d(fp, y_{2n}), D(Tp, y_{2n-1})\}), \\
&\quad \varphi(\{\max d(fp, y_{2n-1}), D(fp, Tp), \\
&\quad d(y_{2n-1}, y_{2n}), d(fp, y_{2n}), D(Tp, y_{2n-1})\})
\end{aligned}$$

Letting n tend to infinity, we get

$$\begin{aligned}
\psi(cD(Tp, fp)) &\leq F(\psi(\max\{0, D(fp, Tp), 0, 0, D(Tp, fp)\}) \\
&\quad \varphi(\max\{0, D(fp, Tp), 0, 0, D(Tp, fp)\}).
\end{aligned}$$

Thus, $D(fp, Tp) = 0$, with implied $fp \in Tp$.

Similarly, as $T(X) \subset f(X)$, there exists $q \in X$ such that $z = fp$.

Since f is T-weakly commuting at coincidence point, it follows that z is a common fixed point of f and T.

Now, we give an example to support Theorem 1.

Example 3 *Let* $(X,d)=([1,+\infty),|.|)$, $fx=x^2$ *and* $Tx=[1,x+3]$ *for all* $x\in X$. *It is easy to verify that for all* $x,y\in X$

Define $\psi,\varphi:[0,+\infty)$ *by* $\psi(x)=3\frac{x}{4},\varphi(x)=2x$ *and* $F(s,t)=\frac{1}{2}s$.

Then, it is easy to verify that the mappings satisfy the conditions of Theorems

$$\begin{aligned} T(X) &\subset f(X).\\ d(fx,fy) &= \left|x^2-y^2\right|.\\ D(fx,Tx) &= \inf(d(x^2,[2,x+3]).\\ H(Tx,Ty) &= |x-y|\leq\frac{1}{2}\left|x^2-y^2\right|=d(fx,fy). \end{aligned}$$

Hence we deduce that

$$\psi(cH(Tx,Ty))\leq F(\psi(M(x,y)),\varphi(M(x,y))),$$

where

$$M(x,y)=\max\{(d(fx,fy)),D(fx,Tx),D(fy,Ty),D(fx,Ty),D(Tx,fy)\}.$$

Therefore, we get that

$$\begin{aligned} \frac{3c}{4}H(Tx,Ty) &\leq \frac{3}{2.4}M(x,y)\\ &\leq \frac{3}{8}(\max\{(d(fx,fy)),D(fx,Tx),D(fy,Ty),D(fx,Ty),D(Tx,fy)\})\\ &\leq \frac{3}{8}d(fx,fy). \end{aligned}$$

Then

$$3cH(Tx,Ty)\leq\frac{3}{2}d(fx,fy).$$

So, if we take $c=2\geq 1$.

Hence, all conditions of Theorem 1 *are satisfied. Moreover,* 1 *is a common fixed point of* f *and* T, *which is their coincidence point also.*

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