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Generalizations of Hermite-Hadamard, Bullen and Simpson inequalities via $h-$convexity [1]

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Abstract

In this paper, the authors established a new identity for differentiable functions, afterward, they obtained some new general inequalities for functions whose first derivatives in absolute value at certain powers are h-convex by using the identity. On the other hand, a general inequality is studied, which gives Hermite-Hadamard, Bullen and Simpson inequalities. Also, they gave some applications for special means for arbitrary positive numbers.

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1 Introduction

1.1 Definitions

A function $\varphi : I \to R$, $I \subseteq R$ is an interval, is said to be a convex function on I if

$$\varphi\left(tu + (1-t)\,v\right) \leq t\varphi\left(u\right) + (1-t)\,\varphi\left(v\right) \tag{1}$$

holds for all $u, v \in I$ and $t \in [0,1]$. If the reversed inequality in (1) holds, then φ is concave.

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Let $s \in (0,1]$. A function $\varphi : (0,\infty] \to [0,\infty]$ is said to be s-convex in the second sense if

$$\varphi(tu + (1-t)v) \leq t^s \varphi(u) + (1-t)^s \varphi(v) \tag{2}$$

for all $u, v \in (0,\infty]$ and $t \in [0,1]$. This class of s-convex functions is usually denoted by K_s^2 [7, Hudzik and Maligranda, 1994].

In 1978, Breckner introduced s-convex functions as a generalization of convex functions in [2]. Also, in that work Breckner proved the important fact that the set valued map is s-convex only if the associated support function is s-convex function in [3]. A number of properties and connections with s-convex in the first sense and its generalizations are discussed in the papers [7, 18, 19, 28]. Of course, s-convexity means just convexity when $s = 1$.

Let $h : J \to \mathbb{R}$ be a nonnegative function, $h \not\equiv 0$. We say that $\varphi : I \to \mathbb{R}$ is an h-convex function, or that φ belongs to the class $SX(h, I)$, if φ is nonnegative and for all $u, v \in I$ and $t \in (0,1)$ we have

$$\varphi(tu + (1-t)v) \leq h(t)\varphi(u) + h(1-t)\varphi(v). \tag{3}$$

If inequality (3) is reversed, then φ is said to be h-concave, i.e. $\varphi \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$ [30, Varošanec, 2007].

1.2 Hermite-Hadamard, Bullen and Simpson Inequalities

If φ is integrable on $[m, n]$, then the average value of φ on $[m, n]$ is

$$\frac{1}{n-m}\int_m^n \varphi(u)du. \tag{4}$$

Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $m, n \in I$ with $m < n$. Then the following double inequality:

$$\varphi\left(\frac{m+n}{2}\right) \leq \frac{1}{n-m}\int_m^n \varphi(u)du \leq \frac{\varphi(m)+\varphi(n)}{2} \tag{5}$$

is known as **Hermite-Hadamard inequality** for convex mappings. For particular choice of the function φ in (5) yields some classical inequalities of means. Both inequalities in (5) hold in reversed direction if φ is concave. The refinement of the second inequality in (5) is due to Bullen as follows:

$$\frac{1}{n-m}\int_m^n \varphi(u)du \leq \frac{1}{2}\left[\varphi\left(\frac{m+n}{2}\right) + \frac{\varphi(m)+\varphi(n)}{2}\right] \leq \frac{\varphi(m)+\varphi(n)}{2} \tag{6}$$

where φ is as above. This (6) integral inequality is well known in the literature as **Bullen Inequality** [27, Pečarić, Proschan and Tong, 1991].

The following inequality is well known in the literature as **Simpson's inequality** [5, Dragomir, Agarwal, and Cerone, 2000];

$$\int_m^n \varphi(u)\,du - \frac{n-m}{3}\left[\frac{\varphi(m)+\varphi(n)}{2} + 2\varphi\left(\frac{m+n}{2}\right)\right] \leq \frac{1}{1280}\left\|\varphi^{(4)}\right\|_\infty (n-m)^5, \tag{7}$$

where the mapping $\varphi : [m,n] \to \mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $\varphi^{(4)}$ to be bounded on (m,n) , that is,

$$\left\|\varphi^{(4)}\right\|_\infty = \sup_{t\in(m,n)}\left|\varphi^{(4)}(t)\right| < \infty. \tag{8}$$

For some recent results in connection with Hermite-Hadamard, Simpson and Bullen types inequalities and their applications we refer to [1]-[32] where further references are given.

In [10], M.Z. Sarıkaya, A. Sağlam and H. Yıldırım established the following Hadamard type inequality for h-convex functions:

Let $\varphi \in SX(h,I)$, $m,n \in I$ and $\varphi \in L_1([m,n])$, then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\varphi\left(\frac{m+n}{2}\right) \leq \frac{1}{n-m}\int_m^n \varphi(u)\,du \leq [\varphi(m)+\varphi(n)]\int_0^1 h(t)\,dt. \tag{9}$$

For recent results and generalizations concerning h-convex functions see [9, 10, 11, 12, 17, 22, 29, 30] and references therein.

In this paper, firstly a new general equation is established for differentiable functions, secondly, we will derive some new general inequalities for functions whose first derivatives in absolute value are $h-$convex and $s-$convex, which not only provides a generalization of previous theorems but also gives some other interesting special results. Then we give some corollaries and remarks for different types of convex functions. Finally, applications to some special means of real numbers are considered.

2 Main Results

Lemma 1 *Let $\varphi : I \subset R \to R$ be a differentiable function on I° (I° is the interior of I) such that $\varphi' \in L^1[m,n]$, where $m,n \in I$ with $m < n$. Then, for any $\alpha,\beta \in \mathbb{R}$ the following equality holds:*

$$
\begin{aligned}
(10)\qquad & \frac{4}{n-m}\int_m^n \varphi(u)\,du + (\alpha-1)\left[\varphi(m)+\varphi(n)\right] + (\beta-\alpha-1)\,\varphi\left(\frac{3m+n}{4}\right) \\
& - 2\beta\varphi\left(\frac{m+n}{2}\right) + (\beta-\alpha-1)\,\varphi\left(\frac{m+3n}{4}\right) \\
& = \frac{n-m}{4}\left\{\int_0^1 (t-\alpha)\,\varphi'\left(ta+(1-t)\,\frac{3m+n}{4}\right)dt\right. \\
& + \int_0^1 (t-\beta)\,\varphi'\left(t\frac{3m+n}{4}+(1-t)\,\frac{m+n}{2}\right)dt \\
& + \int_0^1 (t+\beta-1)\,\varphi'\left(t\frac{m+n}{2}+(1-t)\,\frac{m+3n}{4}\right)dt \\
& \left. + \int_0^1 (t+\alpha-1)\,\varphi'\left(t\frac{m+3n}{4}+(1-t)\,n\right)dt\right\}.
\end{aligned}
$$

Proof. The proof of the lemma is completed by using integration by parts technique. The details are omitted.

Remark 1 ***i)*** *If we take $\alpha=1$, $\beta=2$ in (10), then we get a midpoint type equality.*
ii) *If we take $\alpha=-1$, $\beta=0$ in (10), then we get a trapezoid type equality.*
iii) *If we take$\alpha=0$, $\beta=1$ in (10), then we get a Bullen type equality.*
iv) *If we take $\alpha=\frac{1}{3}$, $\beta=\frac{4}{3}$ in (10), then we get a Simpson type equality.*

Theorem 1 *Let $I\subset[0,\infty)$, $\varphi: I\to R$ be a differentiable function on I° such that $\varphi'\in L^1[m,n]$, where $m,n\in I$ with $m<n$. If $|\varphi'|^q$ is h-convex on $[m,n]$ and $q>1$, then the following inequality hold*

$$
\begin{aligned}
& \left|\frac{4}{n-m}\int_m^n \varphi(u)\,du + (\alpha-1)\left[\varphi(m)+\varphi(n)\right] + (\beta-\alpha-1)\,\varphi\left(\frac{3m+n}{4}\right)\right. \\
& \left. -2\beta\varphi\left(\frac{m+n}{2}\right) + (\beta-\alpha-1)\,\varphi\left(\frac{m+3n}{4}\right)\right| \\
& \le \frac{n-m}{4}\left\{\left(\int_0^1 |t-\alpha|\,dt\right)^{1-\frac{1}{q}}\right.
\end{aligned}
$$

$$\times \left(\left|\varphi'(m)\right|^q \int_0^1 |t-\alpha|\, h(t)\, dt + \left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 |t-\alpha|\, h(1-t)\, dt \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 |t-\beta|\, dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 |t-\beta|\, h(t)\, dt + \left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 |t-\beta|\, h(1-t)\, dt \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 |t+\beta-1|\, dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 |t+\beta-1|\, h(t)\, dt + \left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 |t+\beta-1|\, h(1-t)\, dt \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 |t+\alpha-1|\, dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 |t+\alpha-1|\, h(t)\, dt + \left|\varphi'(n)\right|^q \int_0^1 |t+\alpha-1|\, h(1-t)\, dt \right)^{\frac{1}{q}} \Bigg\}.$$

Proof. By Lemma 1 and using the Hölder inequality, we have

$$\left| \frac{4}{n-m} \int_m^n \varphi(u)\, du + (\alpha-1)\left[\varphi(m)+\varphi(n)\right] + (\beta-\alpha-1)\,\varphi\left(\frac{3m+n}{4}\right) \right.$$
$$\left. -2\beta\varphi\left(\frac{m+n}{2}\right) + (\beta-\alpha-1)\,\varphi\left(\frac{m+3n}{4}\right) \right|$$
$$\leq \frac{n-m}{4} \Bigg\{ \left(\int_0^1 |t-\alpha|\, dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |t-\alpha| \left|\varphi'\left(ta + (1-t)\,\frac{3m+n}{4}\right)\right|^q dt \right)^{\frac{1}{q}}$$
$$+ \left(\int_0^1 |t-\beta|\, dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |t-\beta| \left|\varphi'\left(t\frac{3m+n}{4} + (1-t)\,\frac{m+n}{2}\right)\right|^q dt \right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t+\beta-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t+\beta-1|\left|\varphi'\left(t\frac{m+n}{2}+(1-t)\frac{m+3n}{4}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t+\alpha-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t+\alpha-1|\left|\varphi'\left(t\frac{m+3n}{4}+(1-t)\,n\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}.$$

$$\leq \frac{n-m}{4}\Bigg\{\left(\int_0^1 |t-\alpha|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|\varphi'(m)\right|^q\int_0^1 |t-\alpha|\,h(t)\,dt+\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q\int_0^1 |t-\alpha|\,h(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t-\beta|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q\int_0^1 |t-\beta|\,h(t)\,dt+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\int_0^1 |t-\beta|\,h(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t+\beta-1|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\int_0^1 |t+\beta-1|\,h(t)\,dt+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\int_0^1 |t+\beta-1|\,h(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t+\alpha-1|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\int_0^1 |t+\alpha-1|\,h(t)\,dt+\left|\varphi'(n)\right|^q\int_0^1 |t+\alpha-1|\,h(1-t)\,dt\right)^{\frac{1}{q}}\Bigg\}.$$

$$=\frac{n-m}{4}\Bigg\{\left(\int_0^1 |t-\alpha|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|\varphi'(m)\right|^q\int_0^1 |t-\alpha|\,h(t)\,dt+\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q\int_0^1 |t-\alpha|\,h(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t-\beta|\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q\int_0^1 |t-\beta|\,h(t)\,dt+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\int_0^1 |t-\beta|\,h(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t+\beta-1|\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\int_0^1 |t+\beta-1|\,h(t)\,dt+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\int_0^1 |t+\beta-1|\,h(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 |t+\alpha-1|\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\int_0^1 |t+\alpha-1|\,h(t)\,dt+\left|\varphi'(n)\right|^q\int_0^1 |t+\alpha-1|\,h(1-t)\,dt\right)^{\frac{1}{q}}\Bigg\}.$$

where, if we take $\alpha,\beta\leq 0$, then we have

$$\int_0^1 |t-\alpha|\,dt=\int_0^1 |t+\alpha-1|\,dt=\frac{1}{2}-\alpha,\ \int_0^1 |t-\beta|\,dt=\int_0^1 |t+\beta-1|\,dt=\frac{1}{2}-\beta,$$

if we take $0\leq\alpha,\beta\leq 1$, then we have

$$\int_0^1 |t-\alpha|\,dt=\int_0^1 |t+\alpha-1|\,dt=\alpha^2-\alpha+\frac{1}{2},$$
$$\int_0^1 |t-\beta|\,dt=\int_0^1 |t+\beta-1|\,dt=\beta^2-\beta+\frac{1}{2},$$

if we take $\alpha,\beta\geq 1$, then we have

$$\int_0^1 |t-\alpha|\,dt=\int_0^1 |t+\alpha-1|\,dt=\alpha-\frac{1}{2},\ \int_0^1 |t-\beta|\,dt=\int_0^1 |t+\beta-1|\,dt=\beta-\frac{1}{2}.$$

Thus, the proof is completed.

Corollary 1 *Let $I \subset [0,\infty)$, $\varphi : I \to R$ be a differentiable function on I° such that $\varphi^{'} \in L^{1}[m,n]$, where $m,n \in I$ with $m<n$. If $|\varphi'|^{q}$ is h-convex on $[m,n]$ and we take $\beta = \alpha+1$ in the Theorem 1, then the following inequality hold*

$$
\begin{aligned}
(11)\qquad & \left| \frac{4}{n-m}\int_{m}^{n} \varphi(u)\,du + (\alpha-1)\left[\varphi(m)+\varphi(n)\right] - 2(\alpha+1)\,\varphi\left(\frac{m+n}{2}\right) \right| \\
& \leq \frac{n-m}{4}\Bigg\{ (A_i)^{1-\frac{1}{q}} \left(\left|\varphi^{'}(m)\right|^{q} \int_{0}^{1} |t-\alpha|\, h(t)\,dt \right. \\
& \left. + \left|\varphi^{'}\left(\frac{3m+n}{4}\right)\right|^{q} \int_{0}^{1} |t-\alpha|\, h(1-t)\,dt \right)^{\frac{1}{q}} \\
& + (B_i)^{1-\frac{1}{q}} \left(\left|\varphi^{'}\left(\frac{3m+n}{4}\right)\right|^{q} \int_{0}^{1} |t-\alpha-1|\, h(t)\,dt \right. \\
& \left. + \left|\varphi^{'}\left(\frac{m+n}{2}\right)\right|^{q} \int_{0}^{1} |t-\alpha-1|\, h(1-t)\,dt \right)^{\frac{1}{q}} \\
& + (B_i)^{1-\frac{1}{q}} \left(\left|\varphi^{'}\left(\frac{m+n}{2}\right)\right|^{q} \int_{0}^{1} |t+\alpha|\, h(t)\,dt \right. \\
& \left. + \left|\varphi^{'}\left(\frac{m+3n}{4}\right)\right|^{q} \int_{0}^{1} |t+\alpha|\, h(1-t)\,dt \right)^{\frac{1}{q}} \\
& + (A_i)^{1-\frac{1}{q}} \left(\left|\varphi^{'}\left(\frac{m+3n}{4}\right)\right|^{q} \int_{0}^{1} |t+\alpha-1|\, h(t)\,dt \right. \\
& \left. + \left|\varphi^{'}(n)\right|^{q} \int_{0}^{1} |t+\alpha-1|\, h(1-t)\,dt \right)^{\frac{1}{q}} \Bigg\},
\end{aligned}
$$

where $\alpha \leq -1 \Longrightarrow i=1$, $-1 \leq \alpha \leq 0 \Longrightarrow i=2$, $0 \leq \alpha \leq 1 \Longrightarrow i=3$, $\alpha \geq 1 \Longrightarrow i=4$. Also, where, if we take $\alpha \leq -1$, then we get

$$A_1 = \int_{0}^{1} |t-\alpha|\,dt = \int_{0}^{1} |t+\alpha-1|\,dt = \frac{1}{2}-\alpha,$$

$$B_1 = \int_{0}^{1} |t+\alpha|\,dt = \int_{0}^{1} |t-\alpha-1|\,dt = -\alpha-\frac{1}{2}$$

and, if we take $-1 \leq \alpha \leq 0$*, then we get*

$$A_2 = \int_0^1 |t-\alpha|\,dt = \int_0^1 |t+\alpha-1|\,dt = \frac{1}{2} - \alpha,$$

$$B_2 = \int_0^1 |t+\alpha|\,dt = \int_0^1 |t-\alpha-1|\,dt = \alpha^2 + \alpha + \frac{1}{2}$$

and, if we take $0 \leq \alpha \leq 1$*, then we get*

$$A_3 = \int_0^1 |t-\alpha|\,dt = \int_0^1 |t+\alpha-1|\,dt = \alpha^2 - \alpha + \frac{1}{2},$$

$$B_3 = \int_0^1 |t+\alpha|\,dt = \int_0^1 |t-\alpha-1|\,dt = \alpha + \frac{1}{2}$$

and, if we take $\alpha \geq 1$*, then we get*

$$A_4 = \int_0^1 |t-\alpha|\,dt = \int_0^1 |t+\alpha-1|\,dt = \alpha - \frac{1}{2},$$

$$B_4 = \int_0^1 |t-\alpha-1|\,dt = \int_0^1 |t+\alpha|\,dt = \alpha + \frac{1}{2}.$$

Proof. Inequalities (11) is immediate by setting $\beta = \alpha + 1$ in Theorem 1.

Remark 2 *If we take* $\alpha = 1$ *in (11), then we get a midpoint type inequality*

$$\begin{aligned}
&\left| \frac{1}{n-m} \int_m^n \varphi(u)\,du - \varphi\left(\frac{m+n}{2}\right) \right| \\
&\leq \frac{n-m}{16} \left\{ \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'(m)\right|^q \int_0^1 |t-1|\,h(t)\,dt \right.\right. \\
&\left. + \left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 |t-1|\,h(1-t)\,dt \right)^{\frac{1}{q}} \\
&+ \left(\frac{3}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 |t-2|\,h(t)\,dt \right.
\end{aligned}$$

$$+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 |t-2|\,h(1-t)\,dt\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{3}{2}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 |t+1|\,h(t)\,dt\right.$$
$$+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 |t+1|\,h(1-t)\,dt\Bigg)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 |t|\,h(t)\,dt+\left|\varphi'(n)\right|^q \int_0^1 |t|\,h(1-t)\,dt\right)^{\frac{1}{q}}\right\}.$$

If we take $\alpha=-1$ in (11), then we get a trapezoid type inequality

$$\left|\frac{1}{n-m}\int_m^n \varphi(u)\,du-\frac{\varphi(m)+\varphi(n)}{2}\right|$$
$$\leq \frac{n-m}{16}\left\{\left(\frac{3}{2}\right)^{1-\frac{1}{q}}\left(\left|\varphi'(m)\right|^q \int_0^1 |t+1|\,h(t)\,dt\right.\right.$$
$$+\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 |t+1|\,h(1-t)\,dt\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 |t|\,h(t)\,dt+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 |t|\,h(1-t)\,dt\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 |t-1|\,h(t)\,dt\right.$$
$$+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 |t-1|\,h(1-t)\,dt\Bigg)^{\frac{1}{q}}$$
$$\left.+\left(\frac{3}{2}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 |t-2|\,h(t)\,dt+\left|\varphi'(n)\right|^q \int_0^1 |t-2|\,h(1-t)\,dt\right)^{\frac{1}{q}}\right\}.$$

If we take $\alpha = 0$ in (11), then we get a Bullen type inequality.

$$\left|\frac{1}{n-m}\int_m^n \varphi(u)\,du - \frac{1}{2}\left(\frac{[\varphi(m)+\varphi(n)]}{2} + \varphi\left(\frac{m+n}{2}\right)\right)\right|$$
$$\leq \frac{2^{\frac{1}{q}-1}(n-m)}{16}\left\{\left(\left|\varphi'(m)\right|^q \int_0^1 th(t)\,dt + \left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 th(1-t)\,dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 (1-t)h(t)\,dt + \left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 (1-t)h(1-t)\,dt\right)^{\frac{1}{q}}$$
$$+\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 th(t)\,dt + \left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 th(1-t)\,dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 (1-t)h(t)\,dt + \left|\varphi'(n)\right|^q \int_0^1 (1-t)h(1-t)\,dt\right)^{\frac{1}{q}}\right\}.$$

If we take $\alpha = \frac{1}{3}$ in (11), then we get a Simpson type inequality

$$\left|\frac{1}{n-m}\int_m^n \varphi(u)\,du - \frac{1}{3}\left(\frac{\varphi(m)+\varphi(n)}{2} + 2\varphi\left(\frac{m+n}{2}\right)\right)\right|$$
$$\leq \frac{n-m}{16}\left\{\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left(\left|\varphi'(m)\right|^q \int_0^1 \left|t-\frac{1}{3}\right| h(t)\,dt\right.\right.$$
$$\left.+\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 \left|t-\frac{1}{3}\right| h(1-t)\,dt\right)^{\frac{1}{q}}$$
$$+\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \int_0^1 \left|t-\frac{4}{3}\right| h(t)\,dt\right.$$
$$\left.+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 \left|t-\frac{4}{3}\right| h(1-t)\,dt\right)^{\frac{1}{q}}$$
$$+\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \int_0^1 \left|t+\frac{1}{3}\right| h(t)\,dt\right.$$

$$+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 \left|t+\frac{1}{3}\right| h(1-t)\,dt\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \int_0^1 \left|t-\frac{2}{3}\right| h(t)\,dt+\left|\varphi'(n)\right|^q \int_0^1 \left|t-\frac{2}{3}\right| h(1-t)\,dt\right)^{\frac{1}{q}}\Bigg\}.$$

Corollary 2 *Let $I \subset [0,\infty)$, $\varphi : I \to R$ be a differentiable function on I° such that $\varphi' \in L^1[m,n]$, where $m,n \in I$ with $m<n$. Under the assumption of Corollary 1, if $|\varphi'|^q$ is s-convex in the second sense on $[m,n]$ for some fixed $s \in (0,1]$ and $q>1$, then the following inequality hold:*

$$\begin{aligned}(12)\qquad &\left|\frac{4}{n-m}\int_m^n \varphi(u)\,du+(\alpha-1)[\varphi(m)+\varphi(n)]-2(\alpha+1)\varphi\left(\frac{m+n}{2}\right)\right| \\ \leq &\frac{n-m}{4}\Bigg\{(A_i)^{1-\frac{1}{q}}\left(\left|\varphi'(m)\right|^q C_i+\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q E_i\right)^{\frac{1}{q}} \\ &+(B_i)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q R_i+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q N_i\right)^{\frac{1}{q}} \\ &+(B_i)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q N_i+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q R_i\right)^{\frac{1}{q}} \\ &+(A_i)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q E_i+\left|\varphi'(n)\right|^q C_i\right)^{\frac{1}{q}}\Bigg\}\end{aligned}$$

where A_i, B_i are as in Corollary 1, and $\alpha \leq -1 \Longrightarrow i=1$, $-1 \leq \alpha \leq 0 \Longrightarrow i=2$, $0 \leq \alpha \leq 1 \Longrightarrow i=3$, $\alpha \geq 1 \Longrightarrow i=4$. Also, where, if we take $\alpha \leq -1$, then we get

$$\begin{aligned}C_1:&\quad \int_0^1 |t-\alpha|\,t^s dt=\int_0^1 |t+\alpha-1|\,(1-t)^s\,dt=\tfrac{1}{s^2+3s+2}\,(s-2\alpha-s\alpha+1)\\ E_1:&\quad \int_0^1 |t+\alpha-1|\,t^s dt=\int_0^1 |t-\alpha|\,(1-t)^s\,dt=-\tfrac{2\alpha+s\alpha-1}{s^2+3s+2}\\ R_1:&\quad \int_0^1 |t-\alpha-1|\,t^s dt=\int_0^1 |t+\alpha|\,(1-t)^s\,dt=-\tfrac{2\alpha+s\alpha+1}{s^2+3s+2}\\ N_1:&\quad \int_0^1 |t+\alpha|\,t^s dt=\int_0^1 |t-\alpha-1|\,(1-t)^s\,dt=-\tfrac{1}{s^2+3s+2}\,(s+2\alpha+s\alpha+1),\end{aligned}$$

and, if we take $-1 \leq \alpha \leq 0$*, then we get*

$$C_2: \quad \int_0^1 |t-\alpha|\, t^s dt = \int_0^1 |t+\alpha-1|\,(1-t)^s\, dt = \tfrac{1}{s^2+3s+2}\,(s-2\alpha-s\alpha+1)$$

$$E_2: \quad \int_0^1 |t+\alpha-1|\, t^s dt = \int_0^1 |t-\alpha|\,(1-t)^s\, dt = -\tfrac{2\alpha+s\alpha-1}{s^2+3s+2}$$

$$R_2: \quad \int_0^1 |t-\alpha-1|\, t^s dt = \int_0^1 |t+\alpha|\,(1-t)^s\, dt$$
$$= \tfrac{1}{s^2+3s+2}\left(4\alpha\,(\alpha+1)^s - s\alpha - 2\alpha + 2\,(\alpha+1)^s + 2\alpha^2\,(\alpha+1)^s - 1\right)$$

$$N_2: \quad \int_0^1 |t+\alpha|\, t^s dt = \int_0^1 |t-\alpha-1|\,(1-t)^s\, dt$$
$$= \tfrac{1}{s^2+3s+2}\left(s+2\alpha+s\alpha+2\alpha^2\,(-\alpha)^s+1\right)$$

and, if we take $0 \leq \alpha \leq 1$*, then we get*

$$C_3: \quad \int_0^1 |t-\alpha|\, t^s dt = \int_0^1 |t+\alpha-1|\,(1-t)^s\, dt = \tfrac{1}{s^2+3s+2}\left(s-2\alpha-s\alpha+2\alpha^{s+2}+1\right)$$

$$E_3: \quad \int_0^1 |t+\alpha-1|\, t^s dt = \int_0^1 |t-\alpha|\,(1-t)^s\, dt$$
$$= \tfrac{1}{s^2+3s+2}\left(2\alpha+2\,(1-\alpha)^s + s\alpha + 2\alpha^2\,(1-\alpha)^s - 4\alpha\,(1-\alpha)^s - 1\right)$$

$$R_3: \quad \int_0^1 |t-\alpha-1|\, t^s dt = \int_0^1 |t+\alpha|\,(1-t)^s\, dt = \tfrac{2\alpha+s\alpha+1}{s^2+3s+2}$$

$$N_3: \quad \int_0^1 |t+\alpha|\, t^s dt = \int_0^1 |t-\alpha-1|\,(1-t)^s\, dt = \tfrac{1}{s^2+3s+2}\,(s+2\alpha+s\alpha+1)$$

and, if we take $\alpha \geq 1$*, then we get*

$$C_4: \quad \int_0^1 |t-\alpha|\, t^s dt = \int_0^1 |t+\alpha-1|\,(1-t)^s\, dt = -\tfrac{1}{s^2+3s+2}\,(s-2\alpha-s\alpha+1)$$

$$E_4: \quad \int_0^1 |t+\alpha-1|\, t^s dt = \int_0^1 |t-\alpha|\,(1-t)^s\, dt = \tfrac{2\alpha+s\alpha-1}{s^2+3s+2}$$

$$R_4: \quad \int_0^1 |t-\alpha-1|\, t^s dt = \int_0^1 |t+\alpha|\,(1-t)^s\, dt = \tfrac{2\alpha+s\alpha+1}{s^2+3s+2}$$

$$N_4: \quad \int_0^1 |t+\alpha|\, t^s dt = \int_0^1 |t-\alpha-1|\,(1-t)^s\, dt = \tfrac{1}{s^2+3s+2}\,(s+2\alpha+s\alpha+1)\,.$$

Proof. Inequality (12) is immediate by setting $h(t) = t^s$ in Corollary 1.

Remark 3 *If we take $\alpha = 1$ in (12), then we get a midpoint type inequality*

$$
\begin{aligned}
(13)\quad & \left| \frac{1}{n-m} \int_m^n \varphi(u)\, du - \varphi\left(\frac{m+n}{2}\right) \right| \\
\leq\ & \frac{n-m}{16} \left\{ \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'(m)\right|^q \frac{1}{s^2+3s+2} + \left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \frac{1}{s+2} \right)^{\frac{1}{q}} \right. \\
& + \left(\frac{3}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \frac{s+3}{s^2+3s+2} + \left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \frac{2s+3}{s^2+3s+2} \right)^{\frac{1}{q}} \\
& + \left(\frac{3}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \frac{2s+3}{s^2+3s+2} + \left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \frac{s+3}{s^2+3s+2} \right)^{\frac{1}{q}} \\
& \left. + \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \frac{1}{s+2} + \left|\varphi'(n)\right|^q \frac{1}{s^2+3s+2} \right)^{\frac{1}{q}} \right\}.
\end{aligned}
$$

If we take $\alpha = -1$ in (12), then we get a trapezoid type inequality

$$
\begin{aligned}
(14)\quad & \left| \frac{1}{n-m} \int_m^n \varphi(u)\, du - \frac{\varphi(m)+\varphi(n)}{2} \right| \\
\leq\ & \frac{n-m}{16} \left\{ \left(\frac{3}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'(m)\right|^q \frac{2s+3}{s^2+3s+2} + \left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \frac{s+3}{s^2+3s+2} \right)^{\frac{1}{q}} \right. \\
& + \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \frac{1}{s+2} + \left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \frac{1}{s^2+3s+2} \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q \frac{1}{s^2+3s+2} + \left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \frac{1}{s+2} \right)^{\frac{1}{q}} \\
& \left. + \left(\frac{3}{2}\right)^{1-\frac{1}{q}} \left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q \frac{s+3}{s^2+3s+2} + \left|\varphi'(n)\right|^q \frac{2s+3}{s^2+3s+2} \right)^{\frac{1}{q}} \right\}.
\end{aligned}
$$

If we take $\alpha = 0$ in (12), then we get a Bullen type inequality

$$
\begin{aligned}
(15)\quad & \left| \frac{1}{n-m} \int_m^n \varphi(u)\, du - \frac{1}{2} \left(\frac{\varphi(m)+\varphi(n)}{2} + \varphi\left(\frac{m+n}{2}\right) \right) \right| \\
\leq\ & \frac{2^{1-\frac{1}{q}}(n-m)}{16} \left\{ \left(\left|\varphi'(m)\right|^q \frac{1}{s+2} + \left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q \frac{1}{s^2+3s+2} \right)^{\frac{1}{q}} \right.
\end{aligned}
$$

$$+\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q\frac{1}{s^2+3s+2}+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\frac{1}{s+2}\right)^{\frac{1}{q}}$$
$$+\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\frac{1}{s+2}+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\frac{1}{s^2+3s+2}\right)^{\frac{1}{q}}$$
$$+\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\frac{1}{s^2+3s+2}+\left|\varphi'(n)\right|^q\frac{1}{s+2}\right)^{\frac{1}{q}}\Bigg\}.$$

If we take $\alpha=\frac{1}{3}$ in (12), then we get a Simpson type inequality

(16)
$$\left|\frac{1}{n-m}\int_m^n\varphi(u)\,du-\frac{1}{3}\left(\left[\frac{\varphi(m)+\varphi(n)}{2}\right]+2\varphi\left(\frac{m+n}{2}\right)\right)\right|\leq\frac{n-m}{16}$$
$$\cdot\Bigg\{\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left(\left|\varphi'(m)\right|^q\frac{\left(6s+2\left(\frac{1}{3}\right)^s+3\right)}{9s^2+27s+18}+\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q\frac{\left(3s+8\left(\frac{2}{3}\right)^s-3\right)}{9s^2+27s+18}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{3m+n}{4}\right)\right|^q\frac{(s+5)}{3(s+1)(s+2)}+\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\frac{1}{3}\frac{4s+5}{s^2+3s+2}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+n}{2}\right)\right|^q\frac{1}{3}\frac{4s+5}{s^2+3s+2}+\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\frac{(s+5)}{3(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left(\left|\varphi'\left(\frac{m+3n}{4}\right)\right|^q\frac{\left(3s+8\left(\frac{2}{3}\right)^s-3\right)}{9s^2+27s+18}+\left|\varphi'(n)\right|^q\frac{\left(6s+2\left(\frac{1}{3}\right)^s+3\right)}{9s^2+27s+18}\right)^{\frac{1}{q}}\Bigg\}.$$

3 Applications for positive numbers

We consider the means for arbitrary positive numbers m,n $(m\neq n)$ as follows;

The arithmetic mean:

$$A(m,n)=\frac{m+n}{2}$$

The generalized logarithmic mean:

$$L_p(m,n)=\left[\frac{n^{p+1}-m^{p+1}}{(p+1)(n-m)}\right]^{\frac{1}{p}},\ p\in\mathbb{R}\setminus\{-1,0\}.$$

Now, by using the result of second Section, we give some applications to special means of real numbers.

Proposition 1 *Let $0 < m < n$, $s \in (0,1)$. Then the following inequality hold:*

$$
\begin{aligned}
&\left|L_s^s(m,n) - A^s(m,n)\right| \\
&\leq \frac{n-m}{16}\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|sa^{s-1}\right|^q}{s^2+3s+2} + \frac{\left|sA^{s-1}\left(\frac{3m}{2},\frac{n}{2}\right)\right|^q}{s+2}\right)^{\frac{1}{q}}\right. \\
&+ \left(\frac{3}{2}\right)^{1-\frac{1}{q}}\left(\left|sA^{s-1}\left(\frac{3m}{2},\frac{n}{2}\right)\right|^q \frac{s+3}{s^2+3s+2} + \left|sA^{s-1}(m,n)\right|^q \frac{2s+3}{s^2+3s+2}\right)^{\frac{1}{q}} \\
&+ \left(\frac{3}{2}\right)^{1-\frac{1}{q}}\left(\left|sA^{s-1}(m,n)\right|^q \frac{2s+3}{s^2+3s+2} + \left|sA^{s-1}\left(\frac{m}{2},\frac{3n}{2}\right)\right|^q \frac{s+3}{s^2+3s+2}\right)^{\frac{1}{q}} \\
&\left.+ \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|sA^{s-1}\left(\frac{m}{2},\frac{3n}{2}\right)\right|^q}{s+2} + \frac{\left|sb^{s-1}\right|^q}{s^2+3s+2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Proof. The inequality is derived from (13) applied to the s-convex functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(u) = u^s$, $s \in (0,1)$, $u \in [m,n]$ and $\varphi'(u) = sx^{s-1}$, $s \in (0,1)$, $u \in [m,n]$. The details are disregarded.

Proposition 2 *Let $0 < m < n$, $s \in (0,1)$. Then the following inequality hold:*

$$
\begin{aligned}
&\left|L_s^s(m,n) - A(m^s,n^s)\right| \\
&\leq \frac{n-m}{16}\left\{\left(\frac{3}{2}\right)^{1-\frac{1}{q}}\left(\left|sa^{s-1}\right|^q \frac{2s+3}{s^2+3s+2} + \left|sA^{s-1}\left(\frac{3m}{2},\frac{n}{2}\right)\right|^q \frac{s+3}{s^2+3s+2}\right)^{\frac{1}{q}}\right. \\
&+ \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|sA^{s-1}\left(\frac{3m}{2},\frac{n}{2}\right)\right|^q \frac{1}{s+2} + \left|sA^{s-1}(m,n)\right|^q \frac{1}{s^2+3s+2}\right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|sA^{s-1}(m,n)\right|^q \frac{1}{s^2+3s+2} + \left|sA^{s-1}\left(\frac{m}{2},\frac{3n}{2}\right)\right|^q \frac{1}{s+2}\right)^{\frac{1}{q}} \\
&\left.+ \left(\frac{3}{2}\right)^{1-\frac{1}{q}}\left(\left|sA^{s-1}\left(\frac{m}{2},\frac{3n}{2}\right)\right|^q \frac{s+3}{s^2+3s+2} + \left|sb^{s-1}\right|^q \frac{2s+3}{s^2+3s+2}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Proof. The inequality is derived from (14) applied to the s-convex functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(u) = u^s$, $s \in (0,1)$, $u \in [m,n]$ and $\varphi'(u) = sx^{s-1}$, $s \in (0,1)$, $u \in [m,n]$. The details are disregarded.

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