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On Rathore type operators [1]

Ioan Cristian Buşcu, Vlad Paşca, Andra Seserman

Dedicated to professor Vijay Gupta on the occasion of his 60^{th} birthday

Abstract

V. Gupta introduced recently the Rathore type operators $R_{n,c}$. For them we obtain Voronovskaja type results. We extend the classical Szász-Mirakjan operator and compare the extension with $R_{n,c}$.

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1 Introduction

The Rathore operators were introduced in [8] and studied in several papers (see, e.g., [1], [2], [4], [5], [6], [10] and the references therein). They are defined as

$$R_n f(x) = \int_0^\infty \frac{(n)^{nx}}{\Gamma(nx)} e^{-ny} y^{nx-1} f(y) dy, \ x \geq 0.$$

Here and throughout the paper we consider functions f for which the integrals/series are convergent. A generalization of Rathore operators depending on the parameter $c > 0$ was recently considered in [1] (see also [4]) as follows

$$R_{n,c} f(x) = \int_0^\infty \frac{(nc)^{ncx}}{\Gamma(ncx)} e^{-ncy} y^{ncx-1} f(y) dy.$$

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If $c = 1$, one recovers the operators due to Rathore and for $c = 1/x$, one gets Post-Widder operators [3, (9.1.9)].

Recall the definition of the classical Szász-Mirakjan operators (see [7], [11])

$$S_n f(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \ x \geq 0.$$

We extend these operators as follows

$$S_{nc} f(x) = e^{-ncx} \sum_{k=0}^{\infty} \frac{(ncx)^k}{k!} f\left(\frac{k}{nc}\right), \ x \geq 0, \ c > 0.$$

For $c = 1$ we get Szász-Mirakjan operators.

In this paper we are concerned with a comparison between the operators $R_{n,c}$ and S_{nc}.

2 Moments and central moments

We use the notation $e_m(t) = t^m$, $t \geq 0$, $m = 0, 1, \ldots$. The moments of the operators $R_{n,c}$ were presented in [4].

Lemma 1 *[4] The operators $R_{n,c}$ verify*

$$R_{n,c} e_m(x) = \frac{(ncx)_m}{(nc)^m}, \ m \geq 0.$$

In the same paper [4] one finds the first central moments.

Lemma 2 *[4] If $\mu_{n,m}^{R,c}(x) := (R_{n,c}(e_1 - xe_0)^m)(x)$, then for $x \geq 0$*

$$\mu_{n,0}^{R,c}(x) = 1, \ \mu_{n,1}^{R,c} = 0, \ \mu_{n,2}^{R,c}(x) = \frac{x}{nc}, \ \mu_{n,3}^{R,c}(x) = \frac{2x}{n^2c^2},$$

$$\mu_{n,4}^{R,c}(x) = \frac{6x + 3ncx^2}{n^3c^3} \text{ and } \mu_{n,6}^{R,c}(x) = \frac{15x^3}{n^3c^3} + \frac{130x^2}{n^4c^4} + \frac{120x}{n^5c^5}.$$

Using the moments of the classical Szász-Mirakjan operators it is easy to obtain the following moments of S_{nc}.

Lemma 3 *The operators S_{nc} verify*

a) $S_{nc}(e_0; x) = 1$,

b) $S_{nc}(e_1; x) = x$,

c) $S_{nc}(e_2; x) = x^2 + \frac{x}{nc}$,

d) $S_{nc}(e_3;x) = x^3 + \dfrac{3x^2}{nc} + \dfrac{x}{n^2c^2}$,

e) $S_{nc}(e_4;x) = x^4 + \dfrac{6x^3}{nc} + \dfrac{7x^2}{n^2c^2} + \dfrac{x}{n^3c^3}$,

f) $S_{nc}(e_5;x) = x^5 + \dfrac{10x^4}{nc} + \dfrac{25x^3}{n^2c^2} + \dfrac{15x^2}{n^3c^3} + \dfrac{x}{n^4c^4}$,

g) $S_{nc}(e_6;x) = x^6 + \dfrac{15x^5}{nc} + \dfrac{65x^4}{n^2c^2} + \dfrac{90x^3}{n^3c^3} + \dfrac{31x^2}{n^4c^4} + \dfrac{x}{n^5c^5}$.

Consequently, we can compute the first moments of S_{nc}.

Lemma 4 *If* $\mu_{n,m}^{S,c}(x) := (S_{nc}(e_1 - xe_0)^m)(x)$, *then*

$$\mu_{n,0}^{S,c}(x) = 1,\ \mu_{n,1}^{S,c} = 0,\ \mu_{n,2}^{S,c}(x) = \frac{x}{nc},\ \mu_{n,3}^{S,c}(x) = \frac{x}{n^2c^2},\ \mu_{n,4}^{S,c}(x) = \frac{x(3xnc+1)}{n^3c^3}$$

$$\mu_{n,5}^{S,c}(x) = \frac{x(10xnc+1)}{n^4c^4},\ \mu_{n,6}^{S,c}(x) = \frac{x(15x^2n^2c^2+25xnc+1)}{n^5c^5}.$$

3 Voronovskaja type results

In this section we give Voronovskaja type formulas of first and second orders for the sequences $R_{n,c}$ and S_{nc}.

Denote

$$E := \left\{ f \in C[0,\infty) \;\middle|\; \sup_{t\geq 0} \frac{|f(t)|}{1+t^2} < \infty \right\}.$$

Using central moments and a general approximation theorem for positive linear operators due to Sikkema [9], we obtain the following Voronovskaja type results.

Theorem 1 *Let* $x > 0$ *and* $f \in E$ *such that there exists* $f''(x)$ *finite. Then*

a) $\lim\limits_{n\to\infty} n(R_{n,c}f(x) - f(x)) = \dfrac{x}{2c}f''(x)$,

b) $\lim\limits_{n\to\infty} n(S_{nc}f(x) - f(x)) = \dfrac{x}{2c}f''(x)$.

Remark 1 *A quantitative version of relation a) was given in [4, Th.5].*

Theorem 2 *Let* $x > 0$ *and* $f \in E$ *such that there exists* $f^{(4)}(x)$ *finite. Then*

a) $\lim\limits_{n\to\infty} n^2 \left\{ R_{n,c}f(x) - f(x) - \dfrac{x}{2nc}f''(x) - \dfrac{x}{3n^2c^2}f^{(3)}(x) - \dfrac{x(ncx+2)}{8n^3c^3}f^{(4)}(x) \right\} = 0$,

b) $\lim\limits_{n\to\infty} n^2 \left\{ S_{nc}f(x) - f(x) - \dfrac{x}{2nc}f''(x) - \dfrac{x}{6n^2c^2}f^{(3)}(x) - \dfrac{x(3ncx+1)}{24n^3c^3}f^{(4)} \right\} = 0.$

From Theorem 2 one gets the following result.

Corollary 1 *Under the same hypotheses as in Theorem 2, it follows that*

$$a)\ \lim_{n\to\infty} n\left\{n[R_{n,c}f(x)-f(x)]-\frac{x}{2c}f''(x)\right\}=\frac{x}{3c^2}f^{(3)}(x)+\frac{x^2}{8c^2}f^{(4)}(x),$$

$$b)\ \lim_{n\to\infty} n\left\{n[S_{nc}f(x)-f(x)]-\frac{x}{2c}f''(x)\right\}=\frac{x}{6c^2}f^{(3)}(x)+\frac{x^2}{8c^2}f^{(4)}(x).$$

Corollary 2 *The difference between $R_{n,c}$ and S_{nc} satisfies*

$$\lim_{n\to\infty} n^2\left\{R_{n,c}f(x)-S_{nc}f(x)\right\}=\frac{x}{6c^2}f^{(3)}(x).$$

For similar results involving other sequences of operators see [2, Sect.5].

In [2] the authors introduced the operators

$$\mathcal{A}_n:=R_{n,1}R_{n,1/x},$$

and extended them using a parameter $c>0$ as follows

$$\mathcal{A}_{n,c}(f(t);x):=\mathcal{A}_{n/c}\left(f\left(\frac{t}{c}\right);cx\right).$$

Besides the comparison offered by Corollary 2 we will compare $R_{n,c}$ with the operator $\mathcal{A}_{n,c}$.

The following Voronovskaja type result for the sequence $(\mathcal{A}_{n,c})$ was obtained in [2]:

$$\lim_{n\to\infty} n(\mathcal{A}_{n,c}f(x)-f(x))=\frac{x(1+cx)}{2}f''(x).$$

Combining this results with Theorem 1 a) we get

$$\lim_{n\to\infty} n(\mathcal{A}_{n,c}f(x)-R_{n,c}f(x))=\frac{x(c^2x+c-1)}{2c}f''(x).$$

Remark 2 *From Corollary 2 we see that if $R_{n,c}f\geq S_{nc}f$, $n\geq 1$, then $f^{(3)}\geq 0$.*

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Ioan Cristian Buşcu
Technical University of Cluj-Napoca
Faculty of Automation and Computer Science
Department of Mathematics,
Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania
e-mail: buscu@yahoo.com

Vlad Paşca
Lucian Blaga University of Sibiu
Faculty of Sciences
Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No. 5-7, R-550012 Sibiu, Romania
e-mail: sergiu.pasca@ulbsibiu.ro

Andra Seserman
Technical University of Cluj-Napoca
Faculty of Automation and Computer Science
Department of Mathematics,
Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania
e-mail: campan_aandra@yahoo.com