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Hermite-Hadamard's inequalities for (h, r)-preinvex functions [1]

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Abstract

In this paper, we introduce a new class of generalized convex functions called (h, r)-preinvex functions, and establish some new Hermite-Hadamard type inequalities under (h, r)-preinvexity.

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1 Introduction

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard inequality, which can be state as follows: If f is a convex function on the finite interval $[a, b]$, then

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is concave, then (1.1) holds in the reverse direction (see [35]).

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The Hermite-Hadamard's inequality play an important role in many areas such as in nonlinear analysis, and optimization. The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants of (1.1) have appeared in the literature, we can mention [1, 6, 7, 8, 9, 10, 15, 16, 17, 18, 21, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33, 37, 38, 39, 40, 43], and references cited therein.

In recent years, lot of efforts have been made by researchers to generalize the classical convexity. Hanson in [14], introduced a new class of generalized convex functions, called invex functions. In [4], the authors gave the concept of preinvex function which is special case of invexity. Pini [36], Noor [26, 27], Yang and Li [46] and Weir [44], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

In [7,8], Dragomir et al. established a variante of Hermite-Hadamard inequalities for s-convex function in the second sense, also for Goudunove-Levin functions, P-functions and s-Goudunove-Levin functions

Theorem 1. *[8] Let $f : [0, \infty) \to [0, \infty)$ be an s-convex function in the second sense where $s \in (0, 1]$, and let $a, b \in [0, \infty)$ with $a < b$. If $f \in L([a, b])$, then the following inequalities holds*

$$2^{s-1} f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a} \int_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{s+1}. \tag{1.2}$$

Theorem 2. *[7] Let $f \in Q(I)$, $a, b \in I$ with $a < b$. If $f \in L([a, b])$, then one has the inequality*

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{4}{b-a} \int_a^b f(x)dx. \tag{1.3}$$

Theorem 3. *[7] Let $f \in P(I)$, $a, b \in I$ with $a < b$. If $f \in L([a, b])$, then one has the inequality*

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{2}{b-a} \int_a^b f(x)dx \leq 2\left[f(a) + f(b)\right]. \tag{1.4}$$

In [32], Pachpatte investigate the following Hermite-Hadamard inequality for product of two convex functions

Theorem 4. *[32] Let $f, g : I \to [0, \infty)$ be convex fonctions on $[a, b] \subset I \subset \mathbb{R}$ with $a < b$, then we have*

$$\tfrac{1}{b-a} \int_a^b f(x)g(x)dx \leq \tfrac{1}{3}M(a, b) + \tfrac{1}{6}N(a, b) \tag{1.5}$$

and

$$2f\left(\tfrac{a+b}{2}\right)g\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f(x)g(x)dx + \tfrac{1}{6}M(a,b) + \tfrac{1}{3}N(a,b), \tag{1.6}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

Le Van et al. [16] proved the following variant of Hermite-Hadamard's inequality

Theorem 5. *[16] Let* $f : I \to [0,\infty)$ *be an* (h,r)*-convex function on* $[a,b] \subset I \subset \mathbb{R}$ *with* $a < b$, *and* $0 < r \leq 1$, *then we have*

$$\frac{\left[f\left(\frac{a+b}{2}\right)\right]^r}{2h(\frac{1}{2})} \leq \left(\tfrac{1}{b-a}\int_a^b f(x)dx\right)^r \leq ([f(a)]^r + [f(b)]^r)\left(\int_0^1 [h(t)]^{\frac{1}{r}}\,dt\right)^r. \tag{1.7}$$

In [38], Sarikaya et al. proved the following Hermite-Hadamard's inequality

Theorem 6. *[38] Let* $f : I \to [0,\infty)$ *be an* h*-convex function on* $[a,b] \subset I \subset \mathbb{R}$ *with* $a < b$ *and* $f \in L([a,b])$, *one has*

$$\frac{f\left(\frac{a+b}{2}\right)}{2h(\frac{1}{2})} \leq \tfrac{1}{b-a}\int_a^b f(x)dx \leq [f(a)+f(b)]\int_0^1 h(t)dt. \tag{1.8}$$

Theorem 7. *[38] Let* $f \in SX(h_1,[a,b])$, $g \in SX(h_2,[a,b])$ *with* $a < b$ *be a functions such that* $fg \in L([a,b])$ *and* $h_1h_2 \in L([a,b])$, *then*

$$\tfrac{1}{b-a}\int_a^b f(x)g(x)dx \leq M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt, \tag{1.9}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 8. *[38] Under the hypotesis of Theorem 7, one has*

$$\begin{aligned}&\frac{f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} - \tfrac{1}{b-a}\int_a^b f(x)g(x)dx\\ \leq& M(a,b)\int_0^1 h_1(t)h_2(1-t)dt + N(a,b)\int_0^1 h_1(t)h_2(t)dt.\end{aligned} \tag{1.10}$$

In [23, 29], Noor et al. established the following theorems for product of two preinvex functions

Theorem 9. *[23] Let $f : I = [a, a + \eta(b, a)] \to (0, \infty)$ and $w : I = [a, a + \eta(b, a)] \to \mathbb{R}$ be h_1-preinvex and h_2-preinvex functions respectively with $a < a + \eta(b, a)$ and $h_1(\frac{1}{2}) \neq 0$, $h_2(\frac{1}{2}) \neq 0$, if the bifunction $\eta(.,.)$ satisfies Condition C, then*

$$\begin{aligned}&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{2a+\eta(b,a)}{2}\right) w\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)w(x)dx \\ \leq & M(a,b)\int_0^1 h_1(t)h_2(1-t)dt + N(a,b)\int_0^1 h_1(t)h_2(t)dt,\end{aligned} \tag{1.11}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Theorem 10. *[23] Let $f : I = [a, a + \eta(b, a)] \to (0, \infty)$ and $w : I = [a, a + \eta(b, a)] \to \mathbb{R}$ be h_1-preinvex and h_2-preinvex functions respectively with $a < a+\eta(b, a)$, then we have*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)w(x)dx \leq M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt, \tag{1.12}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Theorem 11. *[29] Let $f, g : K = [a, a + \eta(b, a)] \to (0, \infty)$ be preinvex functions on the interval of real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a+\eta(b, a)$. Then the following inequality holds.*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.13}$$

In [15], Kirmaci et al. established the following Hermite-Hadamard inequality for product of two s-convex functions

Theorem 12. *[15] Let $f, g : [a, b] \subset [0, \infty) \to \mathbb{R}$ be functions such that g and fg are in $L([a, b])$. If f is convex and nonnegative on $[a, b]$ and fg is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then one has*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{s+2}M(a,b) + \frac{1}{(s+1)(s+2)}N(a,b), \tag{1.14}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Theorem 13. *[15] Let $f, g : [a,b] \subset [0,\infty) \to \mathbb{R}$ be functions such that f, g and fg are in $L([a,b])$. If f is s_1-convex and g is s_2-convex on $[a,b]$ for some fixed $s_1, s_2 \in (0,1]$, then we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{1}{s_1+s_2+1}M(a,b) + \beta(s_1+1, s_2+1)N(a,b)\\ &= \frac{1}{s_1+s_2+1}\left[M(a,b) + \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}N(a,b)\right], \end{aligned} \tag{1.15}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 14. *[15] Let $f, g : [a,b] \subset [0,\infty) \to \mathbb{R}$ be functions such that g and fg are in $L([a,b])$. If f is convex and nonnegative on $[a,b]$ and fg is s-convex on $[a,b]$ for some fixed $s \in (0,1]$, then*

$$2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{(s+1)(s+2)}M(a,b) + \frac{1}{s+2}N(a,b), \tag{1.16}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Motivated by the above results, in this paper we first introduce a new concept of generalized convex functions called (h,r)-preinvex function and establish some new Hermite-Hadamard's inequalities for (h,r)-preinvex functions. Several known results are derived.

2 Preliminaries

In this section we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 1. *[35] A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2. *[35] A positive function $f : I \to \mathbb{R}$ is said to be logarithmically convex, if*

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{(1-t)}$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 3. *[5] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 4. *[1] A positive function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-logarithmically convex in the second sense on I for some $s \in (0,1]$, if*

$$f(tx+(1-t)y) \ \leq [f(x)]^{t^s}\,[f(y)]^{(1-t)^s}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 5. *[34] A positive function $f : I \to \mathbb{R}$ is said to be r-convex on I, where $r \geq 0$, if*

$$f(tx+(1-t)y) \leq \begin{cases} \left[tf^r(x)+(1-t)\,f^r(y)\right]^{\frac{1}{r}} \ , & r \neq 0 \\ [f(x)]^{1-t}\,[f(y)]^t\,, & r = 0 \end{cases}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 6. *[7] A nonnegative function $f : I \to \mathbb{R}$ is said to be P-function, if*

$$f\left(tx+(1-t)\,y\right) \leq f(x)+f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 7. *[25] A positive function $f : I \to \mathbb{R}$ is said to be logarithmically P-convex, if*

$$f\left(tx+(1-t)\,y\right) \leq f(x)\,f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 8. *[12] A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function or Q-function, if*

$$f\left(tx+(1-t)\,y\right) \leq \frac{f(x)}{t}+\frac{f(y)}{1-t}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 9. *[25] A positive function $f : I \to \mathbb{R}$ is said to be logarithmically Q-functions, if*

$$f\left(tx+(1-t)\,y\right) \leq [f(x)]^{\frac{1}{t}}\,[f(y)]^{\frac{1}{1-t}}$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 10. *[9] A function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin function in the second sense for $s \in [0,1]$, if*

$$f\left(tx+(1-t)\,y\right) \leq \frac{f(x)}{t^s}+\frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 11. *[24] A positive function $f : I \to \mathbb{R}$ is said to be logarithmically s-Q-function in the second sense, if*

$$f\left(tx+(1-t)\,y\right) \leq \left[f\left(x\right)\right]^{\frac{1}{t^s}} \left[f(y)\right]^{\frac{1}{(1-t)^s}}$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 12. *[41] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function, where $(0,1) \subseteq J$. A nonnegative function $f : I \to \mathbb{R}$ is said to be h-convex function on I, if*

$$f(tx+(1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 13. *[31] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function, where $(0,1) \subseteq J$. A positive function $f : I \to \mathbb{R}$ is called logarithmically h-convex, if*

$$f(tx+(1-t)y) \leq \left[f(x)\right]^{h(t)} \left[f(y)\right]^{h(1-t)}$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Let K be a subset in $\mathbb{R}$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions.

Definition 14. *[44] A set K is said to be invex at x with respect to η, if*

$$x + t\eta\left(y, x\right) \in K$$

holds for all $x, y \in K$ and $t \in [0,1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 15. *[44] A function f on the invex set K is said to be preinvex with respect to η, if*

$$f\left(x + t\eta\left(y, x\right)\right) \leq \left(1-t\right) f\left(x\right) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 16. *[26] A positive function f on the invex set K is said to be logarithmically preinvex with respect to η, if*

$$f\left(x + t\eta\left(y, x\right)\right) \leq \left[f\left(x\right)\right]^{(1-t)} \left[f(y)\right]^{t}$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 17. *[42] A nonnegative function f on the invex set $K \subseteq [0,\infty)$ is said to be s-preinvex in the second sense with respect to η for some fixed $s \in (0,1]$, if*

$$f\left(x + t\eta\left(y, x\right)\right) \leq \left(1-t\right)^s f\left(x\right) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 18. *[43] A positive function f on the invex set $K \subseteq [0,\infty)$ is said to be logarithmically s-preinvex in the second sense with respect to η for some fixed $s \in (0,1]$, if*

$$f\left(x + t\eta\left(y,x\right)\right) \leq \left[f\left(x\right)\right]^{(1-t)^s} \left[f(y)\right]^{t^s}$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 19. *[23] A nonegative function $f : K \to [0,\infty)$ is said to be P-preinvex function with respect to η, if*

$$f\left(x + t\eta\left(y,x\right)\right) \leq f\left(x\right) + f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 20. *[30] A function $f : K \to (0,\infty)$ is said to be logarithmically P-preinvex with respect to η, if*

$$f\left(x + t\eta\left(y,x\right)\right) \leq f\left(x\right) f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 21. *[23] A function $f : K \to (0,\infty)$ is said to be Q-preinvex function with respect to η, if*

$$f\left(x + t\eta\left(y,x\right)\right) \leq \frac{f(x)}{1-t} + \frac{f(y)}{t}$$

holds for all $x, y \in K$ and $t \in (0,1)$.

Definition 22. *[30] A function $f : K \to (0,\infty)$ is said to be logarithmically Q-preinvex with respect to η, if*

$$f\left(x + t\eta\left(y,x\right)\right) \leq \left[f\left(x\right)\right]^{\frac{1}{1-t}} \left[f(y)\right]^{\frac{1}{t}}$$

holds for all $x, y \in K$ and $t \in (0,1)$.

Definition 23. *[28] A function $f : K \to (0,\infty)$ is said to be s-Q-preinvex function with respect to η, if*

$$f\left(x + t\eta\left(y,x\right)\right) \leq \frac{f(x)}{(1-t)^s} + \frac{f(y)}{t^s}$$

holds for all $x, y \in K$, $t \in (0,1)$ and some fixed $s \in (0,1]$.

Definition 24. *[28] A function $f : K \to (0,\infty)$ is said to be logarithmically s-Q-preinvex with respect to η, if*

$$f\left(x + t\eta\left(y,x\right)\right) \leq \left[f\left(x\right)\right]^{\frac{1}{(1-t)^s}} \left[f(y)\right]^{\frac{1}{t^s}}$$

holds for all $x, y \in K$, $t \in (0,1)$ and some fixed $s \in (0,1]$.

Definition 25. *[2] A positive function f on the invex set K is said to be r-preinvex with respect to η, where $r \geq 0$, if*

$$f\left(x+t\eta\left(y,x\right)\right) \leq \begin{cases} \left[\left(1-t\right)f^{r}\left(x\right)+tf^{r}(y)\right]^{\frac{1}{r}} & , \quad r \neq 0 \\ \left[f(x)\right]^{1-t}\left[f(y)\right]^{t}, & r=0 \end{cases}$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 26. *[18] Let $h:\ [0,1] \to \mathbb{R}$ be a nonnegative function $h \neq 0$. A nonnegative function f on the invex set K is said to be h-preinvex function with respect to η, if*

$$f(x+t\eta\left(y,x\right)) \leq h(1-t)f(x)+h(t)f(y)$$

holds for all $x, y \in K$ and $t \in (0,1)$.

Definition 27. *[30] Let $h:\ [0,1] \to \mathbb{R}$ be a nonnegative function $h \neq 0$. A positive function f on the invex set K is said to be logarithmically h-preinvex function with respect to η, if*

$$f(x+t\eta\left(y,x\right)) \leq \left[f(x)\right]^{h(1-t)}\left[f(y)\right]^{h(t)}$$

holds for all $x, y \in K$ and $t \in (0,1)$.

Condition C ([20]) Let $f : K \subset \mathbb{R}$ be an invex set with respect to the bi-function $\eta\left(.,.\right)$ then for any $a, b \in K$ and $t \in [0.1]$, we have

$$\eta\left(a, a+t\eta(b,a)\right) = -t\eta(b,a) \text{ and } \eta\left(b, a+t\eta(b,a)\right) = \left(1-t\right)\eta(b,a).$$

From Condition C, it follows that

$$\eta\left(a+t_2\eta(b,a), a+t_1\eta(b,a)\right) = \left(t_2-t_1\right)\eta\left(b,a\right)$$

holds for any $a, b \in K$ and $t_1, t_2 \in [0.1]$.

Lemma 1. *[19] For $a \geq 0$ and $b \geq 0$, the following algebraic inequalities are true*

$$\left(a+b\right)^{\lambda} \leq 2^{\lambda-1}\left(a^{\lambda}+b^{\lambda}\right), \qquad \text{for } \lambda \geq 1$$

and

$$\left(a+b\right)^{\lambda} \leq a^{\lambda}+b^{\lambda}, \qquad \text{for } 0 \leq \lambda \leq 1.$$

3 Main Results

In what follows, we assume that K is an invex set of $\mathbb{R}$ with respect to the bi-function $\eta(.,.) : K \times K \ \to \mathbb{R}$, $[a, a+\eta(b,a)] \subset K$ with $a, b \in K^{\circ}$ (K°interior of K) and $\eta\left(b,a\right) > 0$, and $h : J \to \mathbb{R}$ be a nonnegative function where $(0,1) \subseteq J \subset \mathbb{R}$.

Definition 28. *A positive function $f : K \subseteq [0,\infty) \to \mathbb{R}$ is said to be (h,r)-preinvex function with respect to η, if*

$$f(a+t\eta(b,a)) \leq \begin{cases} \left(h(1-t)\left[f(a)\right]^r + h(t)\left[f(b)\right]^r\right)^{\frac{1}{r}} & \text{if } r \neq 0 \\ \left[f(a)\right]^{h(1-t)}\left[f(b)\right]^{h(t)} & \text{if } r = 0 \end{cases}$$

holds for all $a,b \in K$, $r \geq 0$, and $t \in (0,1)$.

Remark 1. *Clearly, Definition 28 recapture all definitions cited above for well-chosen values of $\eta(.,.)$, $h(t)$ and r, with the exception of Definition 14.*

Theorem 15. *Let $f : K = [a, a+\eta(b,a)] \to (0,\infty)$ be an (h,r)-preinvex function with respect to η with $h(\frac{1}{2}) \neq 0$ and η satisfies Condition C. If $f \in L([a,a+\eta(b,a)])$ and $h \in L([0,1])$, then the following inequalities:*

$$\begin{cases} \dfrac{f\left(\frac{2a+\eta(b,a)}{2}\right)}{(2h(\frac{1}{2}))^{\frac{1}{r}}} \leq \dfrac{1}{\eta(b,a)}\displaystyle\int_a^{a+\eta(b,a)} f(x)dx \\ \leq \left[\left[f(a)\right]^r + \left[f(b)\right]^r\right]^{\frac{1}{r}} \displaystyle\int_0^1 (h(t))^{\frac{1}{r}}\,dt \quad \text{if } 0 < r \leq 1, \\ \ln\left[f\left(\frac{2a+\eta(b,a)}{2}\right)\right]^{\frac{1}{2h(\frac{1}{2})}} \\ \leq \dfrac{1}{\eta(b,a)}\displaystyle\int_a^{a+\eta(b,a)} \ln f(x)dx \leq \left[\ln f(a) + \ln f(b)\right]\int_0^1 h(t)dt \quad \text{if } r = 0, \end{cases} \tag{3.1}$$

holds for all $0 \leq r \leq 1$.

Proof. In the case where $r = 0$ the prove is similar to that given by Noor et al. in [30]. Assume that $0 < r \leq 1$, we have

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx = \int_0^1 f(a+t\eta(b,a))dt. \tag{3.2}$$

Using the (h,r)-preinvexity of f, we get

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \int_0^1 \left(h(1-t)\left[f(a)\right]^r + h(t)\left[f(b)\right]^r\right)^{\frac{1}{r}} dt. \tag{3.3}$$

From Minkowski's inequality, we deduce

$$\begin{aligned} &\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \\ &\leq \left(\left(\int_0^1 (h(1-t))^{\frac{1}{r}} f(a)dt\right)^r + \left(\int_0^1 (h(t))^{\frac{1}{r}} f(b)dt\right)^r\right)^{\frac{1}{r}} \end{aligned}$$

$$= \left([f(a)]^r \left(\int_0^1 (h(1-t))^{\frac{1}{r}}\, dt \right)^r + [f(b)]^r \left(\int_0^1 (h(t))^{\frac{1}{r}}\, dt \right)^r \right)^{\frac{1}{r}}$$

$$= \left(([f(a)]^r + [f(b)]^r) \left(\int_0^1 (h(t))^{\frac{1}{r}}\, dt \right)^r \right)^{\frac{1}{r}}$$

$$= ([f(a)]^r + [f(b)]^r)^{\frac{1}{r}} \int_0^1 (h(t))^{\frac{1}{r}}\, dt. \tag{3.4}$$

From Condition C, we have

$$f\left(\tfrac{2a+\eta(b,a)}{2}\right) = f\left(\tfrac{a+(1-t)\eta(b,a)+a+t\eta(b,a)}{2}\right). \tag{3.5}$$

Using (h,r)-preinvexity of f, (3.5) becomes

$$f\left(\tfrac{2a+\eta(b,a)}{2}\right) \leq \left[h\left(\tfrac{1}{2}\right)(f(a+(1-t)\eta\,(b,a)))^r + h\left(\tfrac{1}{2}\right)(f(a+t\eta(b,a))^r\right]^{\frac{1}{r}}. \tag{3.6}$$

Integrating both sides of (3.6) with respect to t over $[0,1]$, we get

$$\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)}{\left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}} \leq \int_0^1 [(f(a+(1-t)\eta\,(b,a)))^r + (f(a+t\eta(b,a))^r]^{\frac{1}{r}}\, dt. \tag{3.7}$$

Applying Lemma 1 for (3.7), we obtain

$$\int_0^1 [(f(a+(1-t)\eta\,(b,a)))^r + (f(a+t\eta(b,a))^r]^{\frac{1}{r}}\, dt$$

$$\leq \int_0^1 2^{\frac{1}{r}-1} [f(a+(1-t)\eta\,(b,a)) + f(a+t\eta(b,a))]\, dt$$

$$= 2^{\frac{1}{r}-1} \left[\int_0^1 f(a+(1-t)\eta\,(b,a))dt + \int_0^1 f(a+t\eta(b,a))dt \right]$$

$$= 2^{\frac{1}{r}} \int_0^1 f(a+t\eta(b,a))dt. \tag{3.8}$$

Making the change $a+t\eta(b,a)=x$, (3.8) gives

$$\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)}{\left[2h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}} \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx. \tag{3.9}$$

The desired result follows from (3.4) and (3.9).

Remark 2. *Theorem 15 The inequality* (3.1) *for recapture the inequalities* (1.4),(1.1), (1.2),(1.3) *and* (1.8), *if we take* $r = 1$, $\eta(b,a) = b - a$ *and* $h(t) = 1, h(t) = t, h(t) = t^s, h(t) = \frac{1}{t}$, *and* $h(t) = h(t)$ *respectively. Moreover we obtain inequality* (1.7), *if we take* $\eta(b,a) = b - a$ *and inequality* (1.13), *if we put* $r = 1$.

Theorem 16. *Let* $f : [a, a+\eta(b,a)] \to (0,\infty)$ *be an* (h,r)*-preinvex function with respect to* η *on* $[a, a+\eta(b,a)]$ *with* $h(\frac{1}{2}) \neq 0$ *and* η *satisfies Condition C. If* $f \in L([a, a+\eta(b,a)])$ *and* $h \in L([0,1])$, *then the following inequalities*

$$\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)}{2\left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}} \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq [f(a)+f(b)] \int_0^1 [h(t)]^{\frac{1}{r}}\, dt, \tag{3.10}$$

holds for all $r \geq 1$.

Proof. Since f is (h,r)-preinvex, we have

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \int_0^1 \left(h(1-t)\,[f(a)]^r + h(t)\,[f(b)]^r\right)^{\frac{1}{r}} dt. \tag{3.11}$$

Applying Lemma 1 for (3.11), we get

$$\begin{aligned}\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq & \int_0^1 \left([h(1-t)]^{\frac{1}{r}} f(a) + [h(t)]^{\frac{1}{r}} f(b)\right) dt \\ =& f(a)\int_0^1 [h(1-t)]^{\frac{1}{r}}\, dt + f(b)\int_0^1 [h(t)]^{\frac{1}{r}}\, dt \\ =& [f(a)+f(b)] \int_0^1 [h(t)]^{\frac{1}{r}}\, dt. \end{aligned} \tag{3.12}$$

From Condition C, and (h,r)-preinvexity, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \left[h\left(\frac{1}{2}\right)[f(a+(1-t)\eta(b,a))]^r + h\left(\frac{1}{2}\right)[f(a+t\eta(b,a))]^r\right]^{\frac{1}{r}}. \tag{3.13}$$

From (3.13) and Lemma 1, we get

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} [f(a+(1-t)\eta(b,a)) + f(a+t\eta(b,a))]. \tag{3.14}$$

Now, integrating both sides of (3.14) with respect to t over $[0,1]$, we obtain

$$\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)}{\left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}} \leq \int_0^1 f(a+(1-t)\eta(b,a))dt + \int_0^1 f(a+t\eta(b,a))dt$$

$$
\begin{aligned}
&= 2\int_0^1 f\left(a+t\eta(b,a)\right)dt \\
&= \frac{2}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f(x)dx. \qquad (3.15)
\end{aligned}
$$

The desired result follows from (3.12) and (3.15). The proof is completed.

Theorem 17. *Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ and $g : [a, a+\eta(b,a)] \to \mathbb{R}$ be (h_1, r_1)-preinvex and (h_2, r_2)-preinvex functions with respect to η respectively. If $fg \in L\left([a, a+\eta(b,a)]\right)$ and $h_1 h_2 \in L\left([0,1]\right)$, then the following inequality holds:*

$$
\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \leq & M(a,b,r_1,r_2)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}\,[h_2(t)]^{\frac{1}{r_2}}\,dt \\
& + N(a,b,r_1,r_2)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}\,[h_2(1-t)]^{\frac{1}{r_2}}\,dt,
\end{aligned}
$$

(3.16)

where

$$M(a,b,r_1,r_2) = c\left(r_1,r_2\right)\left[f(a)g(a)+f(b)g(b)\right] \qquad (3.17)$$

$$N(a,b,r_1,r_2) = c\left(r_1,r_2\right)\left[f(a)g(b)+f(b)g(a)\right], \qquad (3.18)$$

and

$$
c\left(r_1,r_2\right) = \begin{cases} 1 & \text{if } r_1, r_2 \geq 1 \\ 2^{\frac{1}{r_1}-1} & \text{if } 0 < r_1 \leq 1 \leq r_2 \\ 2^{\frac{1}{r_2}-1} & \text{if } 0 < r_2 < 1 \leq r_1 \\ 2^{\frac{1}{r_1}+\frac{1}{r_2}-2} & \text{if } 0 < r_1, r_2 \leq 1. \end{cases} \qquad (3.19)
$$

Proof. Since f and g are (h_1, r_1) and (h_2, r_2)-preinvex functions, respectively, we have

$$
\begin{aligned}
f(a+t\eta(b,a))g(a+t\eta(b,a)) \leq & \left(h_1(1-t)\left[f(a)\right]^{r_1} + h_1(t)\left[f(b)\right]^{r_1}\right)^{\frac{1}{r_1}} \\
& \left(h_2(1-t)\left[g(a)\right]^{r_2} + h_2(t)\left[f(b)\right]^{r_2}\right)^{\frac{1}{r_2}}. \qquad (3.20)
\end{aligned}
$$

Applying Lemma 1 for (3.20), we get

$$
\begin{aligned}
f(a+t\eta(b,a))g(a+t\eta(b,a)) \leq & \, c\left(r_1,r_2\right)\left([h_1(1-t)]^{\frac{1}{r_1}}\,f(a) + [h_1(t)]^{\frac{1}{r_1}}\,f(b)\right) \\
& \times \left([h_2(1-t)]^{\frac{1}{r_2}}\,g(a) + [h_2(t)]^{\frac{1}{r_2}}\,f(b)\right), \qquad (3.21)
\end{aligned}
$$

where $c(r_1, r_2)$ is defined as in (3.19).

Now, integrating both sides of (3.21) with respect to t over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\
\leq & c(r_1,r_2)\left[f(a)g(a)\int_0^1 [h_1(1-t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}}\,dt\right. \\
&+ f(b)g(b)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}}\,dt \\
&\left.+ f(a)g(b)\int_0^1 [h_1(1-t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}}\,dt + f(b)g(a)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}}\,dt\right] \\
= & c(r_1,r_2)\left[f(a)g(a)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}}\,dt + f(b)g(b)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}}\,dt\right. \\
&\left.+ f(a)g(b)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}}\,dt + f(b)g(a)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}}\,dt\right] \\
= & c(r_1,r_2)\left[f(a)g(a)+f(b)g(b)\right]\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}}\,dt \\
&+ c(r_1,r_2)\left[f(a)g(b)+f(b)g(a)\right]\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}}\,dt,
\end{aligned}$$

which is the desired result.

Remark 3. *In Theorem 17, for $\eta(b,a) = b-a$ and $r_1 = r_2 = 1$ the inequality* (3.16) *recapture the inequalities* (1.5),(1.14),(1.15) *and*(1.9) *if we take $h_1(t) = h_2(t) = t, h_1(t) = h_2(t) = t^s, (h_1(t), h_2(t)) = (t^{s_1}, t^{s_2})$ $(h_1(t), h_2(t)) = (h_1(t), h_2(t))$ respectively. Also we obtain* (1.12) *only if $r_1 = r_2 = 1$.*

Theorem 18. *Under the assumptions of Theorem 17, and if f and g are $(h_1, 0)$ and $(h_2, 0)$-preinvex functions respectively with $fg \in L([a, a+\eta(b,a)])$ and $h_1, h_2 \in L([0,1])$, then the following inequality holds:*

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \leq & \frac{1}{4}\left(\int_0^1 [f(a)]^{4h_1(1-t)}\,dt + \int_0^1 [g(a)]^{4h_2(1-t)}\,dt\right. \\
&\left.+ \int_0^1 [f(b)]^{4h_1(t)}\,dt + \int_0^1 [f(b)]^{4h_2(t)}\,dt\right). \qquad (3.22)
\end{aligned}$$

Proof. Since f and g are $(h_1, 0)$ and $(h_2, 0)$-preinvex functions, respectively, we have

$$\begin{aligned} &f(a+t\eta(b,a))g(a+t\eta(b,a)) \\ &\leq [f(a)]^{h_1(1-t)}[f(b)]^{h_1(t)}[g(a)]^{h_2(1-t)}[f(b)]^{h_2(t)}. \end{aligned} \tag{3.23}$$

The inequality (3.23) can be restated as follows

$$\begin{aligned} &f(a+t\eta(b,a))g(a+t\eta(b,a)) \\ &\leq \left([f(a)]^{2h_1(1-t)}[g(a)]^{2h_2(1-t)}\right)^{\frac{1}{2}}\left([f(b)]^{2h_1(t)}[f(b)]^{2h_2(t)}\right)^{\frac{1}{2}}, \end{aligned} \tag{3.24}$$

Applying the A-G inequality for (3.24), and then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \tfrac{1}{2}\int_0^1 [f(a)]^{2h_1(1-t)}[g(a)]^{2h_2(1-t)}\,dt + \tfrac{1}{2}\int_0^1 [f(b)]^{2h_1(t)}[f(b)]^{2h_2(t)}\,dt. \end{aligned} \tag{3.25}$$

Now, using Cauchy's inequality for (3.25) it yields

$$\begin{aligned} &\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \tfrac{1}{4}\int_0^1 [f(a)]^{4h_1(1-t)}\,dt + \tfrac{1}{4}\int_0^1 [g(a)]^{4h_2(1-t)}\,dt \\ &\quad + \tfrac{1}{4}\int_0^1 [f(b)]^{4h_1(t)}\,dt + \tfrac{1}{4}\int_0^1 [f(b)]^{4h_2(t)}\,dt, \end{aligned} \tag{3.26}$$

which is the desired result. The proof is completed.

Theorem 19. *Under the assumptions of Theorem 17, and if f and g are (h_1, r) and $(h_2, 0)$-preinvex functions respectively with $r > 0$ and $fg \in L([a, a+\eta(b,a)])$ and $h_1, h_2 \in L([0,1])$, then the following inequality holds:*

$$\begin{aligned} &\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \\ &\leq \tfrac{c(r)[f(a)]^2}{2}\int_0^1 [h_1(1-t)]^{\frac{2}{r}}\,dt + \tfrac{c(r)[f(b)]^2}{2}\int_0^1 [h_1(t)]^{\frac{2}{r}}\,dt \end{aligned}$$

$$+\frac{1}{4}\int_0^1 [g(a)]^{4h_2(1-t)}\,dt+\frac{1}{4}\int_0^1 [f(b)]^{4h_2(t)}\,dt, \tag{3.27}$$

where

$$c\,(r)=\begin{cases} 1 & \text{if } r\geq 2\\ 2^{\frac{2}{r}-1} & \text{if } r\leq 2.\end{cases} \tag{3.28}$$

Proof. Since f and g are (h_1,r) and $(h_2,0)$-preinvex functions respectively, we have

$$\begin{aligned} &f(a+t\eta(b,a))g(a+t\eta(b,a))\\ &\leq \left(h_1(1-t)\,[f(a)]^r+h_1(t)\,[f(b)]^r\right)^{\frac{1}{r}}[g(a)]^{h_2(1-t)}\,[f(b)]^{h_2(t)}. \end{aligned} \tag{3.29}$$

Integrating both sides of (3.29) with respect to t over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt\\ &\leq\int_0^1 [g(a)]^{h_2(1-t)}\,[f(b)]^{h_2(t)}\left(h_1(1-t)\,[f(a)]^r+h_1(t)\,[f(b)]^r\right)^{\frac{1}{r}}dt. \end{aligned} \tag{3.30}$$

Using Cauchy's inequality, (3.30) gives

$$\begin{aligned} &\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt\leq\frac{1}{2}\int_0^1\left(h_1(1-t)\,[f(a)]^r+h_1(t)\,[f(b)]^r\right)^{\frac{2}{r}}dt\\ &+\frac{1}{2}\int_0^1 [g(a)]^{2h_2(1-t)}\,[f(b)]^{2h_2(t)}\,dt. \end{aligned} \tag{3.31}$$

From Lemma 1 and Cauchy's inequality, (3.31) becomes

$$\begin{aligned} &\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt\\ &\leq\frac{1}{2}c\,(r)\int_0^1\left([h_1(1-t)]^{\frac{2}{r}}\,[f(a)]^2+[h_1(t)]^{\frac{2}{r}}\,[f(b)]^2\right)dt\\ &+\frac{1}{4}\int_0^1 [g(a)]^{4h_2(1-t)}\,dt+\frac{1}{4}\int_0^1 [f(b)]^{4h_2(t)}\,dt \end{aligned}$$

$$
\begin{aligned}
=&\tfrac{c(r)[f(a)]^2}{2}\int_0^1 [h_1(1-t)]^{\frac{2}{r}}\,dt+\tfrac{c(r)[f(b)]^2}{2}\int_0^1 [h_1(t)]^{\frac{2}{r}}\,dt\\
&+\tfrac{1}{4}\int_0^1 [g(a)]^{4h_2(1-t)}\,dt+\tfrac{1}{4}\int_0^1 [f(b)]^{4h_2(t)}\,dt,
\end{aligned}
$$

where $c\,(r)$ is defined by (3.28). The proof is completed.

Theorem 20. *Let $f:[a,a+\eta(b,a)]\to(0,\infty)$ and $g:[a,a+\eta(b,a)]\to\mathbb{R}$ be (h_1,r_1)-preinvex and (h_2,r_2)-preinvex functions with respect to η and $h_1(\frac{1}{2})h_2(\frac{1}{2})\neq 0$, $r_1,r_2>0$, and η satisfies Condition C. If $fg\in L([a,a+\eta(b,a)])$ and $h_1h_2\in L([0,1])$, then the following inequality holds:*

$$
\begin{aligned}
&\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)}{2c(r_1,r_2)\left[h_1\left(\frac{1}{2}\right)\right]^{\frac{1}{r_1}}\left[h_2\left(\frac{1}{2}\right)\right]^{\frac{1}{r2}}}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx\\
\le& M(a,b,r_1,r_2)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}\,[h_2(1-t)]^{\frac{1}{r_2}}\,dt\\
&+N(a,b,r_1,r_2)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}\,[h_2(t)]^{\frac{1}{r_2}}\,dt. \qquad (3.32)
\end{aligned}
$$

where $c(r_1,r_2)$, $M(a,b,r_1,r_2)$ and $N(a,b,r_1,r_2)$ are defined as in (3.19), (3.17) *and* (3.18) *respectively.*

Proof. Since f and g are (h_1,r_1) and (h_2,r_2)-preinvex functions, respectively and η satisfies Condition C, we have

$$
\begin{aligned}
&f\left(\tfrac{2a+\eta(b,a)}{2}\right)g\left(\tfrac{2a+\eta(b,a)}{2}\right)\\
=&f\left(\tfrac{a+(1-t)\eta(b,a)+a+t\eta(b,a)}{2}\right)g\left(\tfrac{a+(1-t)\eta(b,a)+a+t\eta(b,a)}{2}\right)\\
\le&\left[h_1\left(\tfrac{1}{2}\right)[f(a+(1-t)\eta\,(b,a))]^{r_1}+h_1\left(\tfrac{1}{2}\right)[f(a+t\eta(b,a)]^{r_1}\right]^{\frac{1}{r_1}}\\
&\times\left[h_2\left(\tfrac{1}{2}\right)[g(a+(1-t)\eta\,(b,a))]^{r_2}+h\left(\tfrac{1}{2}\right)[g(a+t\eta(b,a)]^{r_2}\right]^{\frac{1}{r_2}}. \qquad (3.33)
\end{aligned}
$$

Applying Lemma 1 for (3.33), we get

$$
\begin{aligned}
&f\left(\tfrac{2a+\eta(b,a)}{2}\right)g\left(\tfrac{2a+\eta(b,a)}{2}\right)\\
\le&c(r_1,r_2)\left[h_1\left(\tfrac{1}{2}\right)\right]^{\frac{1}{r_1}}\left[h_2\left(\tfrac{1}{2}\right)\right]^{\frac{1}{r2}}\\
&\times[f(a+(1-t)\eta\,(b,a))+f(a+t\eta(b,a))]\\
&\times[g(a+(1-t)\eta\,(b,a))+g(a+t\eta(b,a))]
\end{aligned}
$$

$$
\begin{aligned}
&= c(r_1,r_2)\left[h_1\left(\tfrac{1}{2}\right)\right]^{\frac{1}{r_1}}\left[h_2\left(\tfrac{1}{2}\right)\right]^{\frac{1}{r_2}} \\
&\quad \times [f(a+t\eta(b,a))g(a+t\eta(b,a)) \\
&\quad + f(a+(1-t)\eta(b,a))g(a+(1-t)\eta(b,a))] \\
&\quad + c(r_1,r_2)\left[h_1\left(\tfrac{1}{2}\right)\right]^{\frac{1}{r_1}}\left[h_2\left(\tfrac{1}{2}\right)\right]^{\frac{1}{r_2}} \\
&\quad \times [f(a+(1-t)\eta(b,a))g(a+t\eta(b,a)) \\
&\quad + f(a+t\eta(b,a))g(a+(1-t)\eta(b,a))]. \qquad (3.34)
\end{aligned}
$$

Dividing both sides of (3.34) by $c(r_1,r_2)\left[h_1\left(\frac{1}{2}\right)\right]^{\frac{1}{r_1}}\left[h_2\left(\frac{1}{2}\right)\right]^{\frac{1}{r_2}}$, and then using the fact that f is (h_1,r_1) and g is (h_2,r_2)-preinvexity, we obtain

$$
\begin{aligned}
&\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)}{c(r_1,r_2)\left[h_1\left(\frac{1}{2}\right)\right]^{\frac{1}{r_1}}\left[h_2\left(\frac{1}{2}\right)\right]^{\frac{1}{r_2}}} \qquad (3.35) \\
&\leq f(a+t\eta(b,a))g(a+t\eta(b,a)) \\
&\quad + f(a+(1-t)\eta(b,a))g(a+(1-t)\eta(b,a)) \\
&\quad + \left([h_1(t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}} + [h_1(1-t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}}\right) M(a,b,r_1,r_2) \\
&\quad + \left([h_1(t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}} + [h_1(1-t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}}\right) N(a,b,r_1,r_2).
\end{aligned}
$$

Integrating both sides of (3.35) with respect to t over $[0,1]$, we get

$$
\begin{aligned}
\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)}{c(r_1,r_2)\left[h_1\left(\frac{1}{2}\right)\right]^{\frac{1}{r_1}}\left[h_2\left(\frac{1}{2}\right)\right]^{\frac{1}{r_2}}} &\leq 2\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\
&\quad + 2M(a,b,r_1,r_2)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(1-t)]^{\frac{1}{r_2}}\,dt \\
&\quad + 2N(a,b,r_1,r_2)\int_0^1 [h_1(t)]^{\frac{1}{r_1}}[h_2(t)]^{\frac{1}{r_2}}\,dt. \qquad (3.36)
\end{aligned}
$$

Multiplying both sides of (3.36) by $\frac{1}{2}$ and making change of variable, we get the desired result.

Remark 4. *In Theorem 20, the inequality* (3.32) *recaptur inequalities* (1.6),(1.10) *and* (1.16)*, if we choose* $\eta(b,a) = b-a$*,* $r_1 = r_2 = 1$*,* $(h_1(t),h_2(t)) = (t,t)$*,* $(h_1(t),h_2(t)) = (h_1(t),h_2(t))$ *and* $(h_1(t),h_2(t)) = (t^s,t^s)$ *respectively. Also we obtain inequality* (1.11) *if we take* $r_1 = r_2 = 1$*.*

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