General Mathematics Vol. 29, No. 2 (2021), 127–136
DOI:10.2478/gm-2021-0019

sciendo

The Fekete-Szego Theorem for Close-to-convex Functions Associated with The Koebe Type Function [1]

Sidik Rathi, Shaharuddin Cik Soh

Abstract

This paper deals with the class S containing functions which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Functions f in S are normalized by $f(0) = 0$ and $f'(0) = 1$ and has the Taylor series expansion of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. In this paper we investigate on the subclass of S of close-to-convex functions denoted as $C_{g_\alpha}(\lambda, \delta)$ where function $f \in C_{g_\alpha}(\lambda, \delta)$ satisfies $\mathrm{Re}\{e^{i\lambda}\frac{zf'(z)}{g_\alpha(z)}\}$ for $|\lambda| < \frac{\pi}{2}$, $\cos(\lambda) > \delta$, $0 \leq \delta < 1$, $0 \leq \alpha \leq 1$ and $g_\alpha = \frac{z}{(1-\alpha z)^2}$. The aim of the present paper is to find the upper bound of the Fekete-Szego functional $|a_3 - \mu {a_2}^2|$ for the class $C_{g_\alpha}(\lambda, \delta)$. The results obtained in this paper is significant in the sense that it can be used in future research in this field, particularly in solving coefficient inequalities such as the Hankel determinant problems and also the Fekete-Szego problems for other subclasses of univalent functions.

2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: univalent functions, Fekete-szego problems, close-to-convex function, Koebe function.

1 Introduction

Let A be the class of analytic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Such functions in A has the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

[1] *Received 30 October, 2021*
Accepted for publication (in revised form) 25 November, 2021

Further, we let S denote the subclass of A that contains the normalized and univalent functions. Also let S^* and C denotes the subclasses of S which are starlike and close-to-convex in U respectively. Analytically, $f \in S^*$ if and only if $f \in A$ satisfy

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in U.$$

Kaplan (1952) introduced the class of close-to-convex, functions, C, i.e. $f \in C$ if and only if there exist function $g \in S^*$ such that

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in U.$$

In this paper we introduce a new generalized subclass of C denoted $C_{g_\alpha}(\lambda, \delta)$ such that $f \in C_{g_\alpha}(\lambda, \delta)$ if and only if $f \in A$ satisfy

$$Re\left\{e^{i\lambda}\frac{zf'(z)}{g_\alpha(z)}\right\} > \delta, \quad z \in U, \tag{2}$$

for $|\lambda| < \frac{\pi}{2}$, $\cos\ (\lambda) > \delta$, $0 \leq \delta < 1$, $0 \leq \alpha \leq 1$ and $g_\alpha(z) = \frac{z}{(1-\alpha z)^2}$. We call $g_\alpha(z)$ in (2) as the Koebe-type function as it is clear that, $g_1(z) = \frac{z}{(1-z)^2}$ is the well-known Koebe function. We also list here some previous study that can be reduced from the class $C_{g_\alpha}(\lambda, \delta)$. In particular, the classes $C_{g_0}(0,0)$, $C_{g_0}(0,\delta)$ for $\delta > 0$, $C_{g_0}(\lambda, 0)$, $C_{g_0}(\lambda, \delta)$ for $|\lambda| < \pi$, $\cos\lambda > \delta$and the class $C_{g_1}(0,0)$ have been studied by MacGregor [21], Goel and Mehrok [17], Silverman and Silvia [9], Mohamad [4] and Kowalczyk and Lecko [2] respectively.

Noonan and Thomas [11] defined the q^{th} Hankel determinant, $H_q(n)$ for $q \geq 1$ and $n \geq 0$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

Previous studies shows that many mathematicians still looking the result on this Hankel determinants for various subclasses of S (see [1, 5, 10, 11, 14, 20]).
$H_2(1) = a_3 - {a_2}^2$ is known as the Fekete-Szego functional. It has been discussed since 1930's until now (see [2, 6, 8, 12, 13, 18, 19, 23]). Fekete and Szego [13] generalized the functional $H_2(1)$and defined the Fekete-Szego problems as an inequality for the coefficient of univalent functions in the unit disc U which represents the upper bound for $\left|a_3 - \mu {a_2}^2\right|$ for real μ. They shows that if $f \in S$, then

$$\left|a_3 - \mu {a_2}^2\right| \leq \begin{cases} 3 - 4\mu & \text{if} \quad \mu \leq 0, \\ 1 + 2\mathrm{e}^{\frac{-2\mu}{1-\mu}} & \text{if} \quad 0 \leq \mu \leq 1 \\ 4\mu - 3 & \text{if} \quad \mu \geq 1. \end{cases},$$

This inequality is sharp in the sense that for each real μ there exists a function in S such that equality hold. This classical theorem has motivated many researchers to solve these upper bounds problem for the class K, S^* and C. Among them, are Keogh and Merkes[6], Koepf [23] and Darus [12]. In particular, Koepf [23] showed that for $f \in C$, the upper bound for $\left|a_3 - \mu {a_2}^2\right|$ with $0 \leq \mu \leq 1$ is given by

$$\left|a_3 - \mu {a_2}^2\right| \leq \begin{cases} 3 - 4\mu & \text{if} \quad \mu \leq \frac{1}{3}, \\ \frac{1}{3} + \frac{4}{9\mu} & \text{if} \quad \frac{1}{3} \leq \mu \leq \frac{2}{3} \\ 1 & \text{if} \quad \frac{2}{3} \leq \mu \leq 1. \end{cases},$$

In this paper we determine the upper bound of $\left|a_3 - \mu {a_2}^2\right|$ for $f \in C_{g_\alpha}\ (\lambda,\delta)$ that follows the condition in (2).

2 Preliminaries

Let P be the class of analytic functions which have the positive real part in U. Such functions $p \in P$ is given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in U, \tag{3}$$

such that $Rep(z) > 0$. The following Lemmas for $p\,(z) \in P$ are required in order to achieve the main theorem of this paper.

Lemma 2.1 (Pommerenke [3]) *If $p \in P$ be given by (3) and $Rep(z) > 0$ for $z \in U$, then $|p_n| \leq 2$ and $\left|p_2 - \frac{{p_1}^2}{2}\right| \leq 2 - \frac{\left|{p_1}^2\right|}{2}$.*

Lemma 2.2 *The power series of $p\,(z)$* given by (3) *converges in U to a function in P if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots \tag{4}$$

and $p_{-k} = \bar{p}_k$, are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \beta_k c_0(e^{it_k} z)$, $c_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

Lemma 2.3 (Libera and Zlotkiewicz [15, 16]) *Let the function $p\,(z) \in P$ given by* (3). *Assuming without restriction that $p_1 > 0$. By rewriting* (4) *for the cases $n = 2$ and $n = 3$*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 \\ p_1 & 2 & p_1 \\ \bar{p}_2 & p_1 & 2 \end{vmatrix} = 8 + 2Re\left({p_1}^2 p_2\right) - 2\left|p_2\right|^2 - 4{p_1}^2 \geq 0$$

which is equivalent to

$$2p_2 = {p_1}^2 + x\left(4 - {p_1}^2\right) \tag{5}$$

for some x , $x \leq 1$. *Then* $D_3 \geq 0$ *is equivalent to*

$$\begin{aligned}\left|\left(4p_3 - 4p_1p_2 + {p_1}^3\right)\left(4 - {p_1}^2\right) + p_1\left(2p_2 - {p_1}^2\right)\right| \\ \leq 2\left(4 - {p_1}^2\right)^2 - 2\left|2p_2 - {p_1}^2\right|\end{aligned}$$

and with (5), *provides the relation*

$$\begin{aligned}4p_3 = {p_1}^3 + 2\left(4 - {p_1}^2\right)p_1x - p_1\left(4 - {p_1}^2\right)x^2 \\ + 2\left(4 - {p_1}^2\right)\left(1 - |x|^2\right)z\end{aligned}$$

for some value of z , $z \leq 1$.

3 Main Results

In order to get the upper bounds for the Fekete-Szego functional $\left|a_3 - \mu {a_2}^2\right|$, first, we obtain the bounds for $|a_2|$ and $|a_3|$ as in the following theorem.

Theorem 1. *If* f *is given by* $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ and $f \in C_{g_\alpha}(\lambda, \delta)$ *satisfying* $Re\left\{e^{i\lambda}\frac{zf'(z)}{g_\alpha(z)}\right\} > \delta$ *for* $z \in U$, $|\lambda| < \frac{\pi}{2}$, $\cos(\lambda) > \delta$, $0 \leq \delta < 1$, $0 \leq \alpha \leq 1$ and $g_\alpha(z) = \frac{z}{(1-\alpha z)^2}$. *Then*

$$|a_2| \leq \alpha + A_{\lambda\delta}$$

and

$$|a_3| \leq \alpha^2 + \frac{1}{3}(4\alpha + 2)A_{\lambda\delta}$$

where

$$A_{\lambda\delta} = \cos\lambda - \delta.$$

Proof. From (2) we can see that $f \in C_{g_\alpha}(\lambda, \delta)$ if and only if

$$f'(z) = \left(A_{\lambda\delta}p(z) + i\sin\lambda + \delta\right)\frac{1}{(1 - \alpha z)^2}e^{-i\lambda}, \tag{6}$$

where $A_{\lambda\delta} = \cos\lambda - \delta$, $p(z)$ as given in (3) and

$$\frac{1}{(1-\alpha z)^2} = \sum_{n=0}^{\infty}(n+1)\,\alpha^n z^n = 1 + 2\alpha z + 3\alpha^2 z^2 + 4\alpha^3 z^3 + 5\alpha^4 z^4 + \ldots.$$

Comparing the coefficients of z and z^2 in (6), gives

$$a_2 = \alpha + \frac{1}{2}p_1A_{\lambda\delta}e^{-i\lambda} \tag{7}$$

and

$$a_3 = \alpha^2 + \frac{1}{3}(2\alpha p_1 + p_2) A_{\lambda\delta} e^{-i\lambda}. \tag{8}$$

By applying the triangle inequality and Lemma 2.1 on (7) and (8), we obtain $|a_2| \leq \alpha + A_{\lambda\delta}$ and $|a_3| \leq \alpha^2 + \frac{1}{3}(4\alpha + 2) A_{\lambda\delta}$ as required.

Now, with the aid of Theorem 1, we give the main theorem of the present paper as follows.

Theorem 2. *If* f *be given by* $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ and $f \in C_{g_\alpha}(\lambda,\delta)$ *satisfying* $Re\left\{e^{i\lambda}\frac{zf'(z)}{g_\alpha(z)}\right\} > \delta$ *for* $z \in U, |\lambda| < \frac{\pi}{2}$, $\cos(\lambda) > \delta$, $0 \leq \delta < 1$, $0 \leq \alpha \leq 1$ and $g_\alpha(z) = \frac{z}{(1-\alpha z)^2}$. *Then,*

$$\left|a_3 - \mu {a_2}^2\right| \leq \begin{cases} \alpha^2(1-\mu) + \frac{2}{3}A_{\lambda\delta}(2\alpha+1) & \\ \qquad - \mu A_{\lambda\delta}(2\alpha + A_{\lambda\delta}) & \text{if } 0 \leq \mu \leq \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)} \\ \frac{2}{3}A_{\lambda\delta} + \frac{4\alpha^2}{9\mu} - \frac{1}{3}\alpha^2 & \text{if } \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)} \leq \mu \leq \frac{2}{3} \\ \frac{7}{9}\alpha^2 - \frac{2}{3}\mu\alpha^2 + \frac{2}{3}A_{\lambda\delta} & \text{if } \frac{2}{3} \leq \mu \leq 1 \end{cases}$$

where $A_{\lambda\delta} = \cos\lambda - \delta$.

Proof. Let $\mu \in \Re$ and using (7) and (8), we have

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &= \left|\alpha^2(1-\mu) + \tfrac{p_1}{3}e^{-i\lambda}A_{\lambda\delta}(2-3\mu)\alpha + \tfrac{p_2}{3}e^{-i\lambda}A_{\lambda\delta} - \tfrac{p_1^2}{4}\mu e^{-2i\lambda}A_{\lambda\delta}^2\right|. \end{aligned} \tag{9}$$

Next, in order to apply Lemma 2.1, we add and minus the term $\frac{1}{3}e^{-i\lambda}A_{\lambda\delta}\frac{1}{2}{p_1}^2$ into (9) and we get,

$$\begin{aligned} \left|a_3 - \mu {a_2}^2\right| \leq{}& \alpha^2|1-\mu| + \tfrac{|p_1|}{3}|2-3\mu|\alpha A_{\lambda\delta} + \tfrac{1}{3}\left(2 - \tfrac{|p_1|^2}{2}\right)A_{\lambda\delta} + \\ & \tfrac{|p_1|^2}{12}\left|2 - 3\mu A_{\lambda\delta}e^{-i\lambda}\right| A_{\lambda\delta}. \end{aligned} \tag{10}$$

Since $\left|2 - 3\mu A_{\lambda\delta}e^{-i\lambda}\right| = \sqrt{4 - 12\mu A_{\lambda\delta}\cos\lambda + 9\mu^2 {A_{\lambda\delta}}^2}$ and $\max\limits_{|\lambda|<\frac{\pi}{2}}\{\cos\lambda\} = 1$, then we have

$$\left|2 - 3\mu A_{\lambda\delta}e^{-i\lambda}\right| \leq 2 - 3\mu A_{\lambda\delta}. \tag{11}$$

Now, by putting $|p_1| = y$in (10) and using (11), we get

$$\begin{aligned} &\left|a_3 - \mu {a_2}^2\right| \\ &\leq \alpha^2|1-\mu| + \tfrac{y|2-3\mu|\alpha A_{\lambda\delta}}{3} + \tfrac{1}{3}\left(2 - \tfrac{y^2}{2}\right)A_{\lambda\delta} + \tfrac{y^2(2-3\mu A_{\lambda\delta})A_{\lambda\delta}}{12}. \end{aligned} \tag{12}$$

For $\mu \leq \frac{2}{3}$, we write (12) as

$$\begin{aligned} &\left|a_3 - \mu {a_2}^2\right| \\ &\leq \alpha^2(1-\mu) + \tfrac{y(2-3\mu)\alpha A_{\lambda\delta}}{3} + \tfrac{1}{3}\left(2 - \tfrac{y^2}{2}\right)A_{\lambda\delta} + \tfrac{y^2(2-3\mu A_{\lambda\delta})A_{\lambda\delta}}{12} = F(y). \end{aligned} \tag{13}$$

Differentiating (13) with respect to y, gives $F'(y) = \frac{2}{3}\alpha A_{\lambda\delta} - \mu\alpha A_{\lambda\delta} - \frac{1}{2}\mu {A_{\lambda\delta}}^2 y$, $F''(y) = -\frac{1}{2}\mu {A_{\lambda\delta}}^2$ and by equating$F'(y) = 0$, we get

$$y = \frac{2\alpha(2-3\mu)}{3\mu A_{\lambda\delta}}. \tag{14}$$

For any point y, it is clear that $F''(y) < 0$. Thus, the second derivative test shows that $F(y)$ have a maximum point and the maximum of $F(y)$ occurs at $y = \frac{2\alpha(2-3\mu)}{3\mu A_{\lambda\delta}}$. Also, since $|p_1| = y$ and by Lemma 2.1, we may write (14) as $0 \leq \frac{2\alpha(2-3\mu)}{3\mu A_{\lambda\delta}} \leq 2$ which implies $\frac{2\alpha}{3(A_{\lambda\delta}+\alpha)} \leq \mu \leq \frac{2}{3}$. Therefore, we will first calculate the upper bounds of the functional $\left|a_3 - \mu {a_2}^2\right|$ corresponds to $\mu = \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}$. Replacing $\mu = \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}$ into (13), gives

$$\begin{aligned}&\left|a_3 - \tfrac{2\alpha}{3(A_{\lambda\delta}+\alpha)}a_2^2\right| \\ &\leq \alpha^2\left(1 - \tfrac{2\alpha}{3(A_{\lambda\delta}+\alpha)}\right) + \tfrac{y}{3}\left(2 - 3\left(\tfrac{2\alpha}{3(A_{\lambda\delta}+\alpha)}\right)\right)\alpha A_{\lambda\delta} + \tfrac{1}{3}A_{\lambda\delta}\left(2 - \tfrac{y^2}{2}\right) \\ &+ \tfrac{y^2}{12}\left(2 - 3\left(\tfrac{2\alpha}{3(A_{\lambda\delta}+\alpha)}\right)A_{\lambda\delta}\right)A_{\lambda\delta}.\end{aligned}$$

Upon further simplification and with the aid of Lemma 2.1, we have

$$\left|a_3 - \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}{a_2}^2\right| \leq \frac{1}{3}\alpha^2 + \frac{2}{3}\alpha A_{\lambda\delta} + \frac{2}{3}A_{\lambda\delta}. \tag{15}$$

Next we find the upper bounds of the functional $\left|a_3 - \mu {a_2}^2\right|$ that corresponds to $\mu \leq \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}$. Without changing the originality of $\left|a_3 - \mu {a_2}^2\right|$, we write

$$\left|a_3 - \mu {a_2}^2\right| = \left|a_3 - \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}{a_2}^2 + \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}{a_2}^2 - \mu {a_2}^2\right|.$$

Then, we have

$$\left|a_3 - \mu {a_2}^2\right| \leq \left|a_3 - \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}{a_2}^2\right| + \left|\frac{2\alpha}{3(A_{\lambda\delta}+\alpha)} - \mu\right|\left|{a_2}^2\right|$$

and for $\mu \leq \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}$, we write

$$\left|a_3 - \mu {a_2}^2\right| \leq \left|a_3 - \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}{a_2}^2\right| + \left(\frac{2\alpha}{3(A_{\lambda\delta}+\alpha)} - \mu\right)\left|{a_2}^2\right|. \tag{16}$$

Now, by applying (15) together with Theorem 1 into (16), gives

$$\left|a_3 - \mu {a_2}^2\right| \leq \frac{1}{3}\alpha^2 + \frac{2}{3}\alpha A_{\lambda\delta} + \frac{2}{3}A_{\lambda\delta} + \frac{2}{3}\alpha^2 + \frac{2}{3}\alpha A_{\lambda\delta} - \mu(\alpha + A_{\lambda\delta})^2$$

and upon simplification, we obtain

$$\left|a_3 - \mu {a_2}^2\right| \leq \alpha^2(1-\mu) + \frac{2}{3}A_{\lambda\delta}(2\alpha + 1) - \mu A_{\lambda\delta}(2\alpha + A_{\lambda\delta}). \tag{17}$$

Next, we find the upper bounds of $\left|a_3-\mu {a_2}^2\right|$ for the case $\frac{2\alpha}{3(A_{\lambda\delta}+\alpha)}\leq\mu\leq\frac{2}{3}$. From (13) we have shown that $F(y)$ have a maximum at $y=\frac{2\alpha(2-3\mu)}{3\mu A_{\lambda\delta}}$ and by using this value of y, we have

$$\begin{aligned} F(y) &= \alpha^2(1-\mu)+\tfrac{2\alpha^2(2-3\mu)^2}{9\mu}+\tfrac{2}{3}A_{\lambda\delta}-\tfrac{2\alpha^2(2-3\mu)^2}{27\mu^2A_{\lambda\delta}}+\tfrac{\alpha^2(2-3\mu A_{\lambda\delta})(2-3\mu)^2}{27\mu^2A_{\lambda\delta}}, \\ &= \alpha^2(1-\mu)+\tfrac{2}{3}A_{\lambda\delta}+\left(\tfrac{2\alpha^2}{9\mu}+\tfrac{\alpha^2(2-3\mu A_{\lambda\delta})}{27\mu^2A_{\lambda\delta}}-\tfrac{2\alpha^2}{27\mu^2A_{\lambda\delta}}\right)(2-3\mu)^2. \end{aligned}$$

Thus, upon further simplification, now (13) becomes

$$\left|a_3-\mu {a_2}^2\right|\leq\left(\frac{4}{9\mu}-\frac{1}{3}\right)\alpha^2+\frac{2}{3}A_{\lambda\delta}. \tag{18}$$

Also, from (13) we can see that by direct substitution with $\mu=\frac{2}{3}$, we obtain

$$\left|a_3-\frac{2}{3}{a_2}^2\right|\leq\frac{1}{3}\alpha^2+\frac{2}{3}A_{\lambda\delta}\,. \tag{19}$$

Next, we determine the upper bounds of $\left|a_3-\mu {a_2}^2\right|$ for $\frac{2}{3}\leq\mu\leq 1$. From (12), for $\mu\in\left[\frac{2}{3},\,1\right]$, we write

$$\begin{aligned} \left|a_3-\mu {a_2}^2\right| &\leq \alpha^2(1-\mu)+\tfrac{y}{3}(3\mu-2)\alpha A_{\lambda\delta}+\tfrac{1}{3}\left(2-\tfrac{y^2}{2}\right)A_{\lambda\delta}+ \\ &\quad \tfrac{y^2}{12}(2-3\mu A_{\lambda\delta})A_{\lambda\delta}=G(y)\,. \end{aligned} \tag{20}$$

Differentiating $G(y)$ with respect to y, gives

$$G'(y)=\mu\alpha A_{\lambda\delta}-\frac{2}{3}\alpha A_{\lambda\delta}-\frac{1}{2}\mu {A_{\lambda\delta}}^2y$$

and

$$G''(y)=-\frac{1}{2}\mu {A_{\lambda\delta}}^2.$$

Setting $G'(y)=0$, we obtain

$$y=\frac{2\alpha(3\mu-2)}{3\mu A_{\lambda\delta}}. \tag{21}$$

Since $G''(y)=-\frac{1}{2}\mu {A_{\lambda\delta}}^2<0$, then the second derivative test shows that, $G(y)$ is maximum at $y=\frac{2\alpha(3\mu-2)}{3\mu A_{\lambda\delta}}$ for $\frac{2}{3}\leq\mu\leq 1$. Lemma 2.1 implies that $0\leq y\leq 2$. Consequently, we have

$$\frac{2}{3}\leq\mu\leq\frac{2\alpha}{3(\alpha-A_{\lambda\delta})}.$$

Now, we calculate the upper bound for $\left|a_3-\mu {a_2}^2\right|$ that corresponds to $\mu=\frac{2\alpha}{3(\alpha-A_{\lambda\delta})}$. Since the maximum of $G(y)$ occurs at $y=\frac{2\alpha(3\mu-2)}{3\mu A_{\lambda\delta}}$, thus, from (20), we obtain

$$\left|a_3-\mu {a_2}^2\right|\leq\frac{2}{3}A_{\lambda\delta}+\frac{4\alpha^2}{9\mu}-\frac{1}{3}\alpha^2. \tag{22}$$

Next, for the case $\mu = 1$, clearly $y = \frac{2\alpha(3\mu-2)}{3\mu A_{\lambda\delta}}$ implies $y = \frac{2\alpha}{3A_{\lambda\delta}}$. Now, by using $\mu = 1$ and $y = \frac{2\alpha}{3A_{\lambda\delta}}$ into (20) we obtain

$$\left|a_3 - {a_2}^2\right| \leq \frac{1}{9}\alpha^2 + \frac{2}{3}A_{\lambda\delta}. \tag{23}$$

Next, we will give the proof for $\frac{2}{3} \leq \mu \leq 1$ as follows: Let

$$a_3 - \mu {a_2}^2 = x\left(a_3 - {a_2}^2\right) + y\left(a_3 - \frac{2}{3}{a_2}^2\right) \tag{24}$$

which can also be written as

$$a_3 - \mu {a_2}^2 = (x+y)\, a_3 - \left(x + \frac{2}{3}y\right) {a_2}^2 \,. \tag{25}$$

Comparing the coefficient of a_3 and ${a_2}^2$ in (25) gives $x+y = 1$and $x+\frac{2}{3}y = \mu$ which implies that

$$x = 3\mu - 2 \tag{26}$$

and

$$y = 3 - 3\mu. \tag{27}$$

Substituting (26) and (27) into (24), we have

$$a_3 - \mu {a_2}^2 = (3\mu - 2)\left(a_3 - {a_2}^2\right) + (3 - 3\mu)\left(a_3 - \frac{2}{3}{a_2}^2\right)$$

and

$$\left|a_3 - \mu {a_2}^2\right| \leq (3\mu - 2)\left|a_3 - {a_2}^2\right| + (3 - 3\mu)\left|a_3 - \frac{2}{3}{a_2}^2\right| \,. \tag{28}$$

Now, by applying (19) and (23) into (28), we have

$$\left|a_3 - \mu {a_2}^2\right| \leq (3\mu - 2)\left(\frac{1}{9}\alpha^2 + \frac{2}{3}A_{\lambda\delta}\right) + (3 - 3\mu)\left(\frac{1}{3}\alpha^2 + \frac{2}{3}A_{\lambda\delta}\right)$$

and upon simplification, we obtain

$$\left|a_3 - \mu {a_2}^2\right| \leq \frac{7}{9}\alpha^2 - \frac{2}{3}\mu\alpha^2 + \frac{2}{3}A_{\lambda\delta}\,. \tag{29}$$

Lastly, by combining (17), (22) and (29) we obtain

$$\left|a_3 - \mu {a_2}^2\right| \leq \begin{cases} \alpha^2(1-\mu) + \frac{2}{3}A_{\lambda\delta}(2\alpha+1) & \\ \qquad - \mu A_{\lambda\delta}(2\alpha + A_{\lambda\delta}) & \text{if } \ 0 \leq \mu \leq \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)} \\ \frac{2}{3}A_{\lambda\delta} + \frac{4\alpha^2}{9\mu} - \frac{1}{3}\alpha^2 & \text{if } \ \frac{2\alpha}{3(A_{\lambda\delta}+\alpha)} \leq \mu \leq \frac{2}{3} \\ \frac{7}{9}\alpha^2 - \frac{2}{3}\mu\alpha^2 + \frac{2}{3}A_{\lambda\delta} & \text{if } \ \frac{2}{3} \leq \mu \leq 1 \end{cases}$$

which are the desired results as in Theorem 2.

References

[1] A. L. P. Hern, A. Janteng, R. Omar, *Hankel Determinant $H_2(3)$ for Certain Subclasses of Univalent Functions*, Mathematics and Statistics, vol. 8, no. 5, 2020, 566-569.

[2] B. Kowalczyk, A. Lecko, *The Fekete-Szegö inequality for close-to-convex functions with respect to a certain starlike function dependent on a real parameter*, Journal of Inequalities and Applications, 2014.

[3] C. Pommerenke, *Univalent functions*, Gottingen: Vandenhoeck and Ruprecht, 1975.

[4] D. Mohamad, *On a class of function whose derivative maps the unit disc into a half plane*, Bulletin of Malaysian Mathematical Sciences Society, vol. 23, 2001, 163-171.

[5] E. Deniz, M. Caglar, H. Orhan, *Second Hankel determinant for bi-starlike and bi-convex functions of order beta*, Appl. Math. Comput., vol. 271, 2015, 301-307.

[6] F. R. Keogh, E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proceeding of the American Mathematical Society, vol. 20, no. 1, 1969, 8-12.

[7] G. M. Goluzin, *Some questions in the theory of univalent functions*, Trudy Mat. Inst. Steklova, vol. 27, 1949, 1-112.

[8] H. Arıkan, H. Orhan, M. C. Aglar, *Fekete-Szego inequality for a subclass of analytic functions defined by Komatu integral operator*, AIMS Mathematics, vol. 5, no. 3, 2020, 1745-1756.

[9] H. Silverman, E. M. Silvia, *On α-close-to-convex Functions*, Publicationes Mathematicae Debrecen, 49, 1996, 532-537.

[10] J. A. Jenkins, *On certain coefficients of univalent functions*, Analytic Functions, Princeton Univ. Press, 1960, 159-194.

[11] J. Noonan, D. K. Thomas, *On the second Hankel determinant of a really mean p-valent functions*, Transactions of the American Mathematical Society, vol. 223, 1976, 337-346.

[12] M. Darus, *The Fekete-Szego problem for close-to-convex functions of the class* $K_{sh}(\alpha, \delta)$, Acta Mathematica Academiae Paedagogicae, 2002.

[13] M. Fekete, G. Szego, *Eine Bemerkung uber ungerade schlichte funktionen*, J. Londan Math. Soc., vol. 8, 1933, 85-89.

[14] P. Zaprawa, *On Hankel determinant $H_2(??)$ for univalent functions*, Results Math, vol. 73, no. 89, 2018.

[15] R. J. Libera, E. J. Zlotkiewicz, *Coefficient bound for the inverse of a function with derivative in positive real part*, Proceeding of the American Mathematical Society, vol. 87, no. 2, 1983, 251-257.

[16] R. J. Libera, E. J. Zlotkiewicz, *Early coefficient of the inverse of a regular convex function*, Proceeding of the American Mathematical Society, vol. 85, no. 2, 1982, 225-230.

[17] R. M. Goel, B. S, Mehrok, *A subclass of univalent function*, Journal of the Australian Mathematical Society, 35, 1983, 1-17.

[18] R. R. London, *Fekete-Szego inequalities for close-to-convex functions*, Proceedings of the American Mathematical Society, vol. 117, no. 4, 1993, 947-950.

[19] S. Altinkaya, S. Yalcin, *The Feketa-Szego problem for a general class of bi-univalent functions satisfying subordinate conditions*, Sahand Communications in Mathematical Analysis, vol. 5, no. 1, 2017, 1-7.

[20] T. Hayami, S. Owa, *Generalized Hankel determinant for certain classes*, Int. J. Math. Anal., vol. 4, no. 52, 2010, 2573-2585.

[21] T. H. MacGregor, *Function whose derivative has a positive real part*, Transactions of the American Mathematical Society, 104, 1962, 532-537.

[22] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Mathematical Journal, vol. 1, no. 2, 1952, 169-185.

[23] W. Koepf, *On the Fekete-Szego problem for close-to-convex functions*, Proceeding of the American Mathematical Society, vol. 101, no. 1, 1987, 89-95.

Sidik Rathi
Universiti Teknologi MARA
Faculty of Computer and Mathematical Sciences
Department of Mathematics
40450 Shah Alam, Selangor, Malaysia
e-mail: sidik@fskm.uitm.edu.my

Shaharuddin Cik Soh
Universiti Teknologi MARA
Faculty of Computer and Mathematical Sciences
Department of Mathematics
40450 Shah Alam, Selangor, Malaysia
e-mail: shahar@fskm.uitm.edu.my