General Mathematics Vol. 29, No. 2 (2021), 137–168
DOI:10.2478/gm-2021-0020

sciendo

Asymptotic Results of a Nonparametric Conditional Quantile Estimator in the Single Functional Index Modeling under Random Censorship [1]

Nadia Kadiri, Abbes Rabhi, Fatima Akkal

Abstract

The main objective of this paper is to estimate non-parametrically the quantiles of a conditional distribution based on the single-index model in the censorship model when the sample is considered as an independent and identically distributed (i.i.d.) random variables. First of all, a kernel type estimator for the conditional cumulative distribution function (*cond-cdf*) is introduced. Afterwards, we give an estimation of the quantiles by inverting this estimated *cond-cdf*, the asymptotic properties are stated when the observations are linked with a single-index structure. Simulation study is also presented to illustrate the validity and finite sample performance of the considered estimator. Finally, the estimation of the functional index via the pseudo-maximum likelihood method is discussed, but not tackled.

2010 Mathematics Subject Classification: 62G20, 62G99, 62M10.
Key words and phrases: Conditional quantile, censored data, functional random variable, kernel estimator, nonparametric estimation, probabilities of small balls, single index model.

1 Introduction

Over the last two decades, functional data modeling was highly considered in the statistical literature. The new generation of electronic devices is now allowing practitioners to have access to data continuously (over time and/or space). This change in the data structure raised several challenging statistical problems in analyzing curve-type data. In practice, one can observe functional data in several fields such as climatology, stock market analysis, signal processing, satellite images analysis, etc.

[1] *Received 4 October, 2020*
Accepted for publication (in revised form) 28 July, 2021

For an overview of the functional data analysis , the readers can refer to Ramsay and Silverman [34, 35], Masry [31], Ferraty and Vieu [19], Ezzahrioui and Ould-Saïd [13], Aspirot [4], Cuevas [10], Hsing and Eubank [25], Goia and Vieu [22] and the references therein.

Modeling of the relationship between two concomitant variables is one of the most relevant tasks in functional data analysis. In this paper, we are interested in using semi-parametric approach to model the conditional distribution of a real-valued response variable given an infinite dimensional (functional) covariate. A dimension reduction approach, based on single index model, is used in this paper to estimate the conditional quantiles whenever the response variable is affected by a right censorship phenomenon.

In multivariate statistics, where the vector of covariates belongs to a high dimensional but finite space, single index model represents one of the well-known semi-parametric models which allows to reduce the dimensionality of the covariate space and, at the same time, gives flexibility in describing the relationship between the response and the covariate through an unknown link function. Indeed, single index model reduces the curse of dimensionality effect known in pure nonparametric estimation methods and it is always seen as a reasonable compromise between nonparametric and parametric models.

Consequently, reducing the dimensionality can be of great interest in practice. For instance, it allows to increase the prediction accuracy and to improve the interpretability of the relationship between a response variable with a vector of covariates. For more details about the advantages of single index models in finite dimensional space setting, the reader can be referred to Härdle and Marron [23], Hristache *et al.* [24], Xia and Härdle [41] and the references therein.

The extension of the single index model to the functional data framework was introduced first in Ferraty *et al.* [16] to estimate semi-parametrically the regression operator where the response variable is real-valued and the covariate is a functional random variable. The single functional index model (SFIM) assumes that a functional explanatory variable acts on a scalar response only through its projection on one functional direction. The SFIM was intensively extended to estimate several statistical parameters describing the shape of the conditional distribution. For instance, Aït Saidi *et al.* [1] were interested in using SFIM to estimate the regression operator and suggest to use a cross-validation procedure allowing the estimated the unknown link function as well as the unknown functional index. Attaoui and Boudiaf [2] and Attaoui and Ling [3] studied, respectively, the estimation of the conditional density and the conditional cumulative distribution function based on a SFIM and assuming that the data satisfy a strong mixing condition. Kadiri *et al.* [26] studied the asymptotic properties of the kernel-type estimator of the conditional quantiles when the response is right-censored and the data is sampled from strong mixing process. Goia and Vieu [21] presented a methodology allowing to approximate in a semi-parametric way the unknown regression operator through a single index approach and by taking possible structural changes into account. Ma [30] proposed a functional single index model to study the link between a scalar response and

multiple functional predictors.

Conditional quantile function is a powerful tool to describe the conditional distribution of a random variable Y given another variable X. From an application point of view, the conditional median, for instance, is an interesting alternative predictor to the conditional mean thanks to its robustness to the presence of outliers. Moreover, conditional quantiles are also useful in providing probabilistic forecasts instead of pointwise forecasts. For the above theoretical and application reasons, the statistical community were interested in the estimation of the conditional quantiles. In a finite-dimensional spaces ($Y \in \mathbb{R}$ and $X \in \mathbb{R}$ or $\mathbb{R}^d, d \geq 2$), a parametric approach was introduced in Koenker and Bassett [27], whereas nonparametric and semi-parametric estimation were introduced, for instance, in Chaudhuri *et al.* [9], Kong and Xia [28] and the references therein.

When the response variable Y is real-valued and the covariate variable X takes values in an infinite-dimensional space, Cardot *et al.* [8] considered a linear model of regression on quantiles and proposed a spline estimator of the functional coefficient that minimizes a penalized L^1-type criterion. Ferraty *et al.* [17] introduced a nonparametric estimator of the conditional quantile, defined as the inverse of the conditional distribution function, when data are dependent. Ezzahrioui and Ould-Saïd [14] established the asymptotic normality of the kernel conditional quantile estimator under an α-mixing assumption. Then, Dabo-Niang and Laksaci [11] provided the consistency in L^p-norm of the conditional quantile estimator for functional dependent data. More recently, the above works were extended to the case where the response variable is not completely observed. For instance, Ould-Saïd and Djabrane [32] established the asymptotic distribution of the conditional quantiles when the response variable is left-truncated and the data satisfy an α-mixing condition.

In this paper we are interested in the semi-parametric estimation of the conditional quantile based on single functional index structure. We suppose that the response variable is right-censored, the covariate is functional. To the best of our knowledge, such framework combining functional data, censored data, with single-index structure was not considered in the statistical literature to estimate the conditional quantiles.

The content of this paper is organized as follows. Section 2 introduces a single functional index model (SFIM) for conditional quantile estimation under random right-censorship assumption. In Sections 3 and 4, we discuss the asymptotic properties, including the pointwise, the uniform almost-complete convergence (with rates) and the asymptotic distribution of the suggested estimator. Section 5 illustrates those asymptotic properties through some simulated data. Finally, Section 6 is devoted to display the technical proofs of the asymptotic results.

In the censoring case, instead of observing the lifetime T, we observe the censored lifetime of items under study. That is, assuming that $(T_i)_{i\geq 1}$ is a stationary sequence of lifetimes which satisfy some kind of dependency and $(C_i)_{i\geq 1}$ is a sequence of i.i.d censoring rv with common unknown continuous *df* G, where $Y_i = \min\{T_i, C_i\}$ and $\delta_i = \mathbf{1}_{T_i \leq C_i}$.

To ensure the identifiability of the model, we suppose that $(T_i)_i$ are independent

of $(C_i)_i$. Let $\{(Y_i, \delta_i, X_i)\}$ be a sequence of strictly stationary random vectors where $X_{i\geq 1}$ is valued in infinite dimensional semi-metric vector space, and Y_i is real valued. To follow the convention in biomedical studies and as indicated before, we assume that $(C_i)_{i\geq 1}$ and $\{(X_i, T_i)_{i\geq 1}\}$ are independent; this condition is plausible whenever the censoring is independent of the patients modality. Furthermore this condition permits to get an unbiased Kernel estimator.

2 The model and the estimates

2.1 The functional nonparametric framework

Consider a random pair (X, T) where T is valued in $\mathbb{R}$ and X is valued in some infinite dimensional Hilbertian space $\mathcal{H}$ with scalar product $< \cdot, \cdot >$. Let $(X_i, T_i)_{i=1,\ldots,n}$ be the statistical sample of pairs which are identically distributed like (X, T), but not necessarily independent. Henceforward, X is called functional random variable *f.r.v.*. Moreover, we consider $d_\theta(\cdot, \cdot)$ a semi-metric associated with the single index $\theta \in \mathcal{H}$ defined by $d_\theta(x_1, x_2) := | < x_1 - x_2, \theta > |$, for x_1 and x_2 in $\mathcal{H}$. Let x be fixed in $\mathcal{H}$ and let $F(\theta, t, x)$ be the conditional cumulative distribution function (*cond-cdf*) of T given $< \theta, X >=< \theta, x >$, specifically:

$$\forall t \in \mathbb{R},\ F(\theta, t, x) = \mathbb{P}(T \leq t | < X, \theta >=< x, \theta >).$$

Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of T given $< \theta, X >$.

Next, in all what follows, we assume only smoothness restrictions for the *cond-cdf* $F(\theta, \cdot, x)$ through nonparametric modelling (Section 2.4). We suppose also that $(X_i, T_i)_{i\in\mathbb{N}}$ is an α-mixing sequence, which is one among the most general mixing structures. The α-mixing condition together with the functional approach allow to deal with continuous time processes (see Section 4 for instance).

In our infinite dimensional purpose, we use the terminology *functional nonparametric*, where the word *functional* referees to the infinite dimensionality of the data and where the word *nonparametric* referees to the infinite dimensionality of the model. Such *functional nonparametric* statistics is also called *doubly infinite dimensional* (see Ferraty and Vieu Ferraty and Vieu [18], for more details). We also use the terminology *operational statistics* since the target object to be estimated (the *cond-cdf* $F(\theta, \cdot, x)$) can be viewed as a nonlinear operator.

2.2 The estimators

The kernel estimator $F_n(\theta, \cdot, x)$ of $F(\theta, \cdot, x)$ is presented as follows:

$$F_n(\theta, t, x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - T_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{1}$$

where K is a kernel function, H a cumulative distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) a sequence of positive real numbers. Note that using similar ideas, Roussas [36] introduced some related estimates but in the special case when X is real, while Samanta [37] produced previous asymptotic study.

Such an estimator is unique as soon as H is an increasing continuous function. Such an approach has been largely used in the case where the variable X is of finite dimension (see *e.g* Whang and Zhao [40], Cai [7], Zhou and Liang [42] or Gannoun *et al.* [20]).

In practice, in particularly medical applications, we can be in the presence of censored variables. This problem is usually modeled by considering a positive C variable called censorship, and the observed random variables are not couples (T_i, X_i), but rather (Y_i, δ_i, X_i) where $Y_i = \min(T_i,\ C_i)$ and $\delta_i = \mathbf{1}_{T_i \leq C_i}$. In the following we will use the notations F_1^X and f_1^X to describe the conditional distribution function and the conditional density C knowing the covariate X.

Such censorship models have been amply studied in the literature for real or multi-dimensional random variables, and in nonparametric frameworks the kernel techniques are particularly used (see Tanner and Wong [38], Padgett [33], Lecoutre and Ould-Saïd [29] and Van Keilegom and Veraverbeke [39], for a necessarily non-exhaustive sample of literature in this area).

The objective of this paragraph is to adapt these ideas under functional random variable X, and build a kernel type estimator of the conditional distribution $F(\theta, \cdot, X)$ adapted for censored samples. Thus we can reformulate the expression (1) as follows:

$$\widetilde{F}(\theta, t, x) = \frac{\displaystyle\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(h_K^{-1}(< x - X_i, \theta >)\right) H\left(h_H^{-1}(t - T_i)\right)}{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}. \tag{2}$$

In practice $\bar{G}(\cdot) = 1 - G(\cdot)$ is unknown, hence it is impossible to use the estimator (2). Then, we replace $\bar{G}(\cdot)$ by its Kaplan and Meier (1958) estimate $\bar{G}_n(\cdot)$ given by

$$\bar{G}_n(t) = 1 - G_n(t) = \begin{cases} \displaystyle\prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbf{1}_{\{Y_{(i)} \leq t\}}}, & \text{if } t < Y_{(n)}; \\ 0, & \text{if } t \geq Y_{(n)}, \end{cases}$$

where $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ are the order statistics of Y_i and $\delta_{(i)}$ is the non-censoring indicator corresponding to $Y_{(i)}$.

Therefore feasible estimator of the conditional distribution function $F(\theta, \cdot, x)$ is

given by

$$\widehat{F}(\theta,t,x)=\frac{\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K\left(h_K^{-1}(<x-X_i,\theta>)\right)H\left(h_H^{-1}(t-T_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}(<x-X_i,\theta>)\right)}. \tag{3}$$

2.3 Assumptions on the functional variable

Let N_x be a fixed neighborhood of x and let $B_\theta(x,h)$ be the ball of center x and radius h, namely $B_\theta(x,h)=\{f\in\mathcal{H}:\ 0<|<x-f,\theta>|<h\}$ and $\mathcal{S}_{\mathbb{R}}$ is a fixed compact of $\mathbb{R}^+$.

For any df L, let $\tau_L:=\sup\{t,\text{ such that }L(t)<1\}$ be its support's right endpoint. Let S be a compact set such that $\zeta_\theta(\gamma,x)\in S\cup(-\infty,\tau]$, where $\tau<\min(\tau_G,\tau_F)$. Assume that, $(C_i)_{i\geq 1}$ are independent and let's consider the following hypotheses:

(H1) $\forall h>0,\ \mathbb{P}(X\in B_\theta(x,h))=\phi_{\theta,x}(h)>0.$

2.4 The nonparametric model

As usually in nonparametric estimation, we suppose that the *cond-cdf* $F(\theta,\cdot,x)$ verifies some smoothness constraints. Let b_1 and b_2 be two positive numbers; such that:

(H2) $\forall(x_1,x_2)\in N_x\times N_x,\ \forall(t_1,t_2)\in\mathcal{S}^2_{\mathbb{R}}$,

(i) $|F(\theta,t_1,x_1)-F(\theta,t_2,x_2)|\leq C_{\theta,x}\left(\|x_1-x_2\|^{b_1}+|t_1-t_2|^{b_2}\right)$,

(ii) $\int_{\mathbb{R}}tf(\theta,t,x)dt<\infty$ for all $\theta,x\in\mathcal{H}$.

(H3) $F(\theta,\cdot,x)$ is l-times continuously differentiable in some neighborhood of $\zeta_\theta(\gamma,x)$.

(H4) $\forall(x_1,x_2)\in N_x\times N_x,\ \forall(t_1,t_2)\in\mathcal{S}^2_{\mathbb{R}}$,

$$|F^{(l)}(\theta,t_1,x_1)-F^{(l)}(\theta,t_2,x_2)|\leq C_{\theta,x}\left(\|x_1-x_2\|^{b_1}+|t_1-t_2|^{b_2}\right),$$

where, for any positive integer l, $F^{(l)}(\theta,z,x)$ denotes its lth derivative (*i.e.* $\left.\frac{\partial^l F(\theta,t,x)}{\partial t^l}\right|_{t=z}$).

3 Asymptotic study

This part of paper is devoted, to the theoretical analysis, we start it by giving the almost complete convergence (*a.co.*) of the estimated conditional distribution function $\widehat{F}(\theta,x,t)$. After that, we will focus on the rate of convergence. Concerning the notations, as soon as possible, C and C' will denote generic constants. Moreover, from now on, h_H (resp. h_K) is a sequence which tends to 0 when n goes to infinity.

3.1 Pointwise almost complete convergence

Let's begin with the statement of an almost complete convergence property [2]. To this end, we need some assumptions concerning the kernel estimator $\widehat{F}(\theta,\cdot,x)$:

(H5) (i) $\forall(t_1,t_2)\in\mathbb{R}^2,\ |H(t_1)-H(t_2)|\leq C|t_1-t_2|$ and $\int|t|^{b_2}H^{(1)}(t)dt<\infty$, where, for all $l\in\mathbb{N}^*$, $H^{(l)}(t)=\left.\frac{d^lH(z)}{dz^l}\right|_{z=t}$ and $\lim_{n\to\infty}n^{\varsigma}h_H=\infty$, for some $\varsigma>0$.

(ii) The support of $H^{(1)}$ is compact and $\forall l\geq j$, $H^{(l)}$ exists and is bounded.

(H6) The restriction of H to the set $\{u\in\mathbb{R},\ H(u)\in(0,1)\}$ is a strictly increasing function.

(H7) K is a positive bounded function with support $[0,1]$ such that $\forall u\in(0,1)\quad 0<K(u)$.

Remark 1 • *(H6) insures the existence of $\widehat{\zeta}_\theta(\gamma,x)$, while (H5) insures its uniqueness.*

• *(H1)-(H4) and (H7) are standard assumptions for the distribution conditional estimation in single functional index model, which have been adopted by Bouchentouf* et al. *[6] for i.i.d case.*

First observe that (3) can be rewritten as:

$$\widehat{F}(\theta,t,x)=\frac{\widehat{F}_N(\theta,t,x)}{\widehat{F}_D(\theta,x)}.$$

Theorem 1 *Suppose that the hypotheses (H1)-(H3), (H5)-(i), (H6) are satisfied and if*

$$\frac{\log n}{n\,\phi_{\theta,x}(h_K)}\underset{n\to\infty}{\longrightarrow}0,$$

then we have

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}(\theta,t,x)-F(\theta,t,x)\right| \;=\; \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,\phi_{\theta,x}(h_K)}}\right).$$

Proof. Consider now, for $i=1,\ldots,n$, in what follows, let's denote:

$$K_i(\theta,x)=K(h_K^{-1}(<x-X_i,\theta>)),\ H_i(t)=H\left(h_H^{-1}\left(t-Y_i\right)\right),\ \bar{G}_i=\bar{G}(Y_i).$$

[2] Recall that a sequence $(S_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable S, if for any $\epsilon>0$, we have $\sum_n\mathbb{P}(|S_n-S|>\epsilon)<\infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, [5])

$$\widehat{F}_N(\theta,t,x) = \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K_i(\theta,x)H_i(t)$$

$$\widetilde{F}_N(\theta,t,x) = \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K_i(\theta,x)H_i(t)$$

$$\widehat{F}_D(\theta,x) = \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)$$

$$\Delta_i(x,\theta) = \frac{K(h_K^{-1}(< x - X_i, \theta >))}{\mathbb{E}K_1(\theta,x)}.$$

The proof is based on the following decomposition, valid for any $t \in \mathcal{S}_{\mathbb{R}}$:

$$\begin{aligned}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}(\theta,t,x) - F(\theta,t,x)\right| \leq\ & \frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widehat{F}_N(\theta,t,x) - \widetilde{F}_N(\theta,t,x)\right|\right\}\\ &+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{F}_N(\theta,t,x) - \mathbb{E}\widetilde{F}_N(\theta,t,x)\right|\right\}\\ &+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\mathbb{E}\widetilde{F}_N(\theta,t,x) - F(\theta,t,x)\right|\right\}\\ &+\frac{F(\theta,t,x)}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|1-\widehat{F}_D(\theta,x)\right|. \qquad (4)\end{aligned}$$

Finally, the proof of this theorem is a direct consequence of the following intermediate results.

Lemma 1 *Assume that either (H5)-(i) is satisfied together with under conditions (H6)-(H7) and if*

$$\left(\frac{\log\log n}{n}\right)^{1/2} = o\left(\phi_{\theta,x}(h_K)\right),$$

we have

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta,t,x)\right| = \mathcal{O}_{a.s.}\left(\frac{\log\log n}{n}\right), \quad \textit{for } l \geq 0,$$

where, for all $t \in \mathbb{R}$,

$$\widehat{F}^{(l)}(\theta,t,x) = \frac{h_H^{-l}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}_n(Y_i)}K(h_K^{-1}(< x - X_i, \theta >))H^{(l)}(h_H^{-1}(t-T_i))}{\sum_{i=1}^{n}K(h_K^{-1}(< x - X_i, \theta >))},$$

$$\widetilde{F}^{(l)}(\theta,t,x) = \frac{h_H^{-l}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K(h_K^{-1}(< x - X_i, \theta >))H^{(l)}(h_H^{-1}(t-T_i))}{\sum_{i=1}^{n}K(h_K^{-1}(< x - X_i, \theta >))}.$$

The following lemma shows the asymptotic bias term of $\widetilde{F}_N(\theta,t,x)$ and $\widehat{F}_D(\theta,x)$ as n tends to infinity.

Lemma 2 *Under hypotheses (H1), (H3) and (H5)-(i), we have as* $n \to \infty$

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\mathbb{E}\left[\widetilde{F}_N(\theta,t,x)\right]-F(\theta,t,x)\right| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right).$$

The following result deals with the variance term of the right-hand side of (4) which is expressed by: $\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{F}_N(\theta,t,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right|\right\}$. For $\widehat{F}_D(\theta,x)-\mathbb{E}\left[\widehat{F}_D(\theta,x)\right]$ the same arguments will be used with a slight difference.

Lemma 3 *Under the assumptions of Theorem 1 and if*

$$\left(\frac{\log\log n}{n}\right)^{1/2} = o\left(\phi_{\theta,x}(h_K)\right),$$

we have

(i) $\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x) = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,\phi_{\theta,x}(h_K)}}\right).$

(ii) $\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,\phi_{\theta,x}(h_K)}}\right).$

We conclude the proof of the Theorem 1 by making use the inequality (4), in conjunction with lemmas Lemma 1, Lemma 2 and Lemma 3.

The proof of these latter will be collected in Section 5.

3.2 Pointwise almost complete rate of convergence

In this section we study the rate of convergence of our conditional distribution estimator $\widehat{F}(\theta,t,x)$. The main objective is the estimation of the conditional cumulative distribution of Y given $< X,\theta >=< x,\theta >$, denoted by $F(\theta,\cdot,x)$.

It is well known that, in nonparametric statistics, this latter provides an alternative approach to study the links between Y and X and it can be also used, in single index modeling, to estimate the functional index θ if it is unknown.

Theorem 2 *Suppose that the hypotheses (H1), (H3)-(H7) are satisfied and if*

$$\lim_{n\to\infty}\frac{\log n}{n\,h_H^{2l-1}\,\phi_{\theta,x}(h_K)} = 0.$$

Then we have

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}^{(l)}(\theta,t,x)-F^{(l)}(\theta,t,x)\right| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H^{2l-1}\,\phi_{\theta,x}(h_K)}}\right).$$

Proof. It is omitted as it very similar to that of Theorem 1, we use again the same kind of decomposition as (4):

$$\begin{aligned}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{F}^{(l)}(\theta,t,x)-F^{(l)}(\theta,t,x)\right| &= \frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widehat{F}_N^{(l)}(\theta,t,x)-\widetilde{F}_N^{(l)}(\theta,t,x)\right|\right\}\\ &+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\widetilde{F}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x)\right|\right\}\\ &+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left\{\left|\mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x)-F^{(l)}(\theta,t,x)\right|\right\}\\ &+\frac{F^{(l)}(\theta,t,x)}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|1-\widehat{F}_D(\theta,x)\right|. \qquad (5)\end{aligned}$$

This proof is very similar to the one of Theorem 1. Indeed the proof of the Theorem 2 by making use the decomposition (5), in conjunction of first part of Lemma 3 with lemmas Lemma 1, Lemma 4 and Lemma 5.

Lemma 4 *Under the hypotheses (H1) and (H4)-(H7) we have*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}|F^{(l)}(\theta,t,x)-\mathbb{E}(\widetilde{F}_N^{(l)}(\theta,t,x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right).$$

Lemma 5 *Under the assumptions of Theorem 2, we have*

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widetilde{F}_N^{(l)}(\theta,t,x)-\mathbb{E}(\widetilde{F}_N^{(l)}(\theta,t,x)|)=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H^{2l-1}\,\phi_{\theta,x}(h_K)}}\right).$$

3.3 Conditional quantile estimation

In this part we study the rate of convergence of our conditional quantile estimator $\widehat{\zeta}_\theta(\gamma,x)$. Now, let $\zeta_\theta(\gamma,x)$ be the γ-order quantile of the distribution of T given $<\theta,X>=<\theta,x>$. From the *cond-cdf* $F(\theta,\cdot,x)$, the general definition of the γ-order quantile is given as:

$$\zeta_\theta(\gamma,x)=\inf\{t\in\mathbb{R}:\ F(\theta,t,x)\geq\gamma\},\qquad\forall\gamma\in(0,1).$$

In order to simplify our framework and to focus on the main interest of our paper (the functional feature of $<\theta,X>$), we assume that $F(\theta,\cdot,x)$ is strictly increasing and continuous in a neighborhood of $\zeta_\theta(\gamma,x)$. This is insuring that the conditional quantile $\zeta_\theta(\gamma,x)$ is uniquely defined by:

$$\zeta_\theta(\gamma,x)\ =\ F^{-1}(\theta,\gamma,x). \qquad (6)$$

As a by-product of (6) and (1), it is easy to derive an estimator $\zeta_{\theta,n}(\gamma,x)$ of $\zeta_\theta(\gamma,x)$:

$$\zeta_{\theta,n}(\gamma,x)\ =\ F_n^{-1}(\theta,\gamma,x).$$

Then a natural estimator of $\zeta_\theta(\gamma, x)$ is given by

$$\begin{aligned}\widehat{\zeta}_\theta(\gamma, x) &= \widehat{F}^{-1}(\theta, \gamma, x) \\ &= \inf\{t \in \mathbb{R} : \ \widehat{F}(\theta, t, x) \geq \gamma\},\end{aligned}$$

which satisfies

$$\widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x)\ , x) = \gamma. \tag{7}$$

Corollary 1 *Under the hypotheses of Theorem 1, thus we have:*

$$\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x) \underset{n \to \infty}{\longrightarrow} 0, \ a.co.$$

Proof.[Proof of Corollary 1]

The proof is based on the pointwise convergence of $\widehat{F}(\theta, \cdot, x)$.

First of all, let's note that because of (H5)-(i) and (H6), $\widehat{F}(\theta, \cdot, x)$ is a continuous and strictly increasing function. So, we have:

$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, \forall y, |\widehat{F}(\theta, t, x) - \widehat{F}\left(\theta, \zeta_\theta(\gamma, x), x\right)| \leq \delta(\epsilon) \Rightarrow |t - \zeta_\theta(\gamma, x)| \leq \epsilon.$$

This leads us to write $\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0$,

$$\begin{aligned}\mathbb{P}\left(|\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x)| > \epsilon\right) &\leq \mathbb{P}\left(|\widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x), x) - \widehat{F}(\theta, \zeta_\theta(\gamma, x), x)| \geq \delta(\epsilon)\right) \\ &= \mathbb{P}\left(|F(\theta, \zeta_\theta(\gamma, x), x) - \widehat{F}(\theta, \zeta_\theta(\gamma, x), x)| \geq \delta(\epsilon)\right),\end{aligned}$$

since (6) and (7) and is implying that $\widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x), x) = \gamma = F(\theta, \zeta_\theta(\gamma, x), x)$.

Moreover, we have,

$$\begin{aligned}|F(\theta, \widehat{\zeta}_\theta(\gamma, x), x) - F(\theta, \zeta_\theta(\gamma, x), x)| &= |F(\theta, \widehat{\zeta}_\theta(\gamma, x), x) - \widehat{F}(\theta, \widehat{\zeta}_\theta(\gamma, x), x)| \\ &\leq \sup_{t \in \mathcal{S}_\mathbb{R}} |\widehat{F}(\theta, t, x) - F(\theta, t, x)|.\end{aligned}$$

The consistency of $\widehat{\zeta}_\theta(\gamma, x)$ follows then immediately from Theorem 1 and the following inequality

$$\sum_{n \geq 1} \mathbb{P}\left(|\zeta_{\theta,n}(\gamma, x) - \zeta_\theta(\gamma, x)| \geq \epsilon\right) \leq \sum_{n \geq 1} \mathbb{P}\left(\sup_{t \in \mathcal{S}_\mathbb{R}} \left|\widehat{F}(\theta, t, x) - F(\theta, t, x)\right| \geq \delta(\epsilon)\right).$$

Because this kind of result is stronger than the previous one, we have to introduce some additional assumptions. As it is usual in conditional quantiles estimation, the rate of convergence can be linked with the flatness of the *cond-cdf* $F(\theta, \cdot, x)$ around the conditional quantile $\zeta_\theta(\gamma, x)$. This is one reason why we introduced hypotheses (H3) and (H4). But a complementary way to take into account this local shape constraint is to suppose that:

$$\text{(H8)}\quad \begin{cases} F^{(m)}(\theta, t_\theta(\alpha), x) = 0, & \text{if; } 1 \leq m < l \\ \text{and } F^{(l)}(\theta, \cdot, x), & \text{is uniformly continuous on } \mathcal{S}_{\mathbb{R}}; \\ \text{such that,} & |F^{(l)}(\theta, t_\theta(\alpha), x)| > C > 0. \end{cases}$$

Because we focus on the local behavior of $F(\theta, \cdot, x)$ around $\zeta_\theta(\gamma, x)$ via its derivatives, that leads us to consider the successive derivatives of $\widehat{F}(\theta, \cdot, x)$ and subsequently some assumptions on the successive derivatives of the cumulative kernel H.

Corollary 2 *Under the hypotheses (H1)-(H8), we have*

$$\widehat{\zeta}_\theta(\gamma, x) - \zeta_\theta(\gamma, x) = \mathcal{O}\left(\left(h_K^{b_1} + h_H^{b_2}\right)^{\frac{1}{l}}\right) + \mathcal{O}_{a.co}\left(\left(\frac{\log n}{n\,\phi_{\theta,x}(h_K)}\right)^{\frac{1}{2l}}\right).$$

Proof.[Proof of Corollary 2] The proof is based on the Taylor expansion of $\widehat{F}(\theta, \cdot, x)$ at $\zeta_\theta(\gamma, x)$ and on the use of (H8):

$$\begin{aligned} & \widehat{F}\left(\theta, \zeta_\theta(\gamma, x), x\right) - \widehat{F}\left(\theta, \widehat{\zeta}_\theta(\gamma, x), x\right) \\ = & \sum_{m=1}^{l-1} \frac{(\zeta_\theta(\gamma, x) - \widehat{\zeta}_\theta(\gamma, x))^{m-1}}{m!} \widehat{F}^{(m)}(\theta, \zeta_\theta(\gamma, x), x) \\ & + \frac{(\zeta_\theta(\gamma, x) - \widehat{\zeta}_\theta(\gamma, x))^{l}}{l!} \widehat{F}^{(l)}(\theta, \zeta_\theta^*, x), \\ = & \sum_{m=1}^{l-1} \frac{(\zeta_\theta(\gamma, x) - \widehat{\zeta}_\theta(\gamma, x))^{m-1}}{m!} \Big(\widehat{F}^{(m)}(\theta, \zeta_\theta(\gamma, x), x) - \\ & \qquad F^{(m)}(\theta, \zeta_\theta(\gamma, x), x)\Big) \\ & + \frac{(\zeta_\theta(\gamma, x) - \widehat{\zeta}_\theta(\gamma, x))^{l}}{l!} \widehat{F}^{(l)}(\theta, \zeta_\theta^*, x), \end{aligned}$$

where $\min(\zeta_\theta(\gamma, x), \widehat{\zeta}_\theta(\gamma, x)) < \zeta_\theta^* < \max(\zeta_\theta(\gamma, x), \widehat{\zeta}_\theta(\gamma, x))$. Suppose now that we have the following result.

Because of Corollary 1, Theorem 2 and (H8), we have:

$$\widehat{F}^{(l)}(\theta, \zeta_\theta^*, x) \underset{n \to \infty}{\longrightarrow} F^{(l)}(\theta, \zeta_\theta(\gamma), x) \neq 0, \ a.co.,$$

then we derive

$$\begin{aligned} \left|\zeta_\theta(\gamma, x) - \widehat{\zeta}_\theta(\gamma, x)\right|^l = & \ \mathcal{O}\left(\widehat{F}(\theta, \zeta_\theta(\gamma, x), x) - F(\theta, \zeta_\theta(\gamma, x), x)\right) \\ & + \mathcal{O}\Big(\sum_{m=1}^{l-1} (\zeta_\theta(\gamma, x) - \widehat{\zeta}_\theta(\gamma, x))^m (\widehat{F}^{(m)}(\theta, \zeta_\theta(\gamma, x), x) - \\ & \qquad F^{(m)}(\theta, \zeta_\theta(\gamma, x), x))\Big), \ a.co. \end{aligned}$$

Now, comparing the convergence rates given in Theorem 1 and Theorem 2, we get

$$\left|\zeta_\theta(\gamma,x)-\widehat{\zeta}_\theta(\gamma,x)\right|^l \;=\; \mathcal{O}\left(\widehat{F}(\theta,t,x)-F(\theta,t,x)\right),\ a.co.$$

Thus, first part of Lemma 3 with lemmas Lemma 4 and Lemma 5 allow us to get the claimed result.

4 Uniform almost complete convergence and rate of convergence

In this section we derive the uniform version of Theorem 1 and Theorem 2. The study of the uniform consistency is a crucial tool for studying the asymptotic properties of all estimates of the functional index if is unknown. In the multivariate case, the uniform consistency is a standard extension of the pointwise one, nevertheless, in the studied case, it requires some additional tools and topological conditions (see Ferraty *et al.* [15]). Consequently, coupled with the conditions introduced antecedently, we need the following ones. Firstly, consider

$$\mathcal{S}_{\mathcal{H}}\subset\bigcup_{k=1}^{d_n^{\mathcal{S}_{\mathcal{H}}}}B_\theta(x_k,r_n)\ \text{ and }\ \Theta_{\mathcal{H}}\subset\bigcup_{q=1}^{d_n^{\Theta_{\mathcal{H}}}}B_\theta(\theta_q,r_n),$$

with x_k (resp. θ_q) $\in\mathcal{H}$ and $r_n, d_n^{\mathcal{S}_{\mathcal{H}}}, d_n^{\Theta_{\mathcal{H}}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity and suppose that $d_n^{\mathcal{S}_{\mathcal{H}}}$, $d_n^{\Theta_{\mathcal{H}}}$ are the minimal numbers of open balls with radius r_n in $\mathcal{H}$, which are required to cover $\mathcal{S}_{\mathcal{H}}$ and $\Theta_{\mathcal{H}}$.

4.1 Conditional distribution estimation

In this subpart we propose to study the uniform almost complete convergence of our estimator (2), to this end, we need to state the following assumptions

(A1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x\in\mathcal{S}_{\mathcal{H}}$ and $\forall\theta\in\Theta_{\mathcal{H}}$,

$$0<C\phi(h)\le\phi_{\theta,x}(h)\le C'\phi(h)<\infty\ \text{ and }\ \exists\eta_0>0,\ \forall\eta<\eta_0,\ \phi'(\eta)<C.$$

(A2) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(x)-K(y)|\le C\|x-y\|.$$

(A3) $\forall(t_1,t_2)\in\mathcal{S}_{\mathbb{R}}\times\mathcal{S}_{\mathbb{R}},\forall(x_1,x_2)\in\mathcal{S}_{\mathcal{H}}\times\mathcal{S}_{\mathcal{H}}$ and $\forall\theta\in\Theta_{\mathcal{H}}$,

$$|F(\theta,t_1,x_1)-F(\theta,t_2,x_2)|\le C\left(\|x_1-x_2\|^{b_1}+|t_1-t_2|^{b_2}\right).$$

(A4) For some $\nu \in (0,1)$, $\lim_{n\to\infty} n^{\nu} h_H = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_\mathcal{H}}$ and $d_n^{\Theta_\mathcal{H}}$ satisfy:

$$\begin{cases} (i)\ \dfrac{(\log n)^2}{n\phi(h_K)} < \log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}} < \dfrac{n\phi(h_K)}{\log n}, \\ (ii)\ \displaystyle\sum_{n=1}^{\infty} n^{1/2b_2} (d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}})^{1-\beta} < \infty \text{ for some } \beta > 1, \\ (iii)\ n\phi(h_K) = \mathcal{O}\left((\log n)^2\right). \end{cases}$$

(A5) $\forall (t_1,t_2) \in \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}, \forall (x_1,x_2) \in \mathcal{S}_\mathcal{H} \times \mathcal{S}_\mathcal{H}$ and $\forall \theta \in \Theta_\mathcal{H}$,

$$|F^{(l)}(\theta,t_1,x_1) - F^{(l)}(\theta,t_2,x_2)| \leq C\left(\|x_1 - x_2\|^{b_1} + |t_1 - t_2|^{b_2}\right).$$

(A6) For some $\nu \in (0,1)$, $\lim_{n\to\infty} n^{\nu} h_H = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{\mathcal{S}_\mathcal{H}}$ and $d_n^{\Theta_\mathcal{H}}$ satisfy:

$$\begin{cases} (i)\ \dfrac{(\log n)^2}{n h_H^{2l-1}\phi(h_K)} < \log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}} < \dfrac{n h_H^{2l-1}\phi(h_K)}{\log n}, \\ (ii)\ \sum_{n=1}^{\infty} n^{(3\nu+1)/2} (d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}})^{1-\beta} < \infty, \quad for \quad some \quad \beta > 1, \\ (iii)\ n h_H^{2l-1}\phi(h_K) = \mathcal{O}\left((\log n)^2\right) \end{cases}$$

In what follows, denote

$$\Lambda_i(x,\theta) = \frac{1}{h_K \phi(h_K)} \mathbf{1}_{B_\theta(x,h) \cup B_\theta(x_{k(x)},h)}(X_i),$$
$$\Omega_i(x,\theta) = \frac{1}{h_K \phi(h_K)} \mathbf{1}_{B_\theta(x_{k(x)},h) \cup B_{\theta_{q(\theta)}}(x_{k(x)},h)}(X_i),$$
$$\Delta_i\left(x_{k(x)}, \theta_{q(\theta)}\right) = \frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)},$$

$$\begin{aligned} \Gamma_i &= \frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)} \frac{\delta_i}{\bar{G}(Y_i)} H\left(h_H^{-1}(v_{k_t} - T_i)\right) \\ &- \mathbb{E}\left(\frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)} \frac{\delta_i}{\bar{G}(Y_i)} H\left(h_H^{-1}(v_{k_t} - T_i)\right)\right) \end{aligned}$$

and

$$\begin{aligned} \Gamma_i^{(l)} &= \frac{1}{h_H^l} \frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)} \frac{\delta_i}{\bar{G}(Y_i)} H^{(l)}\left(h_H^{-1}(v_{k_t} - T_i)\right) \\ &- \frac{1}{h_H^l} \mathbb{E}\left(\frac{K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)}{\mathbb{E}K\left(h_K^{-1} < x_{k(x)} - X_i, \theta_{q(\theta)} >\right)} \frac{\delta_i}{\bar{G}(Y_i)} H^{(l)}\left(h_H^{-1}(v_{k_t} - T_i)\right)\right). \end{aligned}$$

Theorem 3 *Under hypotheses (H1)-(H2), (H5)-(H7) and (A1)-(A4), we have*

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widehat{F}(\theta,t,x)-F(\theta,t,x)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

Proof. Obviously, the proof of these results, can be deduced from the decomposition (5) and the following intermediate results which are only uniform version of Theorem 2.

Lemma 6 *Under the conditions (H1)-(H2) and (H5)-(H7), we have*

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|F^{(l)}(\theta,t,x)-\mathbb{E}\widehat{F}_N^{(l)}(\theta,t,x)\right|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right).$$

Lemma 7 *Under the assumptions of Theorem 3, we have:*

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\left|\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)\right|=\mathcal{O}_{a.co.}\left(\frac{\sqrt{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}}{n\phi(h_K)}\right).$$

Lemma 8 *Under the assumptions of Theorem 3, we have:*

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\widehat{F}_N^{(l)}(\theta,t,x)-\mathbb{E}\widehat{F}_N^{(l)}(\theta,t,x)=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{nh_H^{2l-1}\phi(h_K)}}\right).$$

Corollary 3 *Under the assumptions of Theorem 3, we have:*

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}}|\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x)|\underset{n\to\infty}{\longrightarrow}0,\ a.co.$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}|\widehat{\zeta}_\theta(\gamma,x)-\zeta_\theta(\gamma,x)|=\mathcal{O}\left(\left(h_K^{b_1}+h_H^{b_2}\right)^{\frac{1}{l}}\right)+\mathcal{O}_{a.co.}\left(\left(\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{nh_H^{2l-1}\phi(h_K)}\right)^{\frac{1}{2l}}\right).$$

Remark 2

1. *These results extends Theorem 3 or Theorem 4 given in Bouchentouf* et al. *[6] to the mixing case. The effect of covariance structure for dependence case on the convergence rate is reflected in the last term. Specially, if the functional single-index is fixed, it is easy to prove the following corollary that are similar the one given in Bouchentouf* et al. *[6].*
2. *Theorem 3 are obtained when the functional parameter θ is known. However, if the functional parameter θ is unknown, it would be very important for practical using of the model and would be more interesting.*

5 Simulation study

To study the behavior of our conditional quantiles estimator, we consider in this part two examples of simulation. In the first one, we compare our model CESIM (3) (fonctional single index model with censored data) with that of CNPFDA (8) (censored nonparametric functional data analysis) and in the latter, knowing the distribution of the regression model (the distribution is known and usual), we look to the behavior of our estimator of the conditional distribution function with respect to this distribution.

$$\widehat{F}_n(t,x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}_n(Y_i)} K\left(h_K^{-1} d(x,X_i)\right) H\left(h_H^{-1}(t-T_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(x,X_i)\right)}. \tag{8}$$

Let (X, T) be $E \times \mathbb{R}$-valued random elements, where E is some semi-metric abstract space. Denote by $d(\cdot,\cdot)$ a semi-metric associated with the space E. Suppose now that we observe a sequence $(X_i, T_i)_{i\geq 1}$ of copies of (X, T) that is assumed to be stationary and independent and identically distributed. Let us consider the following regression model where the covariate is a curve and the response is a scalar:

$$T_i = R(X_i) + \epsilon_i,\ i = 1, \ldots, n,$$

where ϵ_i is the error supposed to be generated by an autoregressive model defined by

$$\epsilon_i = \frac{1}{\sqrt{2}}\epsilon_{i-1} + \eta_i,\ i = 1, \ldots, n,$$

with $(\eta_i)_i$ a sequence of i.i.d. random variables normally distributed with a variance equal to 0.1.

The functional covariate X is supposed to be a diffusion process defined on $[0, 2\pi/3]$ and generated by the following equation:

$$X(t) = 2 - \cos\left(W\left(t - \frac{2\pi}{3}\right)\right), \quad t \in]0, 2\pi/3[,$$

where W is an α process generated by $W_i = \frac{2}{9} + \epsilon$, $i = 1, \ldots, 215$ and ϵ are i.i.d $\mathcal{N}(0, 1)$ (standard normal distribution) independently from W_i (The standard normal W_0 is independently generated). We carried out the simulation with a 215-sample of the curve X which is represented by the following graphs (see Figure 1). Here is a nonlinear regression function is considered such that

$$R(X) = \frac{1}{4}\exp\left\{2 - 1/\left(\int_0^1 X'(t)dt\right)^2\right\}.$$

On the other hand, n i.i.d. random variables $(C_i)_i$ are simulated through the exponential distribution $\mathcal{E}(1.5)$.

Given $X = x$, $T \rightsquigarrow \mathcal{N}(R(x), 0.2)$, and thus, the conditional median, the conditional mode and the conditional mean functions will coincide and will be equal to $R(x)$, for any fixed x.

The computation of our estimator is based on the observed data $(X_i, Y_i, \delta_i)_{i=1,\ldots,n}$ where $Y_i = min(T_i, C_i)$; $\delta_i = \mathbb{I}_{\{T_i \leq C_i\}}$ and the single index θ which is unknown and has to be estimated.

In practice this parameter can be selected by cross-validation approach (see Aït Saidi *et al.* [1]).

In this passage it may be that one can select the real-valued function $\theta(t)$ among the eigenfunctions of the covariance operator $\mathbb{E}[(X' - \mathbb{E}X') < X', \cdot >_{\mathcal{H}}]$ where $X(t)$ is a diffusion processes defined on a real interval $[a, b]$ and $X'(t)$ its first derivative (see Attaoui & Ling [3]). So for a chosen training sample $\mathcal{L}$, by applying the principal component analysis (PCA) method, the computation of the eigenvectors of the covariance operator estimated by its empirical covariance operator: $\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X_i' - \mathbb{E}X')\,{}^t(X_i' - \mathbb{E}X')$, will be the one best approximation of our functional parameter θ. Now, let us denote $\theta^\star$the first eigenfunction corresponding to the first higher eigenvalue, which will replace θ during the simulation step.

In practice, some tuning parameters have to be fixed: the kernel $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{2}\left(1 - u^2\right)\mathbb{I}_{[0,1]}$and the cumulative df $H(u) = \displaystyle\int_{-\infty}^{u} \frac{3}{4}\left(1 - z^2\right)\mathbb{I}_{[-1,1]}(z)\,dz$. Take into account of the smoothness of the curves $X_i(t)$(Figure 1), we choose the distance in $\mathcal{H}$ as:

$$d(\chi_1, \chi_2) = \left(\int_0^1 \left(\chi_1'(t) - \chi_2'(t)\right)^2 dt\right)^{1/2},$$

as semi-metric.

In the following graphs, the covariance operator for $\mathcal{L} = \{1, \ldots, 215\}$ gives the discretization of the first eigenfunction θ (presented by a continuous curve), twenty and all the eigenfunctions $\theta_i(t)$(Fig. 2, 3, 4).

Now for simplifying the implementation of our methodology, we take the bandwidths $h_H \sim h_K = h$, where h will be chosen by the cross-validation method on the k-nearest neighbors (see Ferraty and Vieu [19], p. 102) and we denote by $\theta^\star$ the first eigenfunction corresponding to the first higher eigenvalue of the empirical covariance operator:

$$\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X_i' - \mathbb{E}X')\,{}^t(X_i' - \mathbb{E}X'),$$

and we follow the steps:

Step 1. Compute the inner product: $< \theta^*, X_1 >, \ldots, < \theta^*, X_{215} >$, generate independently the variables $\varepsilon_1, \ldots, \varepsilon_{215}$, then simulate the response variables $Y_i = r(< \theta^*, X_i >) + \varepsilon_i$, where $r(< \theta^*, X_i >) = \exp(10(< \theta^*, X_i > -0.05))$ and generate independently the variables $\varepsilon_1, \ldots, \varepsilon_{215}$.

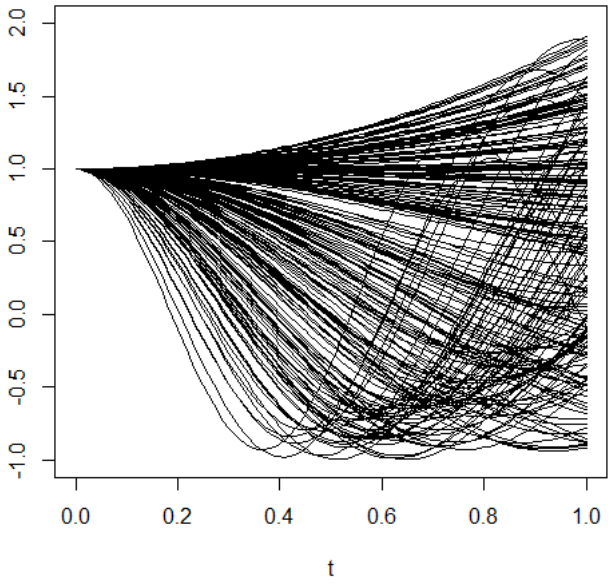


Figure 1: A sample of curves $\{X_i(t),\, t \in [0,1]\}_{i=1,\ldots,215}$

Step 2. For each i in the test sample $\mathcal{J} = 161, \ldots, 215$,we compute: $\widehat{Y}_i = \widehat{\zeta}_{\theta^\star}(\gamma, X_i)$ and $\widehat{Y}_i = \widehat{\zeta}(\gamma, X_i)$, where

$$\zeta(\gamma, \chi) = \inf\{y \in \mathbb{R},\ F^{\chi}(y) \geq \zeta\},$$

and

$$\widehat{F^{\chi}}(y) = \frac{\sum_{i=1}^{n} K\left(h^{-1} d(\chi, X_i)\right)\left(h^{-1}(y - Y_i)\right)}{\sum_{i=1}^{n} K\left(h^{-1} d(\chi, X_i)\right)} \quad \forall y \in \mathbb{R}.$$

Step 3. Finally, we present the results by plotting the predicted values versus the true values and compute the mean squared error (MSE):

$$MSE = \frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} \left(Y_j - \widehat{Y}_j\right)^2.$$

Then using the learning sample to compute the estimator of $\widehat{Y}_i = \widehat{\zeta}_{\theta^*}(\gamma, x)$ and $\widehat{Y}_{ni} = \widehat{\zeta}(\gamma, x)$ for $j = \{151, \ldots, 215\}$.

Finally we show the results by plotting the true values versus the predicted values for the MSE under censored data for both estimators with different censored rate (CR) 3 and 5 which are defined as :

$$FSIM.MSE = \frac{1}{55} \sum_{i=161}^{215} (Y_i - \widehat{Y}_i)^2,$$

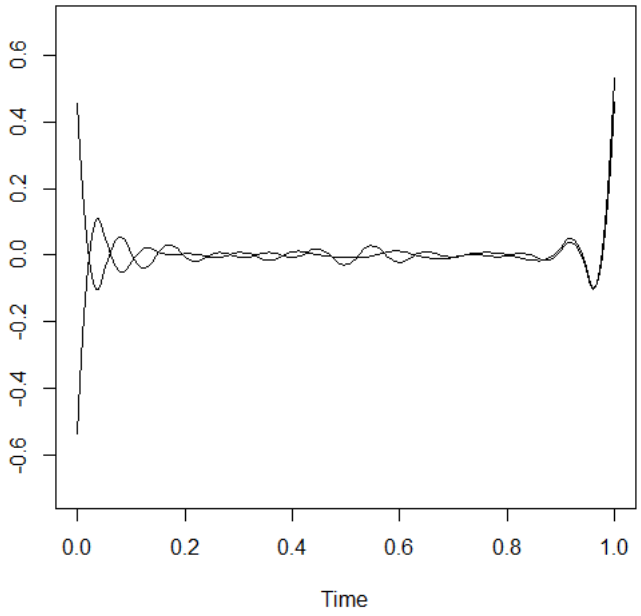


Figure 2: The curves $\theta_{i=1,2}(t_j)$, $t_j \in [0, 1]$

$$NPFDA.MSE = \frac{1}{55}\sum_{i=161}^{215}(Y_i - \widehat{Y}_{ni})^2,$$

respectively.

We see that the sum of mean square error (MSE) of our method (Censored-Single-Index-Method) is less than the one of the Non-Parametric-Functional-Data-Analysis (NPFDA). This is confirmed by the following graphs, when we compare the conditional quantile by censored single index methods (CFSIM) against the conditional quantile by nonparametric functional data analysis (NPFDA). Our estimator is so acceptable.

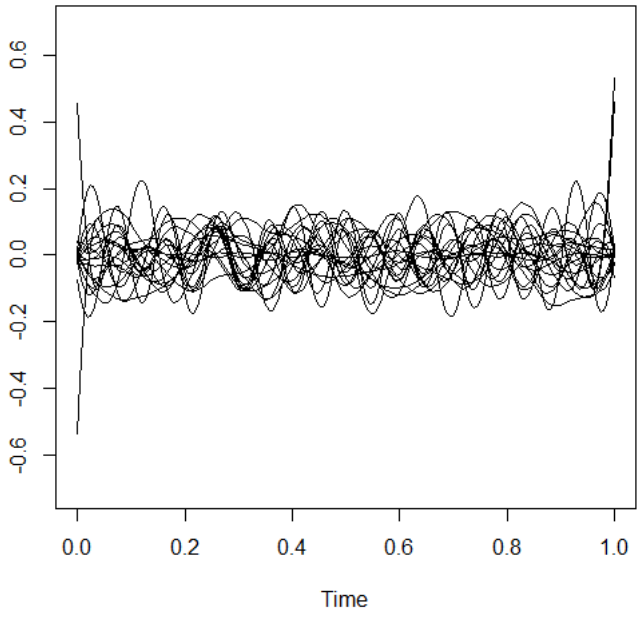


Figure 3: The curves $\theta_{i=1,\ldots,20}(t_j)$, $t_j \in [0, 1]$

6 Proofs of technical lemmas

Proof. [Proof of Lemma 1] Let

$$\begin{aligned}
&|\widehat{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta,t,x)| \\
\leq\ & \frac{h_H^{-l}}{n\mathbb{E}K_1(\theta,x)} \sum_{i=1}^{n} \left| \frac{\delta_i}{\bar{G}_n(Y_i)} K_i(\theta,x) H_i^{(l)}(t) - \frac{\delta_i}{\bar{G}(Y_i)} K_i(\theta,x) H_i^{(l)}(t) \right| \\
\leq\ & \frac{h_H^{-l}}{n\mathbb{E}K_1(\theta,x)} \sum_{i=1}^{n} |\delta_i K_i(\theta,x) H_i(t)| \left| \frac{1}{\bar{G}_n(Y_i)} - \frac{1}{\bar{G}(Y_i)} \right| \\
\leq\ & \frac{h_H^{-l}}{\phi_{\theta,x}(h_K)} \frac{C}{\bar{G}_n(\tau_G)\bar{G}(\tau_G)} \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\bar{G}_n(t) - \bar{G}(t)| \frac{1}{n} \sum_{i=1}^{n} \delta_i |K_i(\theta,x) H_i^{(l)}(t)|.
\end{aligned}$$

In conjunction with the SLLN and the LIL on the censoring law (see Deheuvels and Einmahl [12]), hypotheses (H1), (H6) and $\left(\frac{\log\log n}{n}\right)^{1/2} = o\left(\phi_{\theta,x}(h_K)\right)$ complete the proof.

Proof. [Proof of Lemma 2]

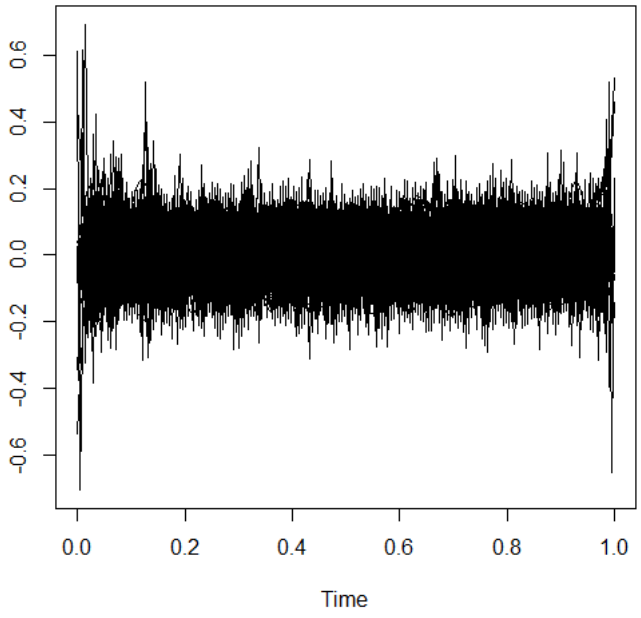


Figure 4: The curves $\theta_{i=1,\ldots,215}(t_j)$, $t_j \in [0, 1]$

The asymptotic behavior of bias term is standard, we have

$$\begin{aligned}
\mathbb{E}\widetilde{F}_N(\theta,t,x) - F(\theta,t,x) &= \frac{1}{\mathbb{E}K_i(x,\theta)}\mathbb{E}\Big(\frac{\delta_i}{\bar{G}(Y_i)}K_i(x,\theta)H_i(t)\Big) - F(\theta,t,x) \\
(9) \quad &= \frac{1}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\Big(\frac{\delta_i}{\bar{G}(Y_i)}K_i(x,\theta)\Big[\mathbb{E}\Big(H_i(t)|<X_1,\theta>\Big)\Big]\Big) - F(\theta,t,x).
\end{aligned}$$

Integrating by parts and using the fact that H is a *cdf* and the use a double conditioning with respect to T_1, we can easily get

$$\begin{aligned}
I &= \mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)|<X_1,\theta>\right) \\
&= \mathbb{E}\left(\mathbb{E}\left[\frac{\mathbf{1}_{T_1\leq C_1}}{\bar{G}(T_1)}H\left(\frac{t-T_1}{h_H}\right)|<X_1,\theta>,T_1\right]\right) \\
&= \mathbb{E}\left(\frac{1}{\bar{G}(T_1)}H\left(\frac{t-T_1}{h_H}\right)\mathbb{E}\left[\mathbf{1}_{T_1\leq C_1}|T_1\right]|<X_1,\theta>\right) \\
&= \mathbb{E}\left[H\left(\frac{t-T_1}{h_H}\right)|<X_1,\theta>\right]
\end{aligned}$$

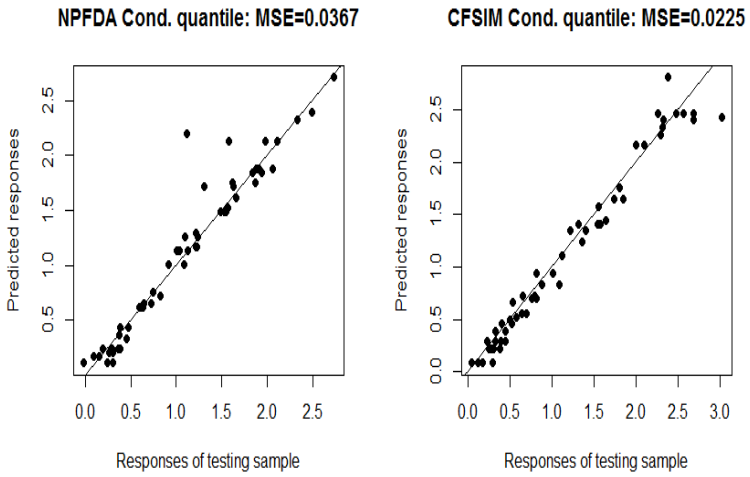


Figure 5: Comparison between NPFDA and CFSIM with CR$\sim$ 3%

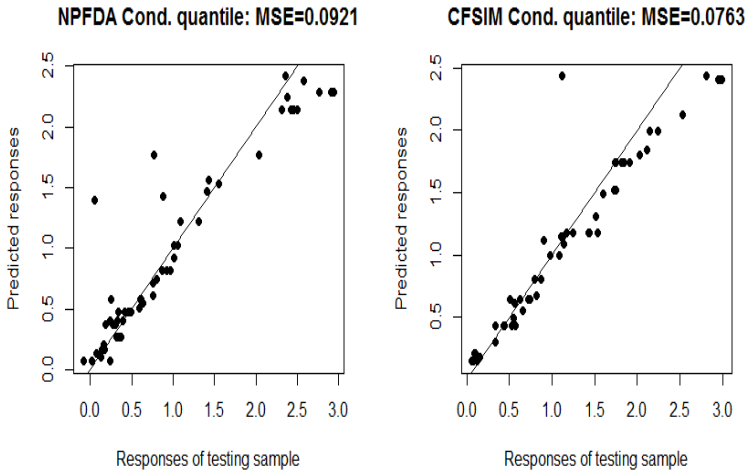


Figure 6: Comparison between NPFDA and CFSIM with CR$\sim$ 18%

$$
\begin{aligned}
&= \int_{\mathbb{R}} H\left(\frac{t-u}{h_H}\right) f(\theta,u,X_1)du,\\
&= \int_{\mathbb{R}} H\left(\frac{t-u}{h_H}\right) dF(\theta,u,X_1),\\
&= \int_{\mathbb{R}} H^{(1)}\left(\frac{t-u}{h_H}\right) F(\theta,u,X_1)du,\\
&= \int_{\mathbb{R}} H^{(1)}(v)F(\theta,t-vh_H,X_1)dv,\\
&= \int_{\mathbb{R}} H^{(1)}(v)\left(F(\theta,t-vh_H,X_1)-F(\theta,t,x)\right)dv + F(\theta,t,x)\int_{\mathbb{R}} H^{(1)}(v)dv,
\end{aligned}
$$

we can write, because of (H2) and (H5)-(i):

$$\begin{aligned} I &\leq C_{x,\theta}\int_{\mathbb{R}} H^{(1)}(v)\left(h_K^{b_1}+|v|^{b_2}h_H^{b_2}\right)dv + F(\theta,t,x) \\ &= \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+F(\theta,t,x). \end{aligned}$$

Combining this last result with (9) allows us to achieve the proof.

Proof. [Proof of Lemma 3]

Using the compactness of $\mathcal{S}_{\mathbb{R}}$, we can write that $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{k=1}^{\tau_n}(z_k-l_n, z_k+l_n)$ with l_n and τ_n can be chosen such that $l_n = C\tau_n^{-1} \sim Cn^{-\varsigma-1/2}$. Taking $k_t = \arg\min_{\{z_1,\ldots,z_{\tau_n}\}}|t-z_k|$.

Thus, we have the following decomposition:

$$\begin{aligned} &\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right| \\ \leq\ &\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t,x)-\widetilde{F}_N(\theta,t_k,x)\right| \\ &+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t_k,x)-\mathbb{E}\widetilde{F}_N(\theta,t_k,x)\right| \\ &+\frac{1}{\widehat{F}_D(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\mathbb{E}\widetilde{F}_N(\theta,t_k,x)-\mathbb{E}\widetilde{F}_N(\theta,t,x)\right| \\ \leq\ &B_1+B_2+B_3. \end{aligned}$$

On the one hand, as the first and the third terms can be treated in the same manner, we deal only with first term. Making use of (H5)-(i) we get

$$\begin{aligned} &\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N(\theta,t,x)-\widetilde{F}_N(\theta,t_k,x)\right| \\ \leq\ &\frac{1}{n\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-\frac{\delta_i}{\bar{G}(Y_i)}H_i(t_k)\right||K_i(\theta,x)| \\ \leq\ &\frac{1}{n\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}H_i(t)-\frac{\delta_i}{\overline{G}_n(Y_i)}H_i(t_k)\right||K_i(\theta,x)| \\ \leq\ &\frac{C}{n\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\frac{|t-t_k|}{h_H}\times\left(\sum_{i=1}^{n}K_i(\theta,x)\left(\frac{1}{\bar{G}(Y_i)}-\frac{1}{\bar{G}_n(Y_i)}\right)\right) \\ \leq\ &\frac{Cl_n}{h_H\overline{G}_n(\tau_G)\bar{G}(\tau_G)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|G_n(t)-G(t)|\widehat{F}_D(\theta,x). \end{aligned}$$

Using $l_n = n^{-\varsigma-1/2}$ we obtain

$$A_1 \leq \frac{Cn^{-\varsigma-1/2}}{h_H\bar{G}_n(\tau_G)\bar{G}(\tau_G)}\left(\frac{\log n\log n}{n}\right)^{1/2},$$

and note that, because of (H6)-(i), we have

$$\frac{l_n}{h_H} = o\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right).$$

Thus, for n large enough, we have

$$B_1 \;=\; \mathcal{O}_{a.co}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right).$$

Following similar arguments, we can write

$$B_3 \;\leq\; B_1.$$

Concerning B_2, let us consider $\varepsilon = \epsilon_0\sqrt{\dfrac{\log n}{n\,\phi_{\theta,x}(h_K)}}$. Since for all $\epsilon_0 > 0$, we have that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\mathbb{E}\widetilde{F}_N(\theta,t_k,x) - \mathbb{E}\widetilde{F}_N(\theta,t,x)\right| > \varepsilon)\right)\\
\leq\;& \mathbb{P}\left(\max_{k\in\{1\ldots\tau_n\}}\left|\mathbb{E}\widetilde{F}_N(\theta,t_k,x) - \mathbb{E}\widetilde{F}_N(\theta,t,x)\right| > \varepsilon\right)\\
\leq\;& \tau_n\mathbb{P}\left(\left|\mathbb{E}\widetilde{F}_N(\theta,t_k,x) - \mathbb{E}\widetilde{F}_N(\theta,t,x)\right| > \varepsilon\right).
\end{aligned}$$

Applying Bernstein's exponential inequality to :

$$\Psi_i = \frac{1}{\mathbb{E}K_1(x,\theta)}\Big[\frac{\delta_i}{\bar{G}(Y_i)}K_i(x,\theta)H_i(t_k) - \mathbb{E}\Big(\frac{\delta_i}{\bar{G}(Y_i)}K_i(x,\theta)H_i(t_k)\Big)\Big].$$

Firstly, it follows from the fact that the Kernel K is bonded and $H \leq 1$, we get

$$\begin{aligned}
\mathbb{P}\left(\left|\widetilde{F}_N(\theta,t_k,x) - \mathbb{E}\widetilde{F}_N(\theta,t_k,x)\right| > \varepsilon\right) \;&\leq\; \mathbb{P}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\Psi_i\right| > \varepsilon\right)\\
&\leq\; 2n^{-c\varepsilon_0^2}.
\end{aligned}$$

Finally, by choosing ε_0 large enough, the proof can be concluded by the use of the Borel-Cantelli Lemma. the result can be easily deduced.

Proof.[Proof of Lemma 4] Using the same arguments in the proof of Lemma 2, replacing $F(\theta,t,x)$ (resp. $\widetilde{F}(\theta,t,x)$) with $F^{(l)}(\theta,t,x)$ (resp. $\widetilde{F}_N^{(l)}(\theta,t,x)$) and considering hypotheses (H4) and (H5) we get:

$$F^{(l)}(\theta,t,x) - \mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x) \;=\; \mathcal{O}\left(h_K^{b_1} + h_K^{b_2}\right).$$

Proof.[Proof of Lemma 5]

To get the asymptotic behavior of the quantity $\widetilde{F}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x)$, we comeback to the proof of Lemma 3, and we replace $F(\theta,t,x)$ (resp. $\widetilde{F}(\theta,t,x)$) with $F^{(l)}(\theta,t,x)$ (resp. $\widetilde{F}_N^{(l)}(\theta,t,x)$).

Note that (H5)-(ii) and (H9) permit to show that

$$\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}H^{(l)}\left(h_H^{-1}(t-Y_i)\right)\frac{\delta_j}{\bar{G}(Y_j)}H^{(l)}\left(h_H^{-1}(t-Y_j)\right)|(X_i,X_j)\right)=\mathcal{O}(h_H^2),$$

while (H4) imply that

$$\mathbb{E}\left(\frac{\delta_i}{\bar{G}(Y_i)}H^{(l)}\left(h_H^{-1}(t_\theta(\gamma)-Y_i)\right)|X_i\right)=\mathcal{O}(h_H).$$

Indeed, it can be found that

$$\widetilde{F}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x)=\frac{1}{n}\sum_{i=1}^{n}\Xi_i(\theta,t,x),$$

where

$$\Xi_i=h_H^{-l}\frac{\delta_i}{\bar{G}(Y_i)}H_i^{(l)}(t)\Delta_i(x,\theta)-\mathbb{E}h_H^{-l}\frac{\delta_i}{\bar{G}(Y_i)}H_i^{(l)}(t)\Delta_i(x,\theta),$$

has zero mean and satisfies $|\Xi_i(\theta,t,x)|\leq Ch_H^{-l}\phi_{\theta,x}^{-1}(h_K)$.

Now, since $H^{(l)}$ is bounded, then it allows us to use directly similar arguments of second part by Lemma 3, we obtain

$$\widetilde{F}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{F}_N^{(l)}(\theta,t,x)=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H^{2l-1}\,\phi_{\theta,x}(h_K)}}\right),$$

which leads directly to the result of Lemma 5

Proof. [Proof of Lemma 6] The proof can be completed following the same steps as Lemma 4.

Proof. [Proof of Lemma 7] Similar to the proof of Lemma 4.4 in Bouchentouf *et al.* [6], it can be completed easily. Here we omit its proof.

Proof. [Proof of Lemma 8] For all $x\in\mathcal{S}_{\mathcal{H}}$ and $\forall\theta\in\Theta_{\mathcal{H}}$, we set $k(x)=\arg\min_{k\in\{1\ldots d_n^{\mathcal{S}_{\mathcal{H}}}\}}\|x-x_k\|$ and $q(\theta)=\arg\min_{m\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}}\|\theta-\theta_q\|$ and by the compact property of $\mathcal{S}_{\mathbb{R}}\subset\mathbb{R}$, we have $\mathcal{S}_{\mathbb{R}}\subset\bigcup_{k=1}^{\tau_n}(v_k-l_n,v_k+l_n)$ with l_n and τ_n can be selected such as $l_n=\mathcal{O}\left(\tau_n^{-1}\right)=\mathcal{O}\left(n^{-(3\nu+1)/2}\right)$. Taking $k_t=\arg\min_{\{v_1,\ldots,v_{\tau_n}\}}|t-v_k|$.

Let us consider the following decomposition

$$\begin{aligned}
&\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widetilde{F}_N^{(l)}(\theta,t,x)-\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta,t,x)\right)\right| \\
\leq\ &\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\Big\{\left|\widetilde{F}_N^{(l)}(\theta,t,x)-\widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})\right| \\
&+\left|\widetilde{F}_N^{(l)}(\theta,t,x)-\widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})\right| \\
&+\left|\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})-\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})\right| \\
&+\left|\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})-\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})\right)\right| \\
&+\left|\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})\right)-\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})\right)\right| \\
&+\left|\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_N^{(l)}(\theta,t,x_{k(x)})\right)\right| \\
&+\left|\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})\right)-\mathbb{E}\left(\widetilde{F}_N^{(l)}(\theta,t,x)\right)\right|\Big\} \\
\leq\ &\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5+\Psi_6+\Psi_7.
\end{aligned}$$

• Concerning Ψ_3 and Ψ_5; by conditions (H5)-(ii) and (A6), boundness of K, we obtain

$$\begin{aligned}
&\left|\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})-\widetilde{F}_N^{(l)}(\theta_{q(\theta)},v_{k_t},x_{k(x)})\right| \\
\leq\ &\frac{1}{nh_H^l\mathbb{E}K_1(\theta,x)}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}K_i(\theta_{q(\theta)},x_{k(x)})\right| \\
&\qquad\left|H^{(l)}\left(\frac{t-Y_i}{h_H}\right)H^{(l)}\left(\frac{v_{k_t}-Y_i}{h_H}\right)\right| \\
\leq\ &\sup_{t\in\mathcal{S}_{\mathbb{R}}}C\frac{|t-v_{k_t}|}{h_H^{l+1}}\Big(\frac{1}{n\mathbb{E}(K_1(\theta_{q(\theta)},x_{k(x)}))} \\
&\qquad\sum_{i=1}^{n}\left|K_i(\theta_{q(\theta)},x_{k(x)})\frac{1}{\bar{G}(Y_i)}\right|\Big) \\
\leq\ &\frac{Cl_n}{\phi(h_K)h_H^{l+1}}=\mathcal{O}\left(\frac{l_n}{h_H^{l+1}\phi(h_K)}\right).
\end{aligned}$$

Now, the fact that $\lim_{n\to\infty}n^{\nu}h_H=\infty$, and choosing $l_n=n^{-(3\nu+1)/2}$ and using the second part of (A6), imply that

$$\frac{l_n}{h_H^{l+1}\phi(h_K)}=o\left(\sqrt{\frac{logn}{nh_H^{2l-1}\phi(h_K)}}\right),$$

as $n\to\infty$, therefore, it follows

$$\Psi_5\leq\Psi_3=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{nh_H^{2l-1}\phi(h_K)}}\right).$$

Concerning Ψ_4, let us consider $\varepsilon = \varepsilon_0\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{nh_H^{2l-1}\phi(h_K)}}$. Since

$$\begin{aligned}
&\mathbb{P}\left(\Psi_4 > \varepsilon_0\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{nh_H^{2l-1}\phi(h_K)}}\right)\\
= \;& \mathbb{P}\left(\max_{q\in\{1\ldots d_n^{\Theta_\mathcal{H}}\}}\max_{k\in\{1\ldots d_n^{\mathcal{S}_\mathcal{H}}\}}\max_{k_t\in\{1,2,\ldots,\tau_n\}}\left|\Gamma_i^{(l)} - \mathbb{E}\Gamma_i^{(l)}\right| > \varepsilon\right)\\
\leq \;& \tau_n d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}\mathbb{P}\left(\left|\Gamma_i^{(l)} - \mathbb{E}\Gamma_i^{(l)}\right| > \varepsilon\right).
\end{aligned}$$

Applying Bernstein's exponential inequality, under (H5) and (H7), to get $\forall q \leq d_n^{\Theta_\mathcal{H}}$, $\forall k \leq d_n^{\mathcal{S}_\mathcal{H}}$ and $\forall k_t \leq \tau_n$,

$$\mathbb{P}\left(|\Gamma_i - \mathbb{E}\Gamma_i| > \varepsilon\right) \leq 2(d_n^{\Theta_\mathcal{H}} d_n^{\mathcal{S}_\mathcal{H}})^{-C\varepsilon_0^2}.$$

Choosing $\tau_n \leq Cn^{(3\nu+1)/2}$, we get

$$\mathbb{P}\left(\Psi_4 > \varepsilon\right) \leq C^{'}\tau_n(d_n^{\Theta_\mathcal{H}} d_n^{\mathcal{S}_\mathcal{H}})^{1-C\varepsilon_0^2}.$$

Putting $C\varepsilon_0^2 = \beta$ and using (A6), to get

$$\Psi_4 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\, h_H^{2l-1}\,\phi(h_K)}}\right). \tag{10}$$

- Concerning Ψ_1 and Ψ_2

$$\begin{aligned}
&\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}|\widetilde{F}_N^{(l)}(\theta,t,x) - \widetilde{F}_N^{(l)}(\theta,t,x_{k(x)})|\\
\leq \;& \frac{1}{nh_H^l\mathbb{E}K_1(\theta,x)}\sup_{\theta\in\Theta_\mathcal{H}}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{t\in\mathcal{S}_\mathbb{R}}\sum_{i=1}^n\left|\frac{\delta_i}{\bar{G}(Y_i)}\right|\left|H_i^{(l)(t)}\right|\\
&\qquad\qquad |(K_i(\theta,x) - K_i(\theta,x_k))|\\
\leq \;& \frac{1}{nh_H^l\phi(h_K)}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\sum_{i=1}^n\left|\Delta_i(x,\theta) - \Delta_i\left(x_{k(x)},\theta\right)\right|\\
\leq \;& \frac{1}{h_H^l\phi(h_K)}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\sum_{i=1}^n\mathbf{1}_{B_\theta(x,h)\cup B_\theta(x_{k(x)},h)}(X_i)\\
\leq \;& \frac{C}{h_H^l}\sup_{x\in\mathcal{S}_\mathcal{H}}\sup_{\theta\in\Theta_\mathcal{H}}\frac{1}{n}\sum_{i=1}^n\Lambda_i(x,\theta).
\end{aligned}$$

Therefore, similar to the arguments for (10), we can get that
$\Psi_1 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{n\, h_H^{2l-1}\,\phi(h_K)}}\right)$.

$$
\begin{aligned}
&\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widetilde{F}_N^{(l)}(\theta,t,x)-\widetilde{F}_N^{(l)}(\theta_{q(\theta)},t,x_{k(x)})|\\
\leq\ & \frac{h_H^{-l}}{n\mathbb{E}K_1(\theta,x)}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}\sum_{i=1}^{n}\left|\frac{\delta_i}{\bar{G}(Y_i)}\right|\left|H_i^{(l)(t)}\right|\\
&\qquad\qquad\left|(K_i(\theta,x_k)-K_i(\theta_{q(\theta)},x_k))\right|\\
\leq\ & \frac{Ch_H^{-l}}{n\phi(h_K)}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\sum_{i=1}^{n}\left|\Delta_i(\theta,x_k)-\Delta_i\left(\theta_{q(\theta)},x_{k(x)}\right)\right|\\
\leq\ & \frac{Ch_H^{-l}}{\phi(h_K)}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{B_\theta(x_k,h)\cup B_{q(\theta)}(x_{k(x)},h)}(X_i)\\
\leq\ & \frac{C}{h_H^{l}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{\theta\in\Theta_{\mathcal{H}}}\frac{1}{n}\sum_{i=1}^{n}\Omega_i(x,\theta).
\end{aligned}
$$

Similar to the deduce of (10), it yields

$$\Psi_2=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\,h_H^{2l-1}\,\phi(h_K)}}\right).$$

On the other hand, since $\Psi_7\leq\Psi_1$ and $\Psi_6\leq\Psi_2$, it also leads to

$$\Psi_6=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\,h_H^{2l-1}\,\phi(h_K)}}\right),$$

and

$$\Psi_7=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\,h_H^{2l-1}\,\phi(h_K)}}\right).$$

Then the proof of Lemma 8 can be completed.

Acknowledgement

Authors are thankful to the anonymous referees for its careful checking and invaluable comments that improved this paper.

References

[1] A. Aït Saidi, F. Ferraty, R. Kassa, P. Vieu, *Cross-validated estimation in the single functional index model*, Statistics, vol. 42, no. 6, 2008, 475-494.

[2] S. Attaoui, M. Boudiaf, *On the nonparametric conditional density and mode estimates in the single functional index model with strongly mixing data*, Sankhyã Indian J. Stat., vol. 76, no. 2, 2014, 356-378.

[3] S. Ataoui, N. Ling, *Asymptotic results of a nonparametric conditional cumulative distribution estimator in the single functional index modeling for time series data with applications*, Metrika: International Journal for Theoretical and Applied Statistics, vol. 79, no. 5, 2016, 485-511.

[4] L. Aspirot, K. Bertin, G. Perera, *Asymptotic Normality of the Nadaraya-Watson Estimator for Non stationary Functional Data and Applications to Telecommunications*, Journal of Nonparametric Statistics, vol. 21, 2009, 535-551.

[5] D. Bosq, J-P. Lecoutre, *Théorie de l'Estimation Fonctionnelle*, ECONOMICA. (N.S.), Paris, 1987.

[6] A. A. Bouchentouf, T. Djebbouri, A. Rabhi, K. Sabri, *Strong uniform consistency rates of some characteristics of the conditional distribution estimator in the functional single-index model*, Appl. Math. (Warsaw), vol. 41, no. 4, 2014, 301-322.

[7] Z. Cai, *Regression quantiles for time series*, Econometric Theory, vol. 18, 2002, 169-192.

[8] H. Cardot, C. Crambes, P. Sarda, *Estimation Spline de Quantiles Conditionnels pour Variables Explicatives Fonctionnelles*, C. R. Math. Paris, vol. 339, 2004, 141-144.

[9] P. Chaudhuri, K. Doksum, A. Samarov, *On Average Derivative Quantile Regression*, Ann. Statist., vol. 25, 1997, 715-744.

[10] A. Cuevas, *A partial overview of the theory of statistics with functional data*, J. Statist. Plann. Inference, vol. 147, 2014, 1-23.

[11] S. Dabo-Niang, A. Laksaci, *Nonparametric Quantile Regression Estimation for Functional Dependent Data*, Comm. Statist. Theory and Methods, vol. 41, 2012, 1254-1268.

[12] P. Deheuvels, J. H. J. Einmahl, *Functional limit laws for the increments of Kaplan-Meier product-limit processes and applications*, Ann. Probab., vol. 28, no. 3, 2000, 1301-1335.

[13] M. Ezzahrioui, E. Ould-Saïd, *Asymptotic Normality of a Nonparametric Estimator of the Conditional Mode Function for Functional Data*, Nonparametr. Stat., vol. 20, 2008, 3-18.

[14] M. Ezzahrioui, E. Ould-Saïd, *Asymptotic results of a nonparametric conditional quantile estimator for functional time series*, Comm. Statist. Theory and Methods, vol. 37, no. 16-17, 2008, 2735-2759.

[15] F. Ferraty, A. Laksaci, A. Tadj, P. Vieu, *Rate of uniform consistency for nonparametric estimates with functional variables*, Statistical Inference for Stochastic Processes, vol. 140, 2010, 335-352.

[16] F. Ferraty, A. Peuch, P. Vieu, *Modèle à indice fonctionnel simple*, C. R. Math. Acad. Sci. Paris., vol. 336, 2003, 1025-1028.

[17] F. Ferraty, A. Rabhi, P. Vieu, *Conditional quantiles for functional dependent data with application to the climatic ElNinõ phenomenon*, Sankhyã B, Special Issue on Quantile Regression and Related Methods, vol. 67, no. 2, 2005, 378-399.

[18] F. Ferraty, P. Vieu, *Functional Nonparametric Statistics: A Double Infinite Dimensional Framework*, Recent advanvces and trends in Nonparametric Statistics, Ed. M. Akritas and D. Politis, Elsevier, 2003.

[19] F. Ferraty, P. Vieu, *Nonparametric Functional Data Analysis: Theory And Practice*, Springer Ser. Statist. Statist. Health Sci. Springer, New York, 2006.

[20] A. Gannoun, J. Saracco, K. Yu, *Nonparametric prediction by conditional median and quantiles*, J. Statist. Plann. Inference, vol. 117, 2003, 207-223.

[21] A. Goia, P. Vieu, *A partitioned Single Functional Index Model*, Computational Statistics, vol. 30, no. 3, 2015, 673-692.

[22] A. Goia, P. Vieu, *An Introduction to Recent Advances in High/Infinite Dimensional Statistics*, J. Mutiv. Anal., vol. 146, 2016, 1-6.

[23] W. Härdle, J. S. Marron, *Optimal Bandwidth Selection in Nonparametric Regression Function Estimation*, The Annals of Statistics, vol. 13, 1985, 1465-1481.

[24] M. Hristache, A. Juditsky, V. Spokoiny, *Direct Estimation of the Index Coeffcient in the Single-index Model*, The Annals of Statistics, vol. 29, 2001, 595-623.

[25] T. Hsing, R. Eubank, *Theoretical Foundations of Functional Data Analysis, With an Introduction to Linear Operators*, Wiley, Chichester, 2015.

[26] N. Kadiri, A. Rabhi, A. A. Bouchentouf, *Strong uniform consistency rates of conditional quantile estimation in the single functional index model under random censorship*, journal Dependence Modeling, vol. 6, no. 1, 2018, 197-227.

[27] R. Koenker, G. Jr. Bassett, *Regression Quantiles*, Econometrica, vol. 46, no. 1, 1978, 33-50.

[28] E. Kong, Y. Xia, *A Single-index Quantile Regression Model and its Estimation*, Econometric Theory, vol. 28, 2012, 730-768.

[29] J-P. Lecoutre, E. Ould-Saïd, *Hazard rate estimation for strong mixing and censored processes*, J. Nonparametr. Stat., vol. 5, 1995, 83-89.

[30] S. Ma, *Estimation and Inference in Functional Single-index Models*, Ann. Inst. Stat. Math., vol. 68, 2016, 181-208.

[31] E. Masry, *Nonparametric Regression Estimation for Dependent Functional Data: Asymptotic Normality*, Stochastic Process. Appl., vol. 115, 2005, 155-177.

[32] E. Ould-Saïd, Y. Djabrane, *Asymptotic Normality of a Kernel Conditional Quantile Estimator under Strong Mixing Hypothesis and Left-truncation*, Comm. Statist. Theory and Methods, vol. 40, 2011, 2605-2627.

[33] W.-J. Padgett, *Nonparametric Estimation Of Density And Hazard Rate Functions When Samples Are Censored*, In P.R. Krishnaiah and C.R. Rao (Eds), Handbook of Statist., Elsevier/North-Holland, Amsterdam, Science Publishers, vol. 7, 1988, 313-331.

[34] J. Ramsay, B. Silverman, *Applied Functional Data Analysis: Methods and Case Studies*, New York, Springer, 2002.

[35] J. Ramsay, B. Silverman, *Functional Aata Analysis*, (2nd-ed.), New York, Springer, 2005.

[36] G. Roussas, *Nonparametric estimation of the transition distribution function of a Markov process*, Ann. Statist., vol. 40, 1969, 1386-1400.

[37] M. Samanta, *Nonparametric estimation of conditional quantiles*, Statist. Probab. Lett., vol. 7, 1989, 407-412.

[38] M. Tanner, W.-H. Wong, *The estimation of the hazard function from randomly censored data by the kernel methods*, Ann. Statist., vol. 11, 1983, 989-993.

[39] I. Van-Keilegom, N. Veraverbeke, *Hazard rate estimation in nonparametric regression with censored data*, Ann. Inst. Statist. Math., vol. 53, 2001, 730-745.

[40] H. Wang, Y. Zhao, *A kernel estimator for conditional t-quantiles for mixing samples and its strong uniform convergence*, (in chinese), Math. Appl. (Wuhan)., vol. 12, 1999, 123-127.

[41] Y. Xia, W. Härdle, *Semi-parametric Estimation of Partially Linear Single-index Models*, Journal of Multivariate Analysis, vol. 97, 2006, 1162-1184.

[42] Y. Zhou, H. Liang, *Asymptotic properties for L_1 norm kernel estimator of conditional median under dependence*, J. Nonparametr. Stat., vol. 15, 2003, 205-219.

Nadia Kadiri
University Djillali LIABES of Sidi Bel Abbes
PO. BOX. 89, Sidi Bel Abbes 22000, Algeria
e-mail: nad.kad06@yahoo.com

Abbes Rabhi
University Djillali LIABES of Sidi Bel Abbes
Faculty of Exact Sciences
Probability and Statistics
PO. BOX. 89, Sidi Bel Abbes 22000, Algeria
e-mail: rabhi_abbes@yahoo.fr

Fatima Akkal
University Djillali LIABES of Sidi Bel Abbes
Faculty of Exact Sciences
Probability and Statistics
PO. BOX. 89, Sidi Bel Abbes 22000, Algeria
e-mail: fatima.akkal@hotmail.com