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Fixed Point Theorems for Integral-type Weak-Contraction Mappings in Modular Metric Spaces[1]

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Abstract

In 2013, Azadifar et al. [3] established fixed point result in integral type contraction in modular metric space and 2020, Chaira et al.[11] established some extensions of Fixed Point Theorems for Weak-Contraction Mapping in Partially Ordered Modular Metric Spaces.

In this paper we have established some common fixed point results in integral type contractions in modular and convex modular metric spaces.

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1 Introduction and Preliminaries

Let (X, d) be a metric space and T be a self map on X. A point $x \in X$ is said to be a fixed point of T if $Tx = x$. A self map T on a metric space (X, d) is called a contraction if there exists $\lambda \in [0, 1)$ such that $d(Tu, Tv) \leq \lambda d(u, v)$, $\forall u, v \in X$. If a contraction principle holds on a complete metric space (X, d), then that mapping has a unique fixed point in X. In 1922, fixed point theorem was first introduced by S.Banach, which was popularly known as Banach contraction Principle(BCP).Since

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then metric spaces have been generalized and fixed point theorems have been established under different contractive conditions. Modular metric space is a nice generalization of metric space. In 1950, Nakano[10] first introduced modular spaces on Linear spaces. In 2010, Chistyakov[18] introduced modular metric spaces or parameterized metric space with the time parameter λ. In 2015, Khamsi, Kozlowski[13] established a fixed point theorem in modular function space.

Let X be a non-empty set and a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, we define as

$$\omega_\tau(x, y) = \omega(\tau, x, y), \qquad \forall \tau > 0; \qquad x, y \in X.$$

Definition 1 *{[16], [1], [13] }. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be modular metric on X if it satisfies the following conditions:*
(a) $\omega_\tau(x, y) = 0$ if and only if $x = y$, for $x, y \in X$ and $\forall\ \tau > 0$;
(b) $\omega_\tau(x, y) = \omega_\tau(y, x)$, for all $x, y \in X$ and $\forall\ \tau > 0$;
(c) $\omega_{\tau+\nu}(x, y) \leq \omega_\tau(x, z) + \omega_\nu(z, y)$, for all $x, y, z \in X$ and $\forall\ \tau, \nu > 0$.

The pair (X, ω) is called modular metric space.
Note: If the condition (m_1) holds for some $\tau > 0$, then ω is said to be regular modular space.
Note: If $\omega_{\tau+\nu}(x, y) \leq \frac{\tau}{\tau+\nu}\omega_\tau(x, z) + \frac{\nu}{\tau+\nu}\omega_\nu(z, y)$ $,\forall\ x, y, z \in X$ and $\forall\ \tau, \nu > 0$, then ω is said to be convex modular.

If $0 < \nu < \tau$, then for a modular metric ω on a set X, the function $\tau \to \omega_\tau(x, y)$ is non-increasing on $(0, \infty)$ for any $x, y \in X$,. So,

$$\omega_\tau(x, y) \leq \omega_{\tau-\nu}(x, x) + \omega_\nu(x, y) = \omega_\nu(x, y).$$

Definition 2 *[7]. Let (X, ω) be a modular metric space and fix $z_0 \in X$. Set*

$$X_\omega = X_\omega(z_0) = \{z \in X : \omega_\tau(z, z_0) = 0 \ \ as \ \ \tau \to \infty\};$$

$$X^*_\omega = X^*_\omega(z_0) = \{z \in X : \omega_\tau(z, z_0) < \infty \ \ for \ \ \tau > 0\};$$

then the two linear spaces X_ω and X^*_ω are said to be modular spaces centered at z_0.

Note: For a metric modular ω on X, if $\omega_\tau(x, y) = \omega_\nu(x, y) < \infty,\ \forall\ \tau, \nu > 0; x, y \in X$, then $\rho(x, y) = \omega_\tau(x, y)$ is a metric on X.

Definition 3 *[1]. Let ω be a modular metric on a set X. Then (i) A sequence $\{x_n\} \subset X_\omega$ is said to be ω-convergent to some $x \in X_\omega$ if and only $\lim_{n\to\infty} \omega_1(x_n, x) = 0$ and x is called the ω- limit of $\{x_n\}$.*
(ii) A sequence $\{x_n\} \subset X_\omega$ is ω-Cauchy if for $m, n \in N$ such that $\lim_{m,n\to\infty} \omega_1(x_m, x_n) = 0$.
(iii) A set $M \subset X_\omega$ is ω- closed if ω- limit of any ω-convergent sequence of M is in M.
(iv)A subset $M \subset X_\omega$ is ω- complete if any ω- Cauchy sequence in M is ω- convergent in M.

Definition 4 *([16], $\triangle_2$-condition). Let ω be a modular metric on X, then ω satisfies the $\triangle_2$- condition or simply ω is $\triangle_2$, if for a given sequence $\{x_n\} \subset X_\omega$ and for $x \in X_\omega$,*

$$\text{for some } \tau > 0, \ \lim_{n\to\infty} \omega_\tau(x_n, x) = 0 \Rightarrow \lim_{n\to\infty} \omega_\tau(x_n, x) = 0, \quad \forall \tau > 0.$$

Definition 5 *([11], $\triangle_2$-type condition). Let ω be a modular metric on X. Then ω satisfies the $\triangle_2$-type condition if there exists a constant $K > 0$ such that*

$$\omega_{\frac{\tau}{2}}(x, y) \leq K\omega_\tau(x, y), \ \forall \ x, y \in X_\omega \text{ and for any } \tau > 0.$$

Lema 1 [11]. If ω satisfies the $\triangle_2$-type condition, then ω satisfies the $\triangle_2$-condition.
Lema 2 [11]. Let $\{x_n\}$ be a sequence in X_ω and $\tau > 0$. If ω satisfies $\triangle_2$-type condition, then $\{x_n\}$ is ω- Cauchy if and only if $\lim_{m,n\to\infty} \omega_\tau(x_m, x_n) = 0$.
Lema 3 [11]. If ω satisfies the $\triangle_2$-type condition, then ω is regular.
Lema 4 [11]. Let (X, ω) be a modular metric space. If a sequence $\{x_n\} \subset X$ is not ω-Cauchy, then there exists $\epsilon > 0$ and two subsequences of integers $\{m_k\}$ and $\{n_k\}$ such that

$$\text{for } m_k > n_k \geq k, \ \ \omega_1(x_{n_k}, x_{m_k}) \geq \epsilon \ \text{ and } \ \omega_1(x_{n_k}, x_{m_k-1}) < \epsilon.$$

Lema 5 [11]. Let (X, ω) be a modular metric space and $p, q \in N^*$ such that ω satisfies $\triangle_2$- type condition. If $\{x_n\}$ is not ω-Cauchy sequence in X, then there exists $\epsilon > 0$ and two subsequences of integers $\{m_k\}$ and $\{n_k\}$ such that

$$for \ m_k > n_k \geq k, \omega_{2^p}(x_{n_k}, x_{m_k}) \geq \epsilon \ \text{ and } \ \omega_{\frac{1}{2^q}}(x_{n_k}, x_{m_k-1}) < \epsilon.$$

Lema 6 [11]. Let (X, ω) be a convex modular metric space such that ω satisfies the $\triangle_2$- condition. If a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} \omega_1(x_n, x_{n+1}) = 0$, then $\{x_n\}$ is a ω- Cauchy.

2 Main Results

From reference [5], we consider the following:
$\Xi = \{\xi | \xi : R^+ \to R^+\}$ is Lebesgue integrable, summable on each compact subset of R^+ such that (a) $\int_0^\epsilon \xi(t) > 0$ for each $\epsilon > 0$, (b) $\int_0^{a+b} \xi(t)dt \leq \int_0^a \xi(t)dt + \int_0^b \xi(t)dt$

2.1 Lema

[5]. Let $\xi \in \Xi$ and $\{s_n\}$ be a non-negative sequence with $\lim_{n\to\infty} s_n = s$, then

$$\lim_{n\to\infty} \int_0^{s_n} \xi(t)dt = \int_0^s \xi(t)dt.$$

2.2 Lema

[5]. Let $\xi \in \Xi$ and $\{s_n\}$ be a non-negative sequence. Then

$$\lim_{n\to\infty} \int_0^{s_n} \xi(t)dt = 0 \text{ if and only if } \lim_{n\to\infty} s_n = 0.$$

Also from reference [20], we consider three class of functions Θ, Ψ and Φ are as follow : (a) $\Theta = \{\theta : [0,\infty) \to [0,\infty)$ such that (i) θ is strictly increasing; (ii) $\lim_{t\to r} \theta(t) > 0$, for $r > 0$ and $\lim_{t\to 0^+} \theta(t) = 0$; (iii) $\theta(t) = 0$ if and only if $t = 0.\}$ (b) $\Psi = \{\psi : [0,\infty) \to [0,\infty)$ such that (i) ψ is non-decreasing; (ii) $\lim_{t\to r} \psi(t) > 0$, for $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0$; (iii) $\psi(t) = 0$ if and only if $t = 0.\}$
and (c) $\Phi = \{\phi : [0,+\infty) \to [0,+\infty)$ such that (i) $\lim_{t\to r} inf \phi(t) > 0$, for each $r > 0$; (ii) $\phi(t) \to 0 \Rightarrow t \to 0$; (iii) $\phi(t) = 0$ if and only $t = 0.\}$

Theorem 1 *Let (X,ω) be a convex modular metric space and F be a non-empty complete subset of X such that ω satisfies the $\triangle_2$-type condition. Let $f, g : (F,\omega) \to (F,\omega)$ be two functions satisfying the following:*

$$\int_0^{\theta_1\{\omega_1(fx,gy)\}} \xi(t)dt \leq \int_0^{\theta_2\{\Omega(x,y)\}} \xi(t)dt - \int_0^{\phi\{\Omega(x,y)\}} \xi(t)dt; \quad . \tag{1}$$

with

$$\int_0^{\theta_1(t)} \xi(z)dz - \int_0^{\theta_2(t)} \xi(z)dz + \int_0^{\phi(t)} \xi(z)dz > 0;$$

$$\forall r > 0, \ \lim_{t\to r} \int_0^{\theta_1(t)} \xi(z)dz - \lim_{t\to r} \int_0^{\theta_2(t)} \xi(z)dz + \lim_{t\to r} inf \int_0^{\phi(t)} \xi(z)dz > 0;$$

where, $\theta_1 \in \Theta, \theta_2 \in \Psi, \phi \in \Phi$ and $\xi \in \Xi$;
and

$$\Omega(x,y) = max\{\omega_1(x,fx), \omega_1(y,gy), \omega_1(x,y), \omega_2(fx,y), \omega_2(x,gy)\}. \quad . \tag{2}$$

Then f and g have a unique common fixed point in F.

Proof. Let x_0 be an arbitrary point in F. We define a sequence $\{x_n\} \subset F$ by

$$x_{2n+1} = fx_{2n} \text{ and } x_{2n+2} = gx_{2n+1}.$$

We consider two cases:
Case-I: For some $n \in N$, $x_n = x_{n+1} \Rightarrow x_{n+1} = x_{n+2}$.
Now for n is even i.e., for $n = 2m, m \in N$, we have

$$x_{2m} = x_{2m+1} \quad . \tag{3}$$

If $x_{2m+1} \neq x_{2m+2}$, then $\omega_1(x_{2m+1}, x_{2m+2}) > 0$,
Now,

$$\begin{aligned}
&\Omega(x_{2m}, x_{2m+1}) \\
&= \max\{\omega_1(x_{2m}, fx_{2m}), \omega_1(x_{2m+1}, gx_{2m+1}), \omega_1(x_{2m}, x_{2m+1}), \omega_2(fx_{2m}, x_{2m+1}), \omega_2(x_{2m}, gx_{2m+1})\} \\
&= \max\{\omega_1(x_{2m}, x_{2m+1}), \omega_1(x_{2m+1}, x_{2m+2}), \omega_1(x_{2m}, x_{2m+1}), \omega_2(x_{2m+1}, x_{2m+1}), \omega_2(x_{2m}, x_{2m+2})\} \\
&\leq \max\{\omega_1(x_{2m}, x_{2m+1}), \omega_1(x_{2m+1}, x_{2m+2}), \omega_1(x_{2m}, x_{2m+1}), \frac{\omega_1(x_{2m}, x_{2m+1}) + \omega_1(x_{2m+1}, x_{2m+2})}{2}\} \\
&= \omega_1(x_{2m+1}, x_{2m+2})
\end{aligned}$$

Hence,

$$\Omega(x_{2m}, x_{2m+1}) = \omega_1(x_{2m+1}, x_{2m+2}) \ . \tag{4}$$

Now from (2.1) we have

$$\begin{aligned}
\int_0^{\theta_1\{\omega_1(x_{2m+1}, x_{2m+2})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(fx_{2m}, gx_{2m+1})\}} \xi(t)dt \\
&\leq \int_0^{\theta_2\{\Omega(x_{2m}, x_{2m+1})\}} \xi(t)dt - \int_0^{\phi\{\Omega(x_{2m}, x_{2m+1})\}} \xi(t)dt \\
&= \int_0^{\theta_2\{\omega_1(x_{2m+1}, x_{2m+2})\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_{2m+1}, x_{2m+2})\}} \xi(t)dt \\
&< \int_0^{\theta_1\{\omega_1(x_{2m+1}, x_{2m+2})\}} \xi(t)dt, \qquad \text{which is a contradiction.}
\end{aligned}$$

Hence,

$$x_{2m} = x_{2m+1} \Rightarrow x_{2m+1} = x_{2m+2}. \ . \tag{5}$$

Now for n is odd, i.e., for $n = 2m + 1, m \in N \cup \{0\}$ we have,

$$x_{2m+1} = x_{2m+2} \ . \tag{6}$$

and

$$\Omega(x_{2m+2}, x_{2m+1}) = \omega_1(x_{2m+2}, x_{2m+3}) \ . \tag{7}$$

$$\begin{aligned}
\int_0^{\theta_1\{\omega_1(x_{2m+3}, x_{2m+2})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(fx_{2m+2}, gx_{2m+1})\}} \xi(t)dt \\
&\leq \int_0^{\theta_2\{\Omega(x_{2m+2}, x_{2m+1})\}} \xi(t)dt - \int_0^{\phi\{\Omega(x_{2m+2}, x_{2m+1})\}} \xi(t)dt \\
&= \int_0^{\theta_2\{\omega_1(x_{2m+2}, x_{2m+3})\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_{2m+2}, x_{2m+3})\}} \xi(t)dt \\
&< \int_0^{\theta_1\{\omega_1(x_{2m+2}, x_{2m+3})\}} \xi(t)dt, \qquad \text{which is a contradiction.}
\end{aligned}$$

Hence,

$$x_{2m+1} = x_{2m+2} \Rightarrow x_{2m+2} = x_{2m+3}. \ . \tag{8}$$

Hence, from (5) and (8) we have

$$x_n = x_{n+1} \Rightarrow x_{n+1} = x_{n+2}. \ . \tag{9}$$

Proceeding in this manner we have, $x_n = x_{n+1} \Rightarrow x_n = x_{n+k}$, for $k = 1, 2, 3, ...$
Therefore, $\{x_n\}$ becomes a constant sequence and hence a ω-Cauchy sequence in F.
Case-II: Assume $x_n \neq x_{n+1}, \quad \forall n \in N$.
Then for n is even i.e., for $n = 2m$, where $m \in N$, we have,
(10)

$$\begin{aligned}
&\Omega(x_{2m}, x_{2m-1}) \\
&= max\{\omega_1(x_{2m}, x_{2m+1}), \omega_1(x_{2m-1}, x_{2m+1}), \omega_1(x_{2m}, x_{2m-1}), \omega_2(x_{2m+1}, x_{2m-1}), \omega_2(x_{2m}, x_{2m})\} \\
&\leq max\{\omega_1(x_{2m}, x_{2m+1}), \omega_1(x_{2m-1}, x_{2m+1}), \omega_1(x_{2m}, x_{2m-1}), \frac{1}{2}[\omega_1(x_{2m+1}, x_{2m}) + \omega_1(x_{2m}, x_{2m-1}
\end{aligned}$$

If $\Omega(x_{2m}, x_{2m-1}) = \omega_1(x_{2m}, x_{2m+1})$, then from (2.1) we have

$$\begin{aligned}
\int_0^{\theta_1\{\omega_1(x_{2m+1}, x_{2m})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(fx_{2m}, gx_{2m-1})\}} \xi(t)dt \\
&\leq \int_0^{\theta_2\{\Omega(x_{2m}, x_{2m-1})\}} \xi(t)dt - \int_0^{\phi\{\Omega(x_{2m}, x_{2m-1})\}} \xi(t)dt \\
&= \int_0^{\theta_2\{\omega_1(x_{2m}, x_{2m+1})\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_{2m}, x_{2m+1})\}} \xi(t)dt \\
&< \int_0^{\theta_1\{\omega_1(x_{2m}, x_{2m+1})\}} \xi(t)dt, \qquad \text{which is a contradiction.}
\end{aligned}$$

Hence,

$$\Omega(x_{2m}, x_{2m-1}) = \omega_1(x_{2m}, x_{2m-1}) \ . \tag{11}$$

Now from (2.1) we have

$$\begin{aligned}
\int_0^{\theta_1\{\omega_1(x_{2m+1}, x_{2m})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(fx_{2m}, gx_{2m-1})\}} \xi(t)dt \\
&\leq \int_0^{\theta_2\{\Omega(x_{2m}, x_{2m-1})\}} \xi(t)dt - \int_0^{\phi\{\Omega(x_{2m}, x_{2m-1})\}} \xi(t)dt \\
&= \int_0^{\theta_2\{\omega_1(x_{2m}, x_{2m-1})\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_{2m}, x_{2m-1})\}} \xi(t)dt \\
&< \int_0^{\theta_1\{\omega_1(x_{2m}, x_{2m-1})\}} \xi(t)dt
\end{aligned}$$

Hence,

$$\int_0^{\theta_1\{\omega_1(x_{2m+1}, x_{2m})\}} \xi(t)dt < \int_0^{\theta_1\{\omega_1(x_{2m}, x_{2m-1})\}} \xi(t)dt \ . \tag{12}$$

Since θ_1 is strictly increasing, so we have

$$\omega_1(x_{2m+1}, x_{2m}) < \omega_1(x_{2m}, x_{2m-1}) \ . \tag{13}$$

For n is odd i.e., for $n = 2m+1, m \in N \cup \{0\}$, by similar way we have,

$$\omega_1(x_{2m+1}, x_{2m+2}) < \omega_1(x_{2m}, x_{2m+1}) \ . \tag{14}$$

From (13) and (14) we have,

$$\omega_1(x_n, x_{n+1}) < \omega_1(x_{n-1}, x_n), \quad \forall \ n = 1, 2, 3, ... \ . \tag{15}$$

Therefore, $\{\omega_1(x_n, x_{n+1})\}$ is monotone decreasing and bounded below, so, convergent.
Let $\lim\limits_{n\to\infty} \omega_1(x_n, x_{n+1}) = r$ and $\lim\limits_{n\to\infty} \int_0^{\theta_1\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt = r^*$, where $r, r^* \geq 0$.
Claim: $r = 0$.
If not then, $r > 0$. Then $\lim\limits_{m\to\infty} \omega_1(x_{2m}, x_{2m+1}) = r$ and $\lim\limits_{m\to\infty} \int_0^{\theta_1\{\omega_1(x_{2m},x_{2m+1})\}} \xi(t)dt = r^*$.
Now from (1), we have

(16)

$$\begin{aligned}\int_0^{\theta_1\{\omega_1(x_{2m+1},x_{2m})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(fx_{2m},gx_{2m-1})\}} \xi(t)dt \\ &\leq \int_0^{\theta_2\{\Omega(x_{2m},x_{2m-1})\}} \xi(t)dt - \int_0^{\phi\{\Omega(x_{2m},x_{2m-1})\}} \xi(t)dt \quad . \\ &= \int_0^{\theta_2\{\omega_1(x_{2m},x_{2m-1})\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_{2m},x_{2m-1})\}} \xi(t)dt\end{aligned}$$

Taking lower limit as $m \to \infty$ on both sides of (16) we have
$\lim\limits_{m\to\infty} \int_0^{\theta_1\{\omega_1(x_{2m+1},x_{2m})\}} \xi(t)dt \leq \lim\limits_{m\to\infty} \int_0^{\theta_2\{\omega_1(x_{2m},x_{2m-1})\}} \xi(t)dt - \lim\limits_{m\to\infty} inf \int_0^{\phi\{\omega_1(x_{2m},x_{2m-1})\}} \xi(t)dt$
or,
$\lim\limits_{m\to\infty} \int_0^{\theta_1\{\omega_1(x_{2m+1},x_{2m})\}} \xi(t)dt < \lim\limits_{m\to\infty} \int_0^{\theta_1\{\omega_1(x_{2m},x_{2m-1})\}} \xi(t)dt$, which gives
$r^* < r^*$, which is a contradiction.
Therefore, $r = 0$.
Hence,

$$\lim_{n\to\infty} \omega_1(x_n, x_{n+1}) = 0. \ . \tag{17}$$

Since (X, ω) is convex modular and F is a ω-complete subset of X satisfying $\triangle_2$ -type condition and $\lim\limits_{n\to\infty} \omega_1(x_n, x_{n+1}) = 0$, so by **Lemma 1.6**, $\{x_n\}$ is ω-Cauchy in F.
Since F is ω- complete, there exists $z \in F$ such that $\lim\limits_{n\to\infty} \omega_1(x_n, z) = 0$.
Therefore, $\lim\limits_{n\to\infty} \omega_1(x_{2n}, z) = 0$.
Claim: $gz = z$. If not, then $\omega_1(gz, z) > 0$.

Now,

(18)
$$\begin{aligned}\Omega(x_{2n},z) &= max\{\omega_1(x_{2n},fx_{2n}),\omega_1(z,gz),\omega_1(x_{2n},z),\omega_2(fx_{2n},z),\omega_2(x_{2n},gz)\}\\ &= \max\{\omega_1(x_{2n},x_{2n+1}),\omega_1(z,gz),\omega_1(x_{2n},z),\omega_2(x_{2n+1},z),\omega_2(x_{2n},gz)\}\end{aligned}.$$

Now from (1), we have

(19)
$$\begin{aligned}\int_0^{\theta_1\{\omega_1(x_{2n+1},gz)\}}\xi(t)dt &= \int_0^{\theta_1\{\omega_1(fx_{2n},gz)\}}\xi(t)dt\\ &\leq \int_0^{\theta_2\{\Omega(x_{2n},z)\}}\xi(t)dt - \int_0^{\phi\{\Omega(x_{2n},z)\}}\xi(t)dt\end{aligned}.$$

Taking lower limit as $n\to\infty$ on both sides of (19), we have
$\lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(x_{2n+1},gz)\}}\xi(t)dt \leq \lim_{n\to\infty}\int_0^{\theta_2\{\Omega(x_{2n},z)\}}\xi(t)dt - \lim_{n\to\infty} inf\int_0^{\phi\{\Omega(x_{2n},z)\}}\xi(t)dt$
or,
$\lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(x_{2n+1},gz)\}}\xi(t)dt \leq \lim_{n\to\infty}\int_0^{\theta_2\{\omega_1(x_{2n},z)\}}\xi(t)dt - \lim_{n\to\infty} inf\int_0^{\phi\{\omega_1(x_{2n},z)\}}\xi(t)dt$
$[As \lim_{n\to\infty}\Omega(x_{2n},z) = \lim_{n\to\infty}\omega_1(x_{2n},gz) = \omega_1(z,gz)]$
or,
$\lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(x_{2n+1},gz)\}}\xi(t)dt < \lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(x_{2n},gz)\}}\xi(t)dt$, which is a contradiction.
$[As \lim_{n\to\infty}\omega_1(x_{2n},z) = \omega_1(z,gz)]$.
Hence, $gz = z$.
Claim: $fz = z$. If not then, $\omega_1(fz,z) > 0$.
Now,

(20)
$$\begin{aligned}&\Omega(z,x_{2n-1})\\ &= max\{\omega_1(z,fz),\omega_1(x_{2n-1},gx_{2n-1}),\omega_1(z,x_{2n-1}),\omega_2(fz,x_{2n-1}),\omega_2(z,gx_{2n-1})\}\\ &= max\{\omega_1(z,fz),\omega_1(x_{2n-1},x_{2n}),\omega_1(z,x_{2n-1}),\omega_2(fz,x_{2n-1}),\omega_2(z,gx_{2n-1})\}\end{aligned}.$$

Now from (1), we have,

(21)
$$\begin{aligned}\int_0^{\theta_1\{\omega_1(fz,x_{2n})\}}\xi(t)dt &= \int_0^{\theta_1\{\omega_1(fz,gx_{2n-1})\}}\xi(t)dt\\ &\leq \int_0^{\theta_2\{\Omega(z,x_{2n-1})\}} - \int_0^{\phi\{\Omega(z,x_{2n-1})\}}\xi(t)dt\end{aligned}.$$

Taking lower limit as $n\to\infty$ on both sides of (21) we have,
$\lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(fz,x_{2n})\}}\xi(t)dt \leq \lim_{n\to\infty}\int_0^{\theta_2\{\Omega(z,x_{2n-1})\}}\xi(t)dt - \lim_{n\to\infty} inf\int_0^{\phi\{\Omega(z,x_{2n-1})\}}\xi(t)dt$
or,
$\lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(fz,x_{2n})\}}\xi(t)dt \leq \lim_{n\to\infty}\int_0^{\theta_2\{\omega_1(z,x_{2n-1})\}}\xi(t)dt - \lim_{n\to\infty} inf\int_0^{\phi\{\omega_1(z,x_{2n-1})\}}\xi(t)dt$
$[As \lim_{n\to\infty}\Omega(z,x_{2n-1}) = \lim_{n\to\infty}\omega_1(fz,x_{2n-1}) = \omega_1(fz,z)]$
or,
$\lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(fz,x_{2n})\}}\xi(t)dt < \lim_{n\to\infty}\int_0^{\theta_1\{\omega_1(fz,x_{2n})\}}\xi(t)dt$, which is a contradiction.
$[As \lim_{n\to\infty}\omega_1(fz,x_{2n}) = \omega_1(fz,z)]$.

Hence, $fz = z$.

Uniqueness: To prove the uniqueness, let us assume that u, v be two fixed points of f and g i.e., $fu = u, gu = u$ and $fv = v, gv = v$ with $u \neq v$.
Now,

$$\begin{aligned}\Omega(u,v) &= \max\{\omega_1(u,fu), \omega_1(v,gv), \omega_1(u,v), \omega_2(fu,v), \omega_2(u,gv)\} \\ &= \max\{\omega_1(u,u), \omega_1(v,v), \omega_1(u,v), \omega_2(u,v), \omega_2(u,v)\} \\ &= \omega_1(u,v)\end{aligned} \tag{22}$$

$[As\ \ \omega_1(u,v) \geq \omega_2(u,v)]$
Now from (2.1) we have

$$\begin{aligned}\int_0^{\theta_1\{\omega_1(u,v)\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(fu,gv)\}} \xi(t)dt \\ &\leq \int_0^{\theta_2\{\Omega(u,v)\}} \xi(t)dt - \int_0^{\phi\{\Omega(u,v)\}} \xi(t)dt \\ &= \int_0^{\theta_2\{\omega_1(u,v)\}} \xi(t)dt - \int_0^{\phi\{\omega_1(u,v)\}} \xi(t)dt \\ &< \int_0^{\theta_1\{\omega_1(u,v)\}} \xi(t)dt, \text{ which is a contradiction.}\end{aligned}$$

Hence, $u = v$. This completes the proof.

Corollary 1 *Let (X, ω) be a convex modular metric space and F be a non-empty complete subset of X such that ω satisfies the $\triangle_2$- type condition. Let f, g be two functions on F such that for all $x, y \in F$ the following hold:*

$$\int_0^{\theta_1\{\omega_1(fx,gy)\}} \xi(t)dt \leq \int_0^{\theta_2\{\Omega(x,y)\}} \xi(t)dt - \int_0^{\phi\{\Omega(x,y)\}} \xi(t)dt; \ . \tag{23}$$

with

$$\int_0^{\theta_1(t)} \xi(z)dz - \int_0^{\theta_2(t)} \xi(z)dz + \int_0^{\phi(t)} \xi(z)dz > 0;$$

$$\forall r > 0, \ \lim_{t\to r} \int_0^{\theta_1(t)} \xi(z)dz - \lim_{t\to r} \int_0^{\theta_2(t)} \xi(z)dz + \lim_{t\to r} inf \int_0^{\phi(t)} \xi(z)dz > 0;$$

where, $\theta_1 \in \Theta, \theta_2 \in \Psi, \phi \in \Phi, \xi \in \Xi$;
and

$$\Omega(x,y) = a_1\omega_1(x,fx) + a_2\omega_1(y,gy) + a_3\omega_1(x,y) + a_4\omega_2(fx,y) + a_5\omega_2(x,gy)\}; \ . \tag{24}$$

where $a_1 + a_2 + a_3 + a_4 + a_5 \leq 1, a_i \geq 0$. Then f and g have a unique common fixed point in F.

Proof. As

(25)

$$\begin{aligned}\Omega(x,y) &= a_1\omega_1(x,fx) + a_2\omega_1(y,gy) + a_3\omega_1(x,y) + a_4\omega_2(fx,y) + a_5\omega_2(x,gy)\} \\ &\leq (a_1+a_2+a_3+a_4+a_5)max\{\omega_1(x,fx),\omega_1(y,gy),\omega_1(x,y),\omega_2(fx,y),\omega_2(x,gy)\} \\ &\leq max\{\omega_1(x,fx),\omega_1(y,gy),\omega_1(x,y),\omega_2(fx,y),\omega_2(x,gy)\}\end{aligned} .$$

Next the proof is similar as in **Theorem 1**.
Note: If ω is convex then **Theorem 1** and **Corollary 1** are aslo hold.

$$[x_0,\ldots,x_n;\ f] = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} . \tag{26}$$

Theorem 2 *Let (X,ω) be a modular metric space such that ω satisfies the $\triangle_2$-type condition on X and F be a non-empty complete subset of X. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of functions on F such that for all $x,y \in F$ the following hold:*

$$\int_0^{\theta_1\{\omega_1(T_ix,T_jy)\}} \xi(t)dt \leq \int_0^{\theta_2\{\Pi(x,y)\}} \xi(t)dt - \int_0^{\phi\{\Pi(x,y)\}} \xi(t)dt; \ . \tag{27}$$

with

$$\int_0^{\theta_1(t)} \xi(z)dz - \int_0^{\theta_2(t)} \xi(z)dz + \int_0^{\phi(t)} \xi(z)dz > 0;$$

$$\forall r>0,\ \lim_{t\to r}\int_0^{\theta_1(t)} \xi(z)dz - \lim_{t\to r}\int_0^{\theta_2(t)} \xi(z)dz + \lim_{t\to r} inf \int_0^{\phi(t)} \xi(z)dz > 0;$$

where, $\theta_1 \in \Theta, \theta_2 \in \Psi, \phi \in \Phi, \xi \in \Xi$;
and

(28)

$$\Pi(x,y) = max\left\{\omega_1(x,y),\omega_1(y,T_jy),\omega_1(T_ix,x),\frac{\{\omega_2(x,T_jy)+\omega_2(T_ix,y)\}}{1+max\{\omega_1(x,y),\omega_1(T_ix,T_jy)\}},\right.$$
$$\left.\frac{\omega_2(y,T_ix)[1+\omega_1(x,y)]}{2\{1+\omega_1(x,y)+\omega_1(T_ix,T_jy)\}},\frac{max\{\omega_2(x,T_jy),\omega_2(T_ix,y)\}}{1+max\{\omega_1(x,y),\omega_1(x,T_ix),\omega_1(y,T_jy)\}}\right\}. .$$

Then the sequence $\{T_i\}_{i=1}^{\infty}$ have a unique fixed point in F. **Proof.** Let x_0 be an arbitrary point in F. We define a sequence $\{x_n\} \subset X$ such that

$$x_{n+1} = T_ix_n \quad \forall \ \ i = 1,2,3,...$$

We consider two cases:
Case-I: For some $n \in N$ $x_n = x_{n+1} \Rightarrow x_{n+1} = x_{n+2}$.
If $x_{n+1} \neq x_{n+2}$, then $\omega_1(x_{n+1},x_{n+2}) > 0$.

Now putting $x = x_n$ and $y = x_{n+1}$ in (27) we have

(29)

$$\begin{aligned}
&\Pi(x_n, x_{n+1}) \\
&= max\Bigg\{\omega_1(x_n, x_{n+1}), \omega_1(x_, T_i x_n), \omega_1(x_{n+1}, T_j x_{n+1}), \frac{\{\omega_2(x_n, T_j x_{n+1}) + \omega_2(T_i x_n, x_{n+1})\}}{1 + max\{\omega_1(x_n, x_{n+1}), \omega_1(T_i x_n, T_j x_{n+1})\}}, \\
&\quad \frac{\omega_2(x_{n+1}, T_i x_n)[1 + \omega_1(x_n, x_{n+1})]}{2\{1 + \omega_1(x_n, x_{n+1}) + \omega_1(T_i x_n, T_j x_{n+1})\}}, \frac{max\{\omega_2(x_n, T_j x_{n+1}), \omega_2(T_i x_n, x_{n+1})\}}{1 + max\{\omega_1(x_n, x_{n+1}), \omega_1(x_n, T_i x_n), \omega_1(x_n, T_j x_{n+1})\}}\Bigg\} \\
&= max\Bigg\{\omega_1(x_n, x_{n+1}), \omega_1(x_, x_{n+1}), \omega_1(x_{n+1}, x_{n+2}), \frac{\{\omega_2(x_n, x_{n+1}) + \omega_2(x_{n+1}, x_{n+1})\}}{1 + max\{\omega_1(x_n, x_{n+1}), \omega_1(x_{n+1}, x_{n+2})\}}, \\
&\quad \frac{\omega_2(x_{n+1}, x_{n+1})[1 + \omega_1(x_n, x_{n+1})]}{2\{1 + \omega_1(x_n, x_{n+1}) + \omega_1(x_{n+1}, x_{n+2})\}}, \frac{max\{\omega_2(x_n, x_{n+2}), \omega_2(x_{n+1}, x_{n+1})\}}{1 + max\{\omega_1(x_n, x_{n+1}), \omega_1(x_n, x_{n+1}), \omega_1(x_{n+1}, x_{n+2})\}}\Bigg\} \\
&= \omega_1(x_{n+1}, x_{n+2})
\end{aligned}$$

Now from (26) we have

(30)

$$\begin{aligned}
\int_0^{\theta_1\{\omega_1(x_{n+1}, x_{n+2})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(T_i x_n, T_j x_{n+1})\}} \xi(t)dt \\
&\leq \int_0^{\theta_2\{\Pi(x_n, x_{n+1})\}} \xi(t)dt - \int_0^{\phi\{\Pi(x_n, x_{n+1})\}} \xi(t)dt \\
&= \int_0^{\theta_2\{\omega_1(x_{n+1}, x_{n+2})\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_{n+1}, x_{n+2})\}} \xi(t)dt \\
&< \int_0^{\theta_1\{\omega_1(x_{n+1}, x_{n+2})\}} \xi(t)dt, \qquad \text{which is a contradiction.}
\end{aligned}$$

Hence, $x_n = x_{n+1} \Rightarrow x_{n+1} = x_{n+2}$. Proceeding in this manner we conclude that

$$x_n = x_{n+1} \Rightarrow x_{n+1} = x_{n+k}, \ for\ k = 1, 2, 3, ...$$

Therefore $\{x_n\}$ becomes a constant sequence and hence a ω-Cauchy in F.

Case-II: Assume $x_n \neq x_{n+1}$, for all $n = 1, 2, 3, ...$

From (27) we have

(31)

$$\begin{aligned}
&\Pi(x_{n-1}, x_n) \\
&= max\Bigg\{\omega_1(x_{n-1}, x_n), \omega_1(x_{n-1}, T_i x_{n-1}), \omega_1(x_n, T_j x_n), \frac{\{\omega_2(x_{n-1}, T_j x_n) + \omega_2(T_i x_{n-1}, x_n)\}}{1 + max\{\omega_1(x_{n-1}, x_n), \omega_1(T_i x_{n-1}, T_j x_n)\}}, \\
&\frac{\omega_2(x_n, T_i x_{n-1})[1 + \omega_1(x_{n-1}, x_n)]}{2\{1 + \omega_1(x_{n-1}, x_n) + \omega_1(T_i x_{n-1}, T_j x_n)\}}, \frac{max\{\omega_2(x_{n-1}, T_j x_n), \omega_2(T_i x_{n-1}, x_n)\}}{1 + max\{\omega_1(x_{n-1}, x_n), \omega_1(x_{n-1}, T_i x_{n-1}), \omega_1(x_{n-1}, T_j x_n)\}}\Bigg\} \\
&= max\Bigg\{\omega_1(x_{n-1}, x_n), \omega_1(x_{n-1}, x_n), \omega_1(x_n, x_{n+1}), \frac{\{\omega_2(x_{n-1}, x_{n+1}) + \omega_2(x_n, x_n)\}}{1 + max\{\omega_1(x_{n-1}, x_n), \omega_1(x_n, x_{n+1})\}}, \\
&\quad \frac{\omega_2(x_n, x_n)[1 + \omega_1(x_{n-1}, x_n)]}{2\{1 + \omega_1(x_{n-1}, x_n) + \omega_1(x_n, x_{n+1})\}}, \frac{max\{\omega_2(x_{n-1}, x_{n+1}), \omega_2(x_n, x_n)\}}{1 + max\{\omega_1(x_{n-1}, x_n), \omega_1(x_{n-1}, x_n), \omega_1(x_n, x_{n+1})\}}\Bigg\}
\end{aligned}$$

If $\omega_1(x_n, x_{n+1}) \geq \omega_1(x_{n-1}, x_n)$, then $\Pi(x_{n-1}, x_n) = \omega_1(x_n, x_{n+1})$ [On simplification]
Now from (30) we have,

(32)
$$\begin{aligned}\int_0^{\theta_1\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(T_ix_{n-1},T_jx_n)\}} \xi(t)dt \\ &\leq \int_0^{\theta_2\{\Pi(x_{n-1},x_n)\}} \xi(t)dt - \int_0^{\phi\{\Pi(x_{n-1},x_n)\}} \xi(t)dt \\ &= \int_0^{\theta_2\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt \\ &< \int_0^{\theta_1\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt, \quad \text{which is a contradiction.}\end{aligned}$$

Hence,

$$\omega_1(x_n, x_{n+1}) < \omega_1(x_{n_1}, x_n) \ . \tag{33}$$

and

$$\Pi(x_{n-1}, x_n) = \omega_1(x_{n-1}, x_n) \ . \tag{34}$$

Therefore the sequence $\{\omega_1(x_n, x_{n+1})\}$ is monotonic decreasing and bounded below and hence ω- convergent.
Let $\lim_{n\to\infty} \omega_1(x_n, x_{n+1}) = l$ and $\lim_{n\to\infty} \int_0^{\theta_1\{\omega_1(x_n,x_{n+1}\}} \xi(t)dt = l^*$, where $l, l^* \geq 0$.
Claim: $l = 0$. If not then, $l > 0$.
Now from (26) we have,

$$\begin{aligned}\int_0^{\theta_1\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt &= \int_0^{\theta_1\{\omega_1(T_ix_{n-1},T_jx_n)\}} \xi(t)dt \\ &\leq \int_0^{\theta_2\{\Pi(x_{n-1},x_n)\}} \xi(t)dt - \int_0^{\phi\{\Pi(x_{n-1},x_n)\}} \xi(t)dt \\ &= \int_0^{\theta_2\{\omega_1(x_{n-1},x_n)\}} \xi(t)dt - \int_0^{\phi\{\omega_1(x_{n-1},x_n)\}} \xi(t)dt\end{aligned} \tag{35}$$

Taking lower limit as $n \to \infty$ on both sides of (34) we have,

$$\lim_{n\to\infty} \int_0^{\theta_1\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt \leq \lim_{n\to\infty} \int_0^{\theta_2\{\omega_1(x_{n-1},x_n)\}} \xi(t)dt - \lim_{n\to\infty} inf \int_0^{\phi\{\omega_1(x_{n-1},x_n)\}} \xi(t)dt$$

or, $\lim_{n\to\infty} \int_0^{\theta_1\{\omega_1(x_n,x_{n+1})\}} \xi(t)dt < \lim_{n\to\infty} \int_0^{\theta_1\{\omega_1(x_{n-1},x_n)\}} \xi(t)dt$
or, $l^* < l^*$ which is a contradiction.
Therefore, $l = 0$.
Hence,

$$\lim_{n\to\infty} \omega_1(x_n, x_{n+1}) = 0. \ . \tag{36}$$

Since ω satisfies $\triangle_2$-type condition so,

$$\lim_{n\to\infty}\omega_1(x_n,x_{n+1})=0\Rightarrow\lim_{n\to\infty}\omega_{\frac{1}{2}}(x_n,x_{n+1})=0. \quad . \tag{37}$$

Now we show that $\{x_n\}$ is ω-Cauchy in F.
If not then, there exists $\epsilon>0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ with $n_k>m_k\geq k$ such that

$$\omega_1(x_{n_k},x_{m_k})\geq\epsilon \text{ and } \omega_{\frac{1}{2}}(x_{n_k-1},x_{m_k})<\epsilon.$$

Now

$$\begin{aligned}\epsilon\leq\omega_1(x_{n_k},x_{m_k})&\leq\omega_{\frac{1}{2}}(x_{n_k},x_{n_k-1})+\omega_{\frac{1}{2}}(x_{n_k-1},x_{m_k})\\&\leq\omega_{\frac{1}{2}}(x_{n_k},x_{n_k-1})+\epsilon\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\omega_1(x_{n_k},x_{m_k})=\epsilon. \quad . \tag{38}$$

Now,
(39)

$$\begin{aligned}&\Pi(x_{n_k},x_{m_k})\\&=max\Bigg\{\omega_1(x_{n_k},x_{m_k}),\omega_1(x_{n_k},x_{n_k+1}),\omega_1(x_{m_k},x_{m_k+1}),\frac{\{\omega_2(x_{n_k},x_{m_k+1})+\omega_2(x_{n_k+1},x_{m_k})\}}{1+max\{\omega_1(x_{n_k},x_{m_k}),\omega_1(x_{n_k+1},x_{m_k+1})\}},\\&\quad\frac{\omega_2(x_{m_k},x_{n_k+1})[1+\omega_1(x_{n_k},x_{m_k})]}{2\{1+\omega_1(x_{n_k},x_{m_k})+\omega_1(x_{n_k+1},x_{m_k+1})\}},\frac{max\{\omega_2(x_{n_k},x_{m_k+1}),\omega_2(x_{n_k+1},x_{m_k})\}}{1+max\{\omega_1(x_{n_k},x_{m_k}),\omega_1(x_{n_k},x_{n_k+1}),\omega_1(x_{m_k},x_{n_k+1})\}}\Bigg\}\end{aligned} \quad .$$

Taking limit as $k\to\infty$ on both sides of (38) we have

$$\lim_{k\to\infty}\Pi(x_{n_k},x_{m_k})=\epsilon \quad . \tag{40}$$

Now from (26) we have,
(41)

$$\int_0^{\theta_1\{\omega_1(x_{n_k+1},x_{m_k+1})\}}\xi(t)dt\leq\int_0^{\theta_2\{(\omega_1(x_{n_k},x_{m_k}))\}}\xi(t)dt-\int_0^{\phi\{\omega_1(x_{n_k},x_{m_k}\}}\xi(t)dt \quad .$$

Taking lower limit as $k\to\infty$ on both sides of (40) we have

$$\begin{aligned}\lim_{k\to\infty}\int_0^{\theta_1\{\omega_1(x_{n_k+1},x_{m_k+1})\}}\xi(t)dt&\leq\lim_{k\to\infty}\int_0^{\theta_2\{(\omega_1(x_{n_k},x_{m_k}))\}}\xi(t)dt-\lim_{k\to\infty}inf\int_0^{\phi\{\omega_1(x_{n_k},x_{m_k})\}}\xi(t)dt\\&<\lim_{k\to\infty}\int_0^{\theta_1\{\omega_1(x_{n_k},x_{m_k})\}}\xi(t)dt,\qquad\text{which is a contradiction.}\end{aligned}$$

$$[\ As\ \lim_{k\to\infty}\Pi(x_{n_k},x_{m_k})=\epsilon=\lim_{n\to\infty}\omega_1(x_{n_k},x_{m_k})]$$

Hence $\{x_n\}$ is ω-Cauchy in F.
Now from the ω-completeness of F, there exists $z\in F$ such that $\lim_{n\to\infty}x_n=z$.

Claim: $T_iz = z, \quad \forall\ i \in N.$
Now
From (27) we have,
(42)

$$\Pi(x_{n-1}, x_n) = max\Bigg\{\omega_1(z, x_n), \omega_1(z, T_iz), \omega_1(x_n, T_jx_n), \frac{\{\omega_2(z, T_jx_n) + \omega_2(T_iz, x_n)\}}{1 + max\{\omega_1(z, x_n), \omega_1(T_iz, T_jx_n)\}},$$
$$\frac{\omega_2(x_n, T_iz)[1 + \omega_1(z, x_n)]}{2\{1 + \omega_1(z, x_n) + \omega_1(T_iz, T_jx_n)\}}, \frac{max\{\omega_2(z, T_jx_n), \omega_2(T_iz, x_n)\}}{1 + max\{\omega_1(z, x_n), \omega_1(z, T_ix_{n-1}), \omega_1(x_n, T_jx_n}$$

Therefore,

$$\lim_{n\to\infty} \Pi(z, x_n) = \omega_1(z, T_iz)\ . \tag{43}$$

Now from (26) we have,

$$\int_0^{\theta_1\{\omega_1(T_iz, x_{n+1})\}} \xi(t)dt = \int_0^{\theta_1\{\omega_1(T_iz, T_jx_n)\}} \xi(t)dt \leq \int_0^{\theta_2\{M(z,x_n)\}} \xi(t)dt - \int_0^{\phi\{M(z,x_n)\}} \xi(t)dt\ . \tag{44}$$

Taking lower limit as $n \to \infty$ on both sides of (43) we have
(45)

$$\lim_{n\to\infty} \int_0^{\theta_1\{\omega_1(T_iz, x_{n+1})\}} \xi(t)dt = \lim_{n\to\infty} \int_0^{\theta_1\{\omega_1(T_iz, T_jx_n)\}} \xi(t)dt \leq \lim_{n\to\infty} \int_0^{\theta_2\{\Pi(z,x_n)\}} \xi(t)dt - \lim_{n\to\infty} inf \int_0^{\phi\{\Pi(z,x_n)\}} \xi(t)dt\ .$$
$$< \lim_{n\to\infty} \int_0^{\theta_1\{\omega_1(x_{n+1}, T_iz)\}} \xi(t)dt, \quad \text{which is a contradiction.}$$

Therefore, $T_iz = z, \quad \forall\ i \in N.$
Therefore, z is a common fixed point of $\{T_i\}_{i=1}^{\infty}$ in F.
Uniqueness: To prove the uniqueness, let us assume that u and v are two common fixed points of $\{T_i\}_{i=1}^{\infty}$ in F i.e., $T_i = u$ and $T_iv = v$ with $u \neq v$, for all $i = 1, 2, 3, ...$
From (27) we have
(46)

$$\Pi(u, v) = max\Bigg\{\omega_1(u, v), \omega_1(u, T_iu), \omega_1(v, T_jv), \frac{\{\omega_2(u, T_jv) + \omega_2(T_iu, v)\}}{1 + max\{\omega_1(u, v), \omega_1(T_iu, T_jv)\}},$$
$$\frac{\omega_2(v, T_iu)[1 + \omega_1(u, v)]}{2\{1 + \omega_1(u, v) + \omega_1(T_iu, T_jv)\}}, \frac{max\{\omega_2(u, T_jv), \omega_2(T_iu, v)\}}{1 + max\{\omega_1(u, v), \omega_1(u, T_iu), \omega_1(v, T_jv)\}}\Bigg\}\ .$$
$$= \omega_1(u, v)$$

Now from (26) we have

$$\begin{aligned}\int_0^{\theta_1\{\omega_1(u,v)\}}\xi(t)dt &= \int_0^{\theta_1\{\omega_1(T_iu,T_jv)\}}\xi(t)dt\\ &\leq \int_0^{\theta_2\{\Pi(u,v)\}}\xi(t)dt-\int_0^{\phi\{\Pi(u,v)\}}\xi(t)dt\\ &= \int_0^{\theta_2\{\omega_1(u,v)\}}\xi(t)dt-\int_0^{\phi\{\omega_1(u,v)\}}\xi(t)dt\\ &< \int_0^{\theta_1\{\omega_1(u,v)\}}\xi(t)dt, \qquad \text{which is a contradiction.}\end{aligned} \tag{47}$$

Hence, $u=v$. This completes the proof.

Corollary 2 *Let (X,ω) be a modular metric space and F be a non-empty complete subset of X such that ω satisfies the $\triangle_2$-type condition on F. Let T_1,T_2 be two functions on F such that for all $x,y\in F$ the following hold:*

$$\int_0^{\theta_1\{\omega_1(T_1x,T_2y)\}}\xi(t)dt \leq \int_0^{\theta_2\{\Pi(x,y)\}}\xi(t)dt-\int_0^{\phi\{\Pi(x,y)\}}\xi(t)dt;$$

with $\int_0^{\theta_1(t)}\xi(z)dz-\int_0^{\theta_2(t)}\xi(z)dz+\int_0^{\phi(t)}\xi(z)dz>0$;
$\forall r>0,\ \lim_{t\to r}\int_0^{\theta_1(t)}\xi(z)dz-\lim_{t\to r}\int_0^{\theta_2(t)}\xi(z)dz+\lim_{t\to r}inf\int_0^{\phi(t)}\xi(z)dz>0$;
where, $\theta_1\in\Theta,\theta_2\in\Psi,\phi\in\Phi,\xi\in\Xi$;
and

$$\Pi(x,y)=max\Bigg\{\omega_1(x,y),\omega_1(y,T_2y),\omega_1(T_1x,x),\frac{\{\omega_2(x,T_2y)+\omega_2(T_1x,y)\}}{1+max\{\omega_1(x,y),\omega_1(T_1x,T_2y)\}},$$
$$\frac{\omega_2(y,T_1x)[1+\omega_1(x,y)]}{2\{1+\omega_1(x,y)+\omega_1(T_1x,T_2y)\}},\frac{max\{\omega_2(x,T_2y),\omega_2(T_1x,y)\}}{1+max\{\omega_1(x,y),\omega_1(x,T_1x),\omega_1(y,T_2y)\}}\Bigg\}.$$

Then T_1,T_2 have a unique common fixed point in F.

Proof. Taking $T_i=T_1$ and $T_j=T_2$ and proceeding same as in **Theorem 2** we obtain the required result.

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