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Extension of the generalized Bézier operators by wavelets [1]

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Dedicated to professor Vijay Gupta on the occasion of his 60^{th} birthday

Abstract

In this paper we introduce a novel extension of generalized Bézier operators by replacing the sample values $f\left(\frac{k}{n}\right)$ with the wavelet expansion of the function f. Using the compactly supported Daubechies wavelets, we construct a wavelet type extension of the generalized Bézier operators defined by Gupta [7]. Moreover, we investigate some properties of these operators in some function spaces.

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1 Introduction

Let $p_{n,k}(t) = \begin{pmatrix} n \\ k \end{pmatrix} t^k(1-t)^{n-k}$ be the Bernstein basis and let $J_{n,k}(t) = \sum_{j=k}^{n} p_{n,j}(t)$, $t \in [0,1]$, be the Bézier basis functions introduced in [3]. For a bounded real valued function f defined on the interval $[0,1]$ ($f \in B[0,1]$) and $\alpha \geq 1$, the Bézier modification $B_{n,\alpha}f$ of the well-known Bernstein operators are defined as

$$(B_{n,\alpha}f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x) \qquad for \quad x \in [0,1], \tag{1}$$

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where $Q_{n,k}^{(\alpha)}(t) = J_{n,k}^{\alpha}(t) - J_{n,k+1}^{\alpha}(t)$ for $t \in [0,1]$ ($J_{n,l}(x) \equiv 0$ if $l > n$). If $\alpha = 1$, then $B_{n,\alpha}f$ reduce to the classical Bernstein operators.

Since Bernstein polynomials and its Bézier variants (1) cannot be used for $L^p(1 \leq p < \infty)$ approximation, to obtain some positive results for these functions, their Kantorovich and Durrmeyer modifications are considered.

The Kantorovich operators K_n, which was defined to solve the approximation problem in the class of integrable functions, are defined by

$$(K_n f)(x) := \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 (n+1)\chi_{nk}(u) f(u) du,$$

where $p_{n,k}(t)$ is again the Bernstein basis, and χ_{nk} is the characteristic function of the interval $[k/(n+1), (k+1)/(n+1)]$ (see e.g. [1], [7], [12], [13] etc.).

It is easy to see that the operators K_n are described in the following way. If F denotes an indefinite integral of f, then $(K_{n-1}f)(x) = \frac{dB_n F(x)}{dx}$. As is well-known, if f is Lebesgue integrable on $[0,1]$ then

$$\lim_{n\to\infty} (K_n f)(x) = f(x) \quad \text{almost everywhere on } [0,1].$$

In 2004, Gupta [7] considered the generalized Kantorovich type operators as;

$$(K_{n,c}f)(x) := n \sum_{k=0}^{\infty} p_{n,k,c}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u) du \tag{2}$$

where f is locally integrable on the interval $[0,\infty)$ and are of polynomial growth as $u \to \infty$, whose particular cases reduce to the well-known Szász-Kantorovich and Baskakov-Kantorovich operators. Indeed, here

$$p_{n,k,c}(x) = (-1)^k \frac{x^k}{k!} \phi_{n,c}^{(k)}(x),$$

and as special cases, $\phi_{n,c}(x) = (1+cx)^{-n/c}$ for $c = 1$, and $\phi_{n,c}(x) = \exp(-nx)$ for $c = 0$, then the generalized Kantorovich type operators (2) turns out to be Baskakov-Kantorovich and Szász-Kantorovich operators, respectively.

As a generalization, Gupta [7] defined the Bézier variant of the aforementioned generalized Kantorovich type operators as follows:

$$\left(K_{n,c}^{\alpha} f\right)(t) := n \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u) du \tag{3}$$

where $Q_{n,k,c}^{(\alpha)}(t) = J_{n,k,c}^{\alpha}(t) - J_{n,k+1,c}^{\alpha}(t)$, $\alpha \geq 1$ and

$$J_{n,k,c}(t) = \sum_{j=k}^{\infty} p_{n,j,c}(t)$$

be the Baskakov basis and Szász basis for $c = 1$ and $c = 0$, respectively. Clearly, if $\alpha = 1$ then the operators $K_{n,c}^{\alpha} f$ (3) reduce to operators (2).

The goal of this study is to introduce a novel extension of generalized Bézier operators by replacing the sample values $f\left(\frac{k}{n}\right)$ with the wavelet expansion of the function f (see [2], [6], [10], [11] and [14]). Using the compactly supported Daubechies wavelets, we construct a wavelet type extension of the generalized Bézier operators defined by Gupta [7]. Moreover, we investigate some properties of these operators in some function spaces. Please also see the author's very recent article on wavelet type Bernstein operators [9].

The new operators are more flexible than the previous ones and they are a natural and novel extensions of the classical generalized Bézier operators, and its Kantorovich and Durrmeyer type operators.

2 Preliminaries and auxiliary results

As usual, we denote by $C[0,\infty)$ the Banach space of continuous and bounded functions $u : [0,\infty) \to \mathbb{R}$ with norm

$$\|u\| = \sup\{|u(x)| : x \in [0,\infty)\}.$$

By $L^p[0,\infty)(1 \leq p \leq \infty)$ we denote the space of Lebesgue measurable functions f satisfying some conditions related with the $p-$th power. The norms of $f \in L^p[0,\infty)(1 \leq p \leq \infty)$ are given by

$$\|f\|_p = \left(\int_0^{\infty} |f(t)|^p \, dt \right)^{1/p} < \infty,$$
$$\|f\|_{\infty} = ess\sup\{|f(x)| : x \in [0,\infty)\}.$$

Now, we give the following fundamental notions and approachs of the construction of the operators.

A Multiresolution Analysis (**MRA**) is an increasing sequence $(Vj)_{j \in Z}$ of closed subspaces of $L^2(\mathbb{R})$ such that the following hold:

i) V_j is a set of all $f \in L^2(\mathbb{R})$ which are constant on 2^{-j} length intervals and

$$... \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset ... \subset V_j \subset V_{j+1} \subset .. \subset L_2(\mathbb{R}),$$

$$\overline{\bigcup_j V_j} = L_2(\mathbb{R}),$$

ii)

$$\begin{aligned} \forall j,k \in \mathbb{Z},\ f(x) \in V_j &\Leftrightarrow f(2x) \in V_{j+1}, \\ \forall k \in \mathbb{Z},\ f(x) \in V_0 &\Leftrightarrow f(x-k) \in V_0 \\ \forall j,k \in \mathbb{Z},\ f(x) \in V_j &\Leftrightarrow f(x-2^{-j}k) \in V_j, \end{aligned}$$

and

iii)

$$\bigcap_j V_j = \{0\}.$$

(see [4], [5]).

A wavelet is a small wave which oscillates and decays in the time domain. In other words, the fundamental idea behind wavelets is to analyze the signals and functions according to scale.

A wavelet basis set starts with two orthogonal functions: the scaling function (or father wavelet) $\phi(t)$ and the wavelet function (or mother wavelet) $\varphi(t)$. By scaling and translation of these two orthogonal functions we obtain a complete basis set. The scaling and wavelet functions, respectively, satisfy

$$\int_{-\infty}^{\infty} \phi(t)dt = 1, \int_{-\infty}^{\infty} \varphi(t)dt = 0.$$

These two functions have finite energy, namely $\phi, \varphi \in L^2(\mathbb{R})$, and orthogonal.

In general, the wavelets refers to the set of family of orthonormal functions of the form

$$\varphi_{a,b}(t) = \frac{1}{\sqrt{a}}\varphi\left(\frac{t-b}{a}\right), \quad a > 0, b \in \mathbb{R}, \tag{4}$$

where φ is the basic wavelet.

The wavelet analysis procedure is to adopt a wavelet prototype function, called an analyzing wavelet (father wavelet) or mother wavelet.

The simplest and easiest wavelet is known as the Haar wavelet defined as;

$$\varphi(x) = \begin{cases} 1 & , \ 0 \le x < \frac{1}{2} \\ -1 & , \ \frac{1}{2} \le x < 1 \\ 0 & , \ e.w. \end{cases}$$

with the corresponding scaling function

$$\phi(t) = \begin{cases} 1 & , \ 0 \le x < 1 \\ 0 & , \ e.w. \end{cases}.$$

Clearly, Haar wavelets constitutes an orthonormal system for the space of square-integrable functions on the real line. Unfortunately, Haar wavelets are not continuously differentiable which somewhat limits their applications. Indeed, since Haar wavelet is neither continuous nor differentiable, it is suitable for representing discrete signals not for representing smooth signals or functions.

In the present study, we consider orthonormal bases of wavelets in $L^2(\mathbb{R})$, and assume that there is a scaling function (father wavelet) $\phi(t)$ whose whose translates $\{\phi(t-n)\}$ are orthogonal and the mother wavelet $\varphi(t)$ based on the father wavelet $\phi(t)$ gives rise to the orthonormal basis $\varphi_{j,k}(t)$ of $L^2(\mathbb{R})$, where

$$\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k). \tag{5}$$

Hence, by using a multiresolution analysis (MRA), each $f \in L^2(\mathbb{R})$ has the following wavelet expansion

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} b_{j,k}\varphi_{j,k}(x),$$

where $b_{j,k}$ are wavelet coefficients defined by

$$b_{j,k} = \langle f(x), \varphi_{j,k}(x)\rangle = 2^{j/2}\int_{\mathbb{R}} f(x)\overline{\varphi(2^j x - k)}dx.$$

Some delicate convergence results about wavelet expansions, please see [10], [14] and [11].

Some of the special cases of a and b, one can obtain from (4) and (5) different type of wavelets, such as Haar wavelet, Franklin system, Meyer wavelets, etc...

Owing to the above definitions, we introduce the compactly supported Daubechies wavelets considered in this paper.

Assume that the scale function (or father wavelet) $\psi \in L_\infty(\mathbb{R})$ and satisfies:

a) ψ is a compactly supported, namely there is a real constant $\lambda > 0$ such that supp $\psi \subset [0, \lambda]$,

b) $\int_{-\infty}^{\infty} \psi(x)dx = 1$,

c) the first N moments of the father wavelet ψ satisfy

$$\int_{-\infty}^{\infty} x^j\psi(x)dx = 0, \qquad j = 1, ..., N.$$

Actually, Daubechies wavelets have strong relations with the properties of continuity and differentaibility. Namely, for an arbitrary fixed integer $N \geq 1$, compactly supported Daubechies wavelet ψ is supported with $[0, 2N-1]$, in addition there exists

a constant $r > 0$ such that for $N \geq 2$, $\psi \in C^{rN}(\mathbb{R})$ and to have a given number of vanishing moments.

In particular, when $N = 1$, then the first Daubechies wavelet ψ will be the classical Haar basis. As N increases, the regularity of the wavelets increase (see [4], [5]).

This means that if we want to use Daubechies wavelets to reconstruct a function, it is more convenient to choose or construct wavelets based on the continuity or differentiability properties of the given function.

Now, we construct the generalized Bézier operators by using the compactly supported Daubechies wavelets.

Definition 1 *Let $f \in L_1[0,\infty)$ and $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies **a)-c)**. Then the wavelet type generalized Bézier operators, constructed by using the compactly supported Daubechies wavelets, are defined by:*

$$(WB^{\alpha}_{n,c}f)(t) := n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t)\int_{\mathbb{R}} f(x)\,\psi(nx-k)dx, \qquad (t\in\mathbb{R}) \tag{6}$$

$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t)\int_{\mathbb{R}} f\left(\frac{x+k}{n}\right)\psi(x)dx, \qquad (t\in\mathbb{R})$$

$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t)\int_{0}^{\lambda} f\left(\frac{x+k}{n}\right)\psi(x)dx, \qquad (t\in\mathbb{R}).$$

Remark 1 *If we choose the father wavelet ψ as the Haar scaling function, namely $\psi(x) = \chi_{[0,1]}(x)$, then clearly our wavelet type operators reduce to the Kantorovich form of the generalized Bézier operators (3) considered and investigated by Gupta in [1], [7] and [8]. Indeed;*

$$(WB^{\alpha}_{n,c}f)(t) = n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t)\int_{\mathbb{R}} f(x)\,\psi(nx-k)dx$$

$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t)\int_{\mathbb{R}} f\left(\frac{u+k}{n}\right)\psi(u)du$$

$$= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t)\int_{0}^{1} f\left(\frac{u+k}{n}\right)\psi(u)du.$$

This means that our new operators constructed by father wavelets are a natural extension of the Kantorovich type of the generalized Bézier operators and also its Durrmeyer type operators.

3 Fundamental and some convergence properties

This section provides the main approximation results of the paper. We have the followings.

We are now ready to establish one of the first main results of this study, which gives a strong relation between Bézier operators, generalized Kantorovich operators (3) and our new operators (6) constructed by wavelets:

Theorem 1 *Let $\psi \in L_\infty(R)$ be a father wavelet satisfies **a)-c)**. Then the moments of wavelet type generalized Bézier operators constructed by using the compactly supported Daubechies wavelets (6) are given by*

$$(WB^{\alpha}_{n,c}x^s)(t) = \sum_{k=0}^{\infty} \frac{k^s}{n^s} Q^{(\alpha)}_{n,k,c}(t), \quad s = 0, 1, ..., M,$$

where M is a positive integer.

Proof. We have

$$\begin{aligned}
(WB^{\alpha}_{n,c}x^s)(t) &= n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} x^s \psi(nx-k)dx \\
&= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} \left(\frac{u+k}{n}\right)^s \psi(u)du \\
&= \frac{1}{n^s}\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} (u+k)^s \psi(u)du \\
&= \frac{1}{n^s}\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} \left[\sum_{i=0}^{s} \binom{s}{i} u^i k^{s-i}\right] \psi(u)du.
\end{aligned}$$

According to **c)**, one has for $i \neq 0$

$$\int_{\mathbb{R}} \left[\sum_{i=0}^{s} \binom{s}{i} u^i k^{s-i}\right] \psi(u)du = 0$$

and for $i = 0$ and from **b)** we have

$$\begin{aligned}
(WB^{\alpha}_{n,c}x^s)(t) &= \frac{1}{n^s}\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_{\mathbb{R}} k^s \psi(u)du \\
&= \sum_{k=0}^{\infty} \frac{k^s}{n^s} Q^{(\alpha)}_{n,k,c}(t).
\end{aligned}$$

Remark 2 *Moreover, the central moments of the wavelet type generalized Bézier operators (6) are given by,*

$$\begin{aligned}(WB_{n,c}^{\alpha}(x-t)^{\beta})(t) &= n\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(t)\int_{\mathbb{R}}(x-t)^{\beta}\psi(nx-k)dx\\ &= \sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(t)\int_{\mathbb{R}}\left(\frac{u+k}{n}-t\right)^{\beta}\psi(u)du\\ &= \frac{1}{n^{\beta}}\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(t)\int_{\mathbb{R}}(u+k-nt)^{\beta}\psi(u)du\\ &= \frac{1}{n^{\beta}}\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(t)\int_{\mathbb{R}}\left[\sum_{i=0}^{\beta}\left(\begin{array}{c}\beta\\ i\end{array}\right)u^{i}(nt-k)^{\beta-i}\right]\psi(u)du.\end{aligned}$$

By the properties ***b)*** *and* ***c)*** *of the compactly supported Daubechies wavelets, we get*

$$(WB_{n,c}^{\alpha}(x-t)^{\beta})(t) = \frac{1}{n^{\beta}}\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(t)(k-nt)^{\beta}.$$

Throughout this work, as in the case of the generalized Bézier operators, the first two central moments of the generalized Bézier operators, constructed by using the compactly supported Daubechies wavelets (6) satisfy for $c=1$ and $c=0$,

$$\begin{aligned}&w_0(t) := (WB_{n,c}^{\alpha}1)(t) = 1,\\ &w_1(t) := (WB_{n,c}^{\alpha}(x-t))(t) = 0,\\ &w_2(t) := (WB_{n,c}^{\alpha}(x-t)^2)(t) = \frac{\alpha t(1+ct)}{n},\end{aligned} \tag{7}$$

and in addition for each $x\in[0,\infty)$

$$(WB_{n,c}^{\alpha}(x-t)^{m})(t) = O\left(n^{-[(m+1)/2]}\right), n\to\infty$$

(see [7]).

Theorem 2 *Let $f\in L_1[0,\infty)$ and let $\psi\in L_{\infty}(\mathbb{R})$ be a father wavelet satisfies* ***a)-c).*** *Then*

$$\lim_{n\to\infty}(WB_{n,c}^{\alpha}f)(t_0) = f(t_0)$$

holds true at each point t_0 of continuity of f.

Proof. So, in view of the definition of the operator (6), one has

$$\begin{aligned}(WB_{n,c}^{\alpha}f)(t_0) - f(t_0) &= n\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(t_0)\int_{\mathbb{R}}f(x)\psi(nx-k)dx - f(t_0)\\ &= \sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(t_0)\int_{\mathbb{R}}f\left(\frac{u+k}{n}\right)\psi(u)du - f(t_0).\end{aligned}$$

By Theorem 1, we know that

$$(WB_{n,c}^{\alpha}1)(t) = 1. \tag{8}$$

Hence we can write

$$\begin{aligned}\left|(WB_{n,c}^{\alpha}f)(t_0) - f(t_0)\right| &= \left|n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\mathbb{R}}(f(x) - f(t_0))\,\psi(nx-k)dx\right| \\ &\leq \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\mathbb{R}}\left|f\left(\frac{u+k}{n}\right) - f(t_0)\right||\psi(u)|\,du\end{aligned}$$

Let us divide the last term into two parts as;

$$\left|(WB_{n,c}^{\alpha}f)(t_0) - f(t_0)\right| \leq P_1 + P_2,$$

where

$$\begin{aligned}P_1 &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\mathbb{R}}\left|f\left(\frac{u+k}{n}\right) - f(t_0)\right||\psi(u)|\,du \\ &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|<\delta}\left|f\left(\frac{u+k}{n}\right) - f(t_0)\right||\psi(u)|\,du\end{aligned}$$

and

$$P_2 = \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|\geq\delta}\left|f\left(\frac{u+k}{n}\right) - f(t_0)\right||\psi(u)|\,du$$

Since t_0 is a continuity point of f, then clearly one has

$$\begin{aligned}P_1 &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|<\delta}\left|f\left(\frac{u+k}{n}\right) - f(t_0)\right||\psi(u)|\,du \\ &\leq \epsilon\,\|\psi\|_{\infty},\end{aligned}$$

and

$$\begin{aligned}P_2 &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|\geq\delta}\left|f\left(\frac{u+k}{n}\right) - f(t_0)\right||\psi(u)|\,du \\ &\leq 2\,\|f\|\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(t_0)\int_{\left|\frac{u+k}{n}-t_0\right|\geq\delta}|\psi(u)|\,du \\ &\leq 2\,\|f\|\,\frac{w_2(t)}{\delta^2 n^2}\,\|\psi\|_{\infty} = O(n^{-3}).\end{aligned}$$

Collecting these estimates we have

$$\lim_{n\to\infty}(WB_{n,c}^{\alpha}f)(t_0) = f(t_0).$$

Corollary 1 *The same arguments of Theorem 2 apply to the case when* $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$. *In this case the convergence is uniform with respect to* $x \in [0,\infty)$, *and hence one has*

$$\lim_{n\to\infty} \left\| \left(WB^{\alpha}_{n,c} f\right) - f \right\|_\infty = 0.$$

Theorem 3 *Let* $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$ *and let* $\psi \in L_\infty(\mathbb{R})$ *be a father wavelet satisfies* ***a)-c)****. Then*

$$\left\| WB^{\alpha}_{n,c} f \right\|_\infty \leq K \left\| f \right\|_\infty,$$

holds true, where $K = \lambda \left\| \psi \right\|_\infty$.

Proof. One has for the operators (6)

$$\begin{aligned} \left| \left(WB^{\alpha}_{n,c} f\right)(t) \right| &= \left| \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_0^{\lambda} f\left(\frac{x+k}{n}\right) \psi(x) dx \right| \\ &\leq \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \int_0^{\lambda} \left| f\left(\frac{x+k}{n}\right) \right| |\psi(x)|\, dx. \end{aligned}$$

By taking the norm of the functions f and ψ, and considering (7), one has

$$\begin{aligned} \left| \left(WB^{\alpha}_{n,c} f\right)(t) \right| &\leq \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(t) \left\| f \right\|_\infty \left\| \psi \right\|_\infty \lambda \\ &\leq \left\| f \right\|_\infty \left\| \psi \right\|_\infty \lambda. \end{aligned}$$

Hence we get

$$\left\| WB^{\alpha}_{n,c} f \right\|_\infty \leq \lambda \left\| \psi \right\|_\infty \left\| f \right\|_\infty.$$

Note 1. Since the compactly supported Daubechies wavelets are also an unconditional orthonormal base of $L^p(\mathbb{R})$, this allows us to investigate the convergence problem on $L^p(\mathbb{R})$ by means of our wavelet type generalized Bézier operators (6).

For $C[0,\infty) \cap L_\infty(\mathbb{R})$, let us consider the following Peetre's $K-$functional:

$$K_2(f,\delta) := \inf_{g \in W^2} \{ \left\| f - g \right\|_\infty + \delta \left\| g'' \right\|_\infty \}, \tag{9}$$

where $\delta > 0$ and $W^2 = \{ g \in C[0,\infty) : g', g'' \in C[0,\infty) \cap L_\infty(\mathbb{R}) \}$. Then there exists an absolute constant $C > 0$ such that

$$C^{-1} \omega_2(f, \sqrt{\delta}) \leq K_2(f,\delta) \leq C \omega_2(f, \sqrt{\delta}), \tag{10}$$

where

$$\omega_2(f, \sqrt{\delta}) := \sup_{0<h\leq\sqrt{\delta}} \sup_{x \in [0,\infty)} \left| \, f(x+2h) - 2f(x+h) + f(x) \, \right| \tag{11}$$

is the second order modulus of smoothness of f.

Theorem 4 *Let $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$ and let $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies **a)-c)**. Then*

$$\lim_{n\to\infty} \left(WB^{\alpha}_{n,c} f\right)(x) = f(x),$$

and

$$\left|\left(WB^{\alpha}_{n,c} f\right)(x) - f(x)\right| \leq (K+1)\, K_2\left(f; \frac{n\alpha x(1+cx)+\lambda^2}{n^2}\right),$$

where $K = \lambda \|\psi\|_\infty$ and $K_2(f;\delta)$ is the Peetre's $K-$functional.

Proof. Let $g \in W^2$. By Taylor's theorem, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-v) g''(v) dv, \;\; t \in [0,\infty).$$

In view of Remark 2 and (8), applying $WB^{\alpha}_{n,c}$ to the both sides of the above equation, we have

$$\begin{aligned}
\left|\left(WB^{\alpha}_{n,c} g\right)(x) - g(x)\right| &= \left|\left(WB^{\alpha}_{n,c}\left(g'(x)(t-x) + \int_x^t (t-v) g''(v) dv\right)\right)(x) - g(x)\right| \\
&\leq \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(x) \int_{\mathbb{R}} \left| \int_x^{\frac{u+k}{n}} \left(\frac{u+k}{n} - v\right) g''(v) dv \right| |\psi(u)|\, du \\
&\leq \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(x) \int_0^{\lambda} \left[\int_x^{\frac{u+k}{n}} \left| \frac{u+k}{n} - v \right| \left|g''(v)\right| dv \right] |\psi(u)|\, du \\
&\leq \lambda \|\psi\|_\infty \left\|g''\right\|_\infty \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(x) \left(\frac{\lambda+k}{n} - x\right)^2 \\
&= \lambda \|\psi\|_\infty \left\|g''\right\|_\infty \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(x) \left[\left(\frac{k}{n} - x\right)^2 + \frac{\lambda^2}{n^2} + 2\frac{\lambda}{n}\left(\frac{k}{n} - x\right)\right] \\
&\leq \lambda \|\psi\|_\infty \left\|g''\right\|_\infty \left[\frac{\alpha x(1+cx)}{n} + \frac{\lambda^2}{n^2}\right] \\
&= \frac{\lambda \|\psi\|_\infty \|g''\|_\infty}{n^2} \left[n\alpha x(1+cx) + \lambda^2\right].
\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in W^2$ and using (9), we get

$$\begin{aligned}
\left|\left(WB^{\alpha}_{n,c} f\right)(x) - f(x)\right| &\leq \inf_{g\in W^2} \{\left\|WB^{\alpha}_{n,c}(f-g)\right\|_\infty + \|f-g\|_\infty + \left|\left(WB^{\alpha}_{n,c} g\right)(x) - g(x)\right|\} \\
&\leq \inf_{g\in W^2} \{(\lambda\|\psi\|_\infty + 1)\|f-g\|_\infty + \frac{\lambda\|\psi\|_\infty \left[n\alpha x(1+cx)+\lambda^2\right]}{n^2} \left\|g''\right\|_\infty\} \\
&\leq (K+1) \inf_{g\in W^2} \{\|f-g\|_\infty + \frac{n\alpha x(1+cx)+\lambda^2}{n^2} \left\|g''\right\|_\infty\} \\
&= (K+1)\, K_2\left(f; \frac{n\alpha x(1+cx)+\lambda^2}{n^2}\right),
\end{aligned}$$

here $K = \lambda \|\psi\|_\infty$.

Theorem 5 *Let $f \in C[0,\infty) \cap L_\infty(\mathbb{R})$, $\psi \in L_\infty(\mathbb{R})$ be a father wavelet satisfies **a)-c)** and $\eta \in (0,2)$ be fixed real number. Then*

$$\omega_2(f;t) = \mathcal{O}(t^\eta) \Rightarrow \left|\left(WB_{n,c}^{\alpha} f\right)(x) - f(x)\right| = \mathcal{O}(1/n)^\eta$$

holds true.

Proof. In view of the relation (10) between modulus of smoothess (11) and Peetre's $K-$functional (9), we have from Theorem 4

$$\begin{aligned}
\left|\left(WB_{n,c}^{\alpha} f\right)(x) - f(x)\right| &\leq (K+1)\, K_2\left(f; \frac{n\alpha x(1+cx)+\lambda^2}{n^2}\right) \\
&\leq (K+1)\, C\omega_2\left(f; \sqrt{\frac{n\alpha x(1+cx)+\lambda^2}{n^2}}\right) \\
&\leq (K+1)\, C\left(\frac{n\alpha x(1+cx)+\lambda^2}{n^2}\right)^{\eta/2}.
\end{aligned}$$

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