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Some Properties of a Family of Univalent Functions Defined by a Generalized Opoola Differential Operator [1]

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Abstract

In this investigation, we introduce a new q-differential operator that generalizes the well-known Opoola differential operator. Using the operator, we define a family of q-bounded turning functions. The new class is denoted by $\mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$. Afterwards, inclusion, characterisation, growth, distortion, covering, linear combination and neighborhood properties of functions $f \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$ are presented.

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1 Introduction and Definitions

Here, we let $\mathfrak{A}$ represent the family of *normalized analytic functions* of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad f(0) = 0 = f'(0) - 1, \ z \in \mathfrak{U} := \{z \in \mathbb{C}, |z| < 1\} \tag{1}$$

and we let $\mathfrak{S} \subset \mathfrak{A}$ be the family of *analytic and univalent functions* in $\mathfrak{U}$.

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Let $\mathfrak{P}$ denote the family of analytic functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \quad p(0) = 1, \ \mathcal{R}e\, p(z) > 0, \ z \in \mathfrak{U}. \tag{2}$$

$\mathfrak{P}$ is known as the family of *functions with positive real parts in* $\mathfrak{U}$ or *Carathéodory functions*, see [6].

Let $\mathfrak{R}(\lambda) \subset \mathfrak{S}$ represent the family of functions such that $\mathcal{R}ef'(z) > \lambda$ $(0 \leqslant \lambda < 1)$, see [13]. $f(z)$ in this case is called *bounded turning function of order* λ. If $\lambda = 0$, then $\mathfrak{R}(0) = \mathfrak{R}$ is simply refer to as the family of *bounded turning functions* studied by MacGregor [9].

Jackson [7] (see also [3, 8]) introduced the concept of q-differential operator defined by

$$\begin{cases} \mathcal{D}_q f(0) = f'(0) \\ \mathcal{D}_q f(z) = \frac{f(z)-f(qz)}{(1-q)z}, \ z \neq 0 \\ \mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)) \end{cases} \tag{3}$$

where $0 < q < 1$. Now by (1) and (3),

$$\begin{cases} \mathcal{D}_q f(z) = 1 + \sum\limits_{k=2}^{\infty} [k]_q a_k z^{k-1} \\ \mathcal{D}_q^2 f(z) = \sum\limits_{k=2}^{\infty} [k]_q [k-1]_q a_k z^{k-2} \end{cases} \tag{4}$$

where

$$[k]_q = \frac{1-q^k}{1-q} \quad \text{and} \quad \lim_{q\uparrow 1} [k]_q = k. \tag{5}$$

Using the idea of Opoola [10] and the q-operator in (4), we generalize the Opoola differential operator as follows.

Definition 1 *For functions* $f(z) \in \mathfrak{A}$, *define the linear operator* $\mathfrak{D}_q^{n,\mu,\beta,\tau} : \mathfrak{A} \longrightarrow \mathfrak{A}$ *by*

$$\mathfrak{D}_q^{0,\mu,\beta,\tau} f(z) = f(z) \tag{6}$$

$$\mathfrak{D}_q^{1,\mu,\beta,\tau} f(z) = z\tau \mathcal{D}_q f(z) - z\tau(\beta - \mu) + (1 + (\beta - \mu - 1)\tau) f(z) = d_{q,t} f(z) \tag{7}$$

$$\mathfrak{D}_q^{2,\mu,\beta,\tau} f(z) = d_{q,t}(\mathfrak{D}_q^{1,\mu,\beta,\tau} f(z))$$

so that

$$\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z) = d_{q,t}(\mathfrak{D}_q^{n-1,\mu,\beta,\tau} f(z))$$

which corresponds to

$$\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z) = z + \sum_{k=2}^{\infty} (1 + ([k]_q + \beta - \mu - 1)\tau)^n a_k z^k, \quad z \in \mathfrak{U} \tag{8}$$

where the declarations in (10) *hold true.*

Remark 1 *The following characteristics hold in* (8).

1. $\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{0,\mu,\beta,\tau}f(z)=\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{n,\mu,\beta,0}f(z)=\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{0,\mu,\beta,0}f(z)=f(z)\in\mathfrak{A}$ *in* (1),
2. $\mathfrak{D}_q^{n,\beta,\beta,1}f(z)=\mathfrak{D}_q^{n,\mu,\mu,1}f(z)=\mathfrak{D}_q^{n}f(z)$ *is the Sălăgean's q-differential operator studied by Adwan and Aouf [1],*
3. $\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{n,\beta,\beta,1}f(z)=\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{n,\mu,\mu,1}f(z)=\mathfrak{D}_q^{n}f(z)$ *is the Sălăgean's differential operator introduced in [12],*
4. $\mathfrak{D}_q^{n,\beta,\beta,\tau}f(z)=\mathfrak{D}_q^{n,\mu,\mu,\tau}f(z)=\mathfrak{D}_q^{n,\tau}f(z)$ *is the Al-Oboudi's q-differential operator,*
5. $\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{n,\beta,\beta,\tau}f(z)=\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{n,\mu,\mu,\tau}f(z)=\mathfrak{D}_q^{n.\tau}f(z)$ *is the Al-Oboudi's differential operator introduced in [2],*
6. $\lim\limits_{q\uparrow 1}\mathfrak{D}_q^{n,\mu,\beta,\tau}=\mathfrak{D}_q^{n,\mu,\beta,\tau}f(z)$ *is the Opoola's differential operator introduced in [10].*

Now using the operator $\mathfrak{D}_q^{n,\mu,\beta,\tau}:\mathfrak{A}\longrightarrow\mathfrak{A}$, the class $\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda)$ is defined as follows.

Definition 2 *A function $f(z)\in\mathfrak{A}$ is said to be a member of the family $\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda)$ if*

$$\mathcal{R}e\{\mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau}f(z))\}>\lambda,\quad z\in\mathfrak{U} \tag{9}$$

where

$$\begin{cases}\tau\geqslant 0,\ 0\leqslant\mu\leqslant\beta,\ n\in\mathbb{N}\cup\{0\},\ 0\leqslant\lambda<1,\\ 0<q<1,\ [k]_q=\frac{1-q^k}{1-q}\ \text{and}\ \lim\limits_{q\uparrow 1}[k]_q=k.\end{cases} \tag{10}$$

Remark 2 *We note the following subfamilies of $\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda)$.*

1. $\lim\limits_{q\uparrow 1}\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda)=\mathfrak{R}^n(\mu,\beta,\tau;\lambda)$, *the family of functions studied by Opoola [10],*
2. $\lim\limits_{q\uparrow 1}\mathfrak{R}_q^n(\beta,\beta,\tau;\lambda)=\lim\limits_{q\uparrow 1}\mathfrak{R}_q^n(\mu,\mu,\tau;\lambda)=\mathfrak{R}^n(\tau;\lambda)$, *the family of functions studied by Al-Oboudi [2],*
3. $\lim\limits_{q\uparrow 1}\mathfrak{R}_q^0(\mu,\beta,\tau;\lambda)=\lim\limits_{q\uparrow 1}\mathfrak{R}_q^n(\mu,\beta,0;\lambda)=\lim\limits_{q\uparrow 1}\mathfrak{R}_q^0(\mu,\beta,0;\lambda)=\mathfrak{R}(\lambda)$, *the family of functions studied by Tuan and Anh [13],*
4. $\lim\limits_{q\uparrow 1}\mathfrak{R}_q^0(\mu,\beta,\tau;0)=\lim\limits_{q\uparrow 1}\mathfrak{R}_q^n(\mu,\beta,0;0)=\lim\limits_{q\uparrow 1}\mathfrak{R}_q^0(\mu,\beta,0;0)=\mathfrak{R}$, *the family of functions studied by MacGregor [9].*

By studying the family $\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda)$, we therefore investigate some of its geometric properties such as inclusion, characterisation, growth, distortion, covering, linear combination and neighborhood properties. Our results generalize many known ones.

2 Preliminary Lemmas

We shall apply the following lemmas in proving our results. These present lemmas are the q-analogous versions of the referenced authors' work.

Lemma 1 (Noshiro-Warschawski's Lemma [6]) *If $f(z)$ is analytic in the domain Δ and $\mathcal{R}e(\mathcal{D}_q f(z)) > 0$ in Δ, then f is univalent in Δ.*

Lemma 2 ([4]) *If $u = u_1 + u_2 i$, $\nu = \nu_1 + \nu_2 i$ and $\Phi(u,\nu) : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a complex function such that*

1. *$\Phi(u,\nu)$ is continuous in a domain $\Delta \subset \mathbb{C}^2$,*
2. *$(1,0) \in \Delta$ and $\mathcal{R}e(\Phi(1,0)) > 0$,*
3. *$\mathcal{R}e(\Phi(\lambda + (1-\lambda)u_2 i, \nu_1)) \leqslant \lambda \quad (0 \leqslant \lambda < 1)$*
   if $(\lambda + (1-\lambda)u_2 i, \nu_1) \in \Delta$ and $\nu_1 \leqslant -\frac{1}{2}(1-\lambda)(1+u_2^2)$
   and
   $\mathcal{R}e(\Phi(\lambda + (1-\lambda)u_2 i, \nu_1)) \geqslant \lambda \quad (\lambda > 1)$
   if $(\lambda + (1-\lambda)u_2 i, \nu_1) \in \Delta$ and $\nu_1 \geqslant \frac{1}{2}(1-\lambda)(1+u_2^2)$.

Hence, if $p(z) \in \mathfrak{P}$ for $(p(z), z\mathcal{D}_q p(z)) \in \Delta$ and $\mathcal{R}e(\Phi(p(z), z\mathcal{D}_q p(z))) > \lambda$ for $z \in \mathfrak{U}$, then $\mathcal{R}e(p(z)) > \lambda$ in $\mathfrak{U}$.

3 Main Results

Henceforth, unless otherwise declared, it is assumed that all parameters are as declared in (10). Our results are presented as follows.

Theorem 1 $\mathfrak{R}_q^{n+1}(\mu,\beta,\tau;\lambda) \subset \mathfrak{R}_q^n(\mu,\beta,\tau;\lambda)$.

Proof. Suppose that $f(z) \in \mathfrak{R}_q^{n+1}(\mu,\beta,\tau;\lambda)$, then using (7),

$$\mathfrak{D}_q^{n+1,\mu,\beta,\tau} f(z) = z\tau\mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z)) - z\tau(\beta-\mu) + (1+(\beta-\mu-1)\tau)\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z)$$

so that

$$\mathcal{D}_q(\mathfrak{D}_q^{n+1,\mu,\beta,\tau} f(z)) = z\tau\mathcal{D}_q^2(\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z)) + (1+\tau(\beta-\mu))\mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z)) - \tau(\beta-\mu)$$

and

$$\mathcal{D}_q(\mathfrak{D}_q^{n+1,\mu,\beta,\tau} f(z)) = z\tau\mathcal{D}_q p(z) + (1+\tau(\beta-\mu))p(z) - \tau(\beta-\mu) \tag{11}$$

where $p(z) = \mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z))$. Now consider RHS of (11) and define the complex-valued function

$$\Phi(u,\nu) = (1+\tau(\beta-\mu))p(z) + z\tau\mathcal{D}_q p(z) - \tau(\beta-\mu) = (1+\tau(\beta-\mu))u + \tau\nu - \tau(\beta-\mu)$$

so that in view of Lemma 2, it is clear that condition 1 holds true. For condition 2,

$$\Phi(1,0) = 1 + \tau(\beta-\mu) - \tau(\beta-\mu) = 1 \implies \mathcal{R}e\,\Phi(1,0) > 0.$$

For condition 3,

$$\Phi(\lambda + (1-\lambda)u_2 i, \nu_1) = \lambda + \tau(\beta-\mu)(\lambda-\tau) + \tau\nu_1 + (1-\lambda)(1+\tau(\beta-\mu))u_2 i$$

implies that

$$\begin{aligned}\mathcal{R}e\,\Phi(\lambda + (1-\lambda)u_2 i, \nu_1) &= \lambda + \tau(\beta-\mu)(\lambda-1) + \tau\nu_1 \\ &= \lambda - \tau(\beta-\mu)(1-\lambda) - \frac{1}{2}\tau(1-\lambda)(1+u_2^2) \leqslant \lambda\end{aligned}$$

for $\nu_1 \leqslant -\frac{1}{2}(1-\lambda)(1+u_2^2)$. Thus since $\Phi(u,\nu)$ satisfies conditions $1-3$ in Lemma 2, therefore,

$$\mathcal{R}e\,p(z) = \mathcal{R}e\,\mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau} f(z)) > \lambda \quad (0 \leqslant \lambda < 1)$$

which implies the theorem.

Theorem 2 $\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda) \subset \mathfrak{R}_q(\lambda)$, *the family of q-bounded turning functions of order* λ.

Proof. From Theorem 1, it can easily be seen that for each $n \in \mathbb{N} \cup \{0\}$,

$$\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda) \subset \mathfrak{R}_q^{n-1}(\mu,\beta,\tau;\lambda) \subset \cdots\cdots \subset \mathfrak{R}_q^0(\mu,\beta,\tau;\lambda).$$

Observe that by (6) and (3),

$$\mathcal{D}_q(\mathfrak{D}_q^{0,\mu,\beta,\tau} f(z)) = \mathcal{D}_q f(z)$$

so that

$$\mathcal{R}e\{\mathcal{D}_q(\mathfrak{D}_q^{0,\mu,\beta,\tau} f(z))\} = \mathcal{R}e\,\mathcal{D}_q f(z) > \lambda, \quad 0 \leqslant \lambda < 1$$

which implies that

$$f(z) \in \mathfrak{R}_q^0(\mu,\beta,\tau;\lambda) = \mathfrak{R}_q(\lambda),$$

hence the proof is complete.

Corollary 1 *By Lemma 1 and Theorem 2,* $f(z) \in \mathfrak{R}_q^n(\mu,\beta,\tau;\lambda)$ *are univalent functions.*

Theorem 3 *Let $f(z) \in \mathfrak{A}$, then $f \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$ if, and only if*

$$\sum_{k=2}^{\infty}(1+([k]_q+\beta-\mu-1)\tau)^n[k]_q|a_k| \leqslant 1-\lambda$$

with extremal function

$$f(z) = z + \frac{1-\lambda}{(1+([k]_q+\beta-\mu-1)\tau)^n[k]_q}z^k, \quad k=\{2,3,4,\cdots\}. \tag{12}$$

Proof. If $f(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$, then it is sufficient to show that

$$|\mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau}f(z)) - 1| < 1-\lambda, \quad 0 \leqslant \lambda < 1$$

which implies that

$$\left|\sum_{k=2}^{\infty}(1+([k]_q+\beta-\mu-1)\tau)^n[k]_q a_k z^{k-1}\right| < 1-\lambda. \tag{13}$$

Now since $\mathcal{R}e\, z \leqslant |z| < 1$ and if we allow for $z \longrightarrow 1$ on the real axis, then it implies that

$$\sum_{k=2}^{\infty}(1+([k]_q+\beta-\mu-1)\tau)^n[k]_q|a_k| \leqslant 1-\lambda. \tag{14}$$

Conversely, suppose (14) holds and allowing $|z| = 1$, then (13) simplifies to

$$\sum_{k=2}^{\infty}(1+([k]_q+\beta-\mu-1)\tau)^n[k]_q|a_k| \leqslant 1-\lambda \tag{15}$$

and by maximum modulus theorem, the proof is complete.

Corollary 2 *Let $f \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$, then*

$$|a_k| \leqslant \frac{1-\lambda}{(1+([k]_q+\beta-\mu-1)\tau)^n[k]_q}, \quad k=\{2,3,\ldots\}.$$

Theorem 4 *Let $f(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$, then*

$$r - r^2\frac{1-\lambda}{(1+(q+\beta-\mu)\tau)^n(1+q)} \leqslant |f(z)| \leqslant r + r^2\frac{1-\lambda}{(1+(q+\beta-\mu)\tau)^n(1+q)}$$

with extremal function

$$f(z) = z + \frac{1-\lambda}{(1+(q+\beta-\mu)\tau)^n(1+q)}z^2. \tag{16}$$

Proof. From Theorem 3 and since $\phi = (1 + ([k]_q + \beta - \mu - 1)\tau)$ is an increasing function, then

$$\sum_{k=2}^{\infty} |a_k| \leqslant \frac{1-\lambda}{(1 + ([2]_q + \beta - \mu - 1)\tau)^n [2]_q}. \tag{17}$$

Also from (1),

$$|f(z)| = \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \leqslant r + r^2 \sum_{k=2}^{\infty} |a_k|, \quad |z| = r < 1 \tag{18}$$

or

$$|f(z)| = \left| z - \sum_{k=2}^{\infty} a_k z^k \right| \geqslant r - r^2 \sum_{k=2}^{\infty} |a_k|, \quad |z| = r < 1. \tag{19}$$

Thus, putting (17) into both (18) and (19) and using (5) complete the proof.

Corollary 3 *Let $f(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$, then*

$$1 - 2r\frac{1-\lambda}{(1 + (q + \beta - \mu)\tau)^n (1+q)} \leqslant |f'(z)| \leqslant 1 + 2r\frac{1-\lambda}{(1 + (q + \beta - \mu)\tau)^n (1+q)}.$$

with (16) *as extremal function.*

Theorem 5 *Let $w = f(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$, then $f(z)$ maps the unit disk $\mathfrak{U}$ onto a domain that covers the disk*

$$|w| < \frac{(1 + (q + \beta - \mu)\tau)^n (1+q) - (1-\lambda)}{(1 + (q + \beta - \mu)\tau)^n (1+q)}.$$

The result is sharp for the function in (16).

Proof. From Theorem 4, if for $z \in \mathfrak{U}$, $|z| = r \longrightarrow 1$, then

$$|w| < 1 - \frac{1-\lambda}{(1 + (q + \beta - \mu)\tau)^n (1+q)}$$

and simplifying the inequality completes the proof.

Theorem 6 *$\mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$ is a convex set.*

Proof. Let $g_1(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ and $g_2(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$ be in $\mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$, then it suffices to show that

$$G(z) = t_1 g_1(z) + t_2 g_2(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda), \quad 0 \leqslant t_1, t_2 \leqslant 1, \ t_1 + t_2 = 1. \tag{20}$$

Now (20) simplifies to

$$G(z) = z + \sum_{k=2}^{\infty}(t_1 a_k + t_2 b_k)z^k$$

so that

$$\mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau}G(z)) = 1 + \sum_{k=2}^{\infty}(1 + ([k]_q + \beta - \mu - 1)\tau)^n [k]_q (t_1 a_k + t_2 b_k)z^{k-1}.$$

Now

$$\begin{aligned}\mathcal{R}e\{\mathcal{D}_q(\mathfrak{D}_q^{n,\mu,\beta,\tau}G(z))\} =& \mathcal{R}e\bigg(1 + \sum_{k=2}^{\infty}(1 + ([k]_q + \beta - \mu - 1)\tau)^n [k]_q t_1 a_k z^{k-1}\bigg) \\ &+ \mathcal{R}e\bigg(1 + \sum_{k=2}^{\infty}(1 + ([k]_q + \beta - \mu - 1)\tau)^n [k]_q t_2 b_k z^{k-1}\bigg) \\ >& 1 + \lambda(t_1 + t_2) - (t_1 + t_2)\end{aligned}$$

and since $t_1 + t_2 = 1$, hence the proof is complete.

(q,γ)-Neighborhood of Analytic Functions

In this investigation, we introduce the (q,γ)-neighborhood of $f \in \mathfrak{A}$ as follows. Let $0 < q < 1$, then the (q,γ)-neighborhood of $f \in \mathfrak{A}$ is defined by

$$\mathcal{N}_{q,\gamma}(f;g) = \left\{g(z) : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathfrak{A},\ \sum_{k=2}^{\infty}[k]_q |a_k - b_k| \leqslant \gamma,\ \gamma \geqslant 0\right\} \tag{21}$$

and in particular, for the identity function $\xi(z) = z \in \mathfrak{A}$, the (q,γ)-neighborhood is defined by

$$\mathcal{N}_{q,\gamma}(\xi;g) = \left\{g(z) : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathfrak{A},\ \sum_{k=2}^{\infty}[k]_q |b_k| \leqslant \gamma,\ \gamma \geqslant 0\right\}. \tag{22}$$

The idea of neighborhood property originated from the work of Goodman [5] and was later generalized by Ruscheweyh [11]. Note that $\lim\limits_{q\uparrow 1}\mathcal{N}_{q,\gamma}(\xi;g) = \mathcal{N}_{\gamma}(\xi;g)$ and $\lim\limits_{q\uparrow 1}\mathcal{N}_{q,\gamma}(f;g) = \mathcal{N}_{\gamma}(f;g)$.

Theorem 7 *Let*

$$\gamma = \frac{1-\lambda}{(1 + (q + \beta - \mu)\tau)^n},$$

then $\mathfrak{R}_q^n(\mu,\beta,\tau;\lambda) \subset \mathcal{N}_{q,\gamma}(\xi;g)$.

Proof. For $f(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$ and from Theorem 3, we have

$$\sum_{k=2}^{\infty}[k]_q|a_k| \leqslant \frac{1-\lambda}{(1+([2]_q+\beta-\mu-1)\tau)^n}$$

and by using (5), we have

$$\sum_{k=2}^{\infty}[k]_q|a_k| \leqslant \frac{1-\lambda}{(1+(q+\beta-\mu)\tau)^n} = \gamma \tag{23}$$

so that by (22), the proof is complete.

Theorem 8 *Let $g(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$ and let*

$$\sigma = 1 - \frac{\gamma(1+(q+\beta-\mu)\tau)^n}{(1+q)(1+(q+\beta-\mu)\tau)^n - (1-\lambda)}, \tag{24}$$

then $\mathcal{N}_{q,\gamma}(g) \subset \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$.

Proof. Suppose that $f(z) \in \mathcal{N}_{q,\gamma}(g)$, then from (22) and using (5),

$$\sum_{k=2}^{\infty}[k]_q|a_k - b_k| \leqslant \gamma \implies \sum_{k=2}^{\infty}|a_k - b_k| \leqslant \frac{\gamma}{1+q}. \tag{25}$$

Also, since $g(z) \in \mathfrak{R}_q^n(\mu, \beta, \tau; \lambda)$, then from (23) and using (5),

$$\sum_{k=2}^{\infty}[k]_q|b_k| \leqslant \frac{1-\lambda}{(1+(q+\beta-\mu)\tau)^n} \implies \sum_{k=2}^{\infty}|b_k| \leqslant \frac{1-\lambda}{(1+q)(1+(q+\beta-\mu)\tau)^n}. \tag{26}$$

To proof this result, it suffices to show, by using (25) and (26) that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=2}^{\infty}|a_k - b_k|}{1 - \sum_{k=2}^{\infty}|b_k|} \leqslant \frac{\gamma(1+(q+\beta-\mu)\tau)^n}{(1+q)(1+(q+\beta-\mu)\tau)^n - (1-\lambda)} = 1 - \sigma.$$

where for σ in (24), the proof is complete.

Remark 3 *Letting $q \uparrow 1$ and $\beta = \mu$ in some of our Theorems, then we will have some of the results of Al-Oboudi [2] and letting $q \uparrow 1$ in some of our Theorems, then we will have some of the results of Opoola [10].*

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