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New Subclasses of Bi-Univalent Functions based on the Fibonacci Numbers [1]

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Abstract

In this work, by using the Al-Oboudi differential operator and the rule of subordination, we introduced the new subclasses $D^{n,\rho}_{\Sigma,\delta}(\Phi)$ and $F^{n,\alpha}_{\Sigma,\delta}(\Phi)$ of the bi-univalent functions. Likewise, we use the Fibonacci numbers to derive the initial coefficients bounds for $|a_2|$ and $|a_3|$ of the bi-univalent function subclasses.

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1 Introduction

Let $D = \{z : z \in C \text{ and } |z| < 1\}$ be the open unit disc in the complex plane C and A be the class of the analytic function class in D normalised by the condition $f(0) = f'(0) - 1 = 0$, with f in A has the following Taylor series expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Then, let S be the subclass of A in D.

Based on the Koebe-One Quarter Theorem [16], it confirms that every univalent function f in A has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z (z \in D)$$

and

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$$f(f^{-1}(w)) = w(|w| < r_0(f) : r_0(f) \geq \tfrac{1}{4}).$$

Moreover, the f^{-1} is given by

$$(2) \quad g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

Let Σ denote the class of the bi-univalent functions in D. A function f given by (1) in A is said to be bi-univalent in D if both f and f^{-1} are univalent in D. The remarkable work and history on Σ can be seen from the studies done by [34], [17], [6], [13], and [23]. Furthermore, the study on the initial coefficients $|a_2|$ and $|a_3|$ of the bi-univalent functions had received high attraction from many researchers [3], [25], [4], [30], and [14]. In the study of the Geometric Function Theory, determining of the bounds for $|a_k|$ is essential. However, the estimation of the general coefficients $|a_k|(k \in N\backslash\{1,2\}; N = \{1,2,3,\cdots\})$ is still an open problem.

Let P denote the class of functions of the form

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + ..., z \in D,$$

which are analytic with $R\{p(z)\} > 0$. Here, p is called a Caratheodory function [9]. It is well known that the following correspondence between the class P and the class of Schwarz functions w exists where $p \in P$ if and only if $p(z) = \dfrac{1+w(z)}{(1-w(z)}$. Let $P(\beta), 0 \leq \beta < 1$, denote the class of analytic functions p in D with $p(0) = 1$ and $R\{p(z)\} > \beta$.

It is well known that the Fibonacci sequence denoted by $\{F_n\}$ is such that each number is the sum of the two preceding ones, starting from 0 and 1; that is $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}, n \geq 1$. It is also well-known that we can write

$$F_n = \frac{\varphi^n - \Lambda^n}{\sqrt{5}} = \frac{(1-\Lambda)^n - \Lambda^n}{\sqrt{5}}, (n \in N),$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887...$$

called the golden ratio and

$$\Lambda = \frac{1-\sqrt{5}}{2} = 1 - \varphi = -\frac{1}{\varphi} \approx -0.6180339887...$$

For the history and properties of the Fibonacci numbers and golden ratio, one may refer to the book by [5]. We next recall the following function

$$\Phi(z) = \frac{1+\Lambda^2z^2}{1-\Lambda z-\Lambda^2z^2}, (z \in D),$$

introduced by [33].

To learn more about Fibonacci numbers, we may refer to [22], [31], [19], [25], and [2]. In [22], the authors determined the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ involving λ-pseudo bi-starlike functions and Sakaguchi functions related to shell-like curves connected with Fibonacci numbers. Other few papers that also introduced and investigated new subclasses with Sakaguchi functions related to Fibonacci numbers are [18], [24] and [35]. Defining new subclasses of analytic bi-univalent functions and investigating certain geometric properties associated with the Fibonacci numbers are also among the studies that have received much attention from researchers (see [2], [25], [30], [14], [31], [20], [15]).

In the study of Geometric Function Theory, the operators of normalised analytic functions are very popular, namely for Differential and Integral. Many articles discuss operators and the new generalisation of various authors. For instance, Ruscheweyh [26] led the way in the differential operator introduced in 1975. It is then followed by Sălăgean [32], giving another version of differential and integral operator. Many properties have been discussed and studied by many researchers for these two operators. Twenty-one years later, Al-Oboudi [1] generalised Sălăgean operator, followed by Eker and Seker [12] and Eker and Owa [11]. In Shaba et al. [29], the authors introduced a new subclass of p-valent functions with negative coefficient defined by Hadamard product associated with generalised Al-Oboudi differential operator. They established the coefficient inequalities of the Taylor-Mclaurin series in this study. For more reading involving another study on using the Al-Oboudi operator, one can refer to [28], [36], [27], [7], and [8]. We can see that many types of functions have been investigated related to Al-Oboudi differential operators, such as meromorphic bi-univalent functions [28], p-valently starlike and convex functions with complex order [27], as well as other types of multivalent functions [36].

In this work, motivated by [2] and [14], we establish two new subclasses $D^{n,\rho}_{\Sigma,\delta}(\Phi)$ and $F^{n,\alpha}_{\Sigma,\delta}(\Phi)$ of the bi-univalent functions that were related with the Al-Oboudi differential operator. Moreover, the initial coefficients $|a_2|$ and $|a_3|$ of the bi-univalent function classes are derived by using the Fibonacci numbers.

2 Methods

In this paper, several definitions and lemmas are used.

Definition 1 *[21] An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z), (z \in D)$, if an analytic function w exists in D with $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. If g is univalent in D, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$ where :*

$$w(z) = \sum_{n=1}^{\infty} c_n z^n (w(0) = 0, |w(z)| < 1).$$

Lemma 1 *(Caratheodory Lemma) Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$, then*

$$|c_n| \leq 2, \tag{3}$$

for $n \geq 1$.

Definition 2 *For* $f \in A$, *the subsequent generalised Sălăgean operator is proposed by Al-Oboudi [1] :*

$$D_\delta^0 f(z) = f(z),$$

$$D_\delta^1 f(z) = (1-\delta)f(z) + \delta z f'(z) = D_\delta f(z) \ (\delta \geq 0), \tag{4}$$

$$D_\delta^n f(z) = D_\delta(D_\delta^{n-1} f(z)) \ (n \in N), \tag{5}$$

from (4) and (5), we have

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k \ (n \in N),$$

with f *that is given by (1) and* $D_\delta^n f(0) = 0$. *The Sălăgean's differential operator [32] will be obtained if* $\delta = 1$.

Let Σ denote the class of the bi-univalent functions in D.

Definition 3 *A function* $f \in \Sigma$ *is said to be in the class :*

$$D_{\Sigma,\delta}^{n,\rho}(\Phi)(\rho \geq 1, \delta \geq 0, n \in N, z, w \in D),$$

if the following subordination holds :

$$\left[(1-\rho)\frac{D_\delta^n f(z)}{z} + \rho\,(D_\delta^n f(z))'\right] \prec \Phi(z) = \frac{1+\Lambda^2 z^2}{1-\Lambda z - \Lambda^2 z^2},$$

$$\left[(1-\rho)\frac{D_\delta^n g(w)}{w} + \rho\,(D_\delta^n g(w))'\right] \prec \Phi(w) = \frac{1+\Lambda^2 w^2}{1-\Lambda w - \Lambda^2 w^2},$$

where g *is given by (2) and* $\Lambda = \dfrac{1-\sqrt{5}}{2} \approx -0.618$.

Remark 1 *The function* $\Phi(z)$ *is univalent in the disk* $|z| < \dfrac{3-\sqrt{5}}{2} \approx -0.38$ *but not univalent in* D. *For instance,* $\Phi(0) = \Phi(-\dfrac{1}{2\Lambda})$ *and* $\Phi\left(e^{\pm i arccos(1/4)}\right) = \dfrac{\sqrt{5}}{5}$. *Besides, it is stated as :* $\dfrac{1}{|\Lambda|} = \dfrac{|\Lambda|}{1-|\Lambda|}$. *It fulfills the golden section as* $|\Lambda|$ *divides* $[0,1]$ *([10]).*

Definition 4 *A function $f \in \Sigma$ is said to be in the class*

$$F_{\Sigma,\delta}^{n,\alpha}(\Phi)(\delta \geq 0,\ 0 \leq \alpha \leq 1,\ n \in N,\ z, w \in D).$$

Then, the following subordination holds:

$$\alpha\left(1+\frac{zD_\delta^n f''(z)}{D_\delta^n f'(z)}\right)+(1-\alpha)\left(\frac{zD_\delta^n f'(z)}{D_\delta^n f(z)}\right) \prec \Phi(z)=\frac{1+\Lambda^2 z^2}{1-\Lambda z-\Lambda^2 z^2},$$

$$\alpha\left(1+\frac{wD_\delta^n g''(w)}{D_\delta^n g'(w)}\right)+(1-\alpha)\left(\frac{wD_\delta^n g'(w)}{D_\delta^n g(w)}\right) \prec \Phi(w)=\frac{1+\Lambda^2 w^2}{1-\Lambda w-\Lambda^2 w^2},$$

where $\Lambda=\dfrac{1-\sqrt{5}}{2} \approx -0.618$ *and g is given by (2).*

A valuable relationship between the Fibonacci numbers and the function $\Phi(z)$ is shown by Dziok et al. [10]. Let $\{F_n\}$ represent the sequence of the Fibonacci numbers

$$F_0=0, F_1=1, F_{n+2}=F_n+F_{n+1}\ (n \in N=\{0,1,2,\dots\}),$$

where

$$F_n=\frac{(1-\Lambda)^n-\Lambda^n}{\sqrt{5}}, \Lambda=\frac{1-\sqrt{5}}{2}.$$

Then

$$\Phi(z)=1+\sum_{n=1}^{\infty}\Phi_n z^n,$$

$$\Phi(z)=1+(F_0+F_2)\Lambda z+(F_1+F_3)\Lambda^2 z^2+\sum_{n=3}^{\infty}(F_{n-3}+F_{n-2}+F_{n-1}+F_n)\Lambda^n z^n.$$

Hence, the coefficients of Φ_n fulfill

$$\Phi_n=\begin{cases}\Lambda & (\text{n=1})\\ 3\Lambda^2 & (\text{n=2})\\ \Lambda\Phi_{n-1}+\Lambda^2\Phi_{n-2} & (\text{n=3,4,}\dots).\end{cases} \tag{6}$$

3 Results

We obtained the initial coefficients that are bound in $D^{n,\rho}_{\Sigma,\delta}(\Phi)$ and $F^{n,\alpha}_{\Sigma,\delta}(\Phi)$.

Theorem 1 *Let* $f \in D^{n,\rho}_{\Sigma,\delta}(\Phi), (\rho \geq 1, \delta \geq 0,\ n \in N)$, *then*

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{|\Lambda([1+2\delta]^n(1+2\rho) - 3[1+\delta]^{2n}(1+\rho)^2) + [1+\delta]^{2n}(1+\rho)^2|}},$$

and

$$|a_3| \leq \frac{\Lambda^2}{[1+\delta]^{2n}(1+\rho)^2} + \frac{|\Lambda|}{[1+2\delta]^n(1+2\rho)}.$$

Proof. Suppose that $f \in D^{n,\rho}_{\Sigma,\delta}(\Phi)$. By the relation of subordination, for the analytic functions u, v where $u(0) = v(0) = 0,\ |u(z)| < 1,\ |v(w)| < 1,\ (z, w \in D)$, we have

$$\left[(1-\rho)\frac{D^n_\delta f(z)}{z} + \rho(D^n_\delta f(z))'\right] = \Phi(u(z)), \tag{7}$$

$$\left[(1-\rho)\frac{D^n_\delta g(w)}{w} + \rho(D^n_\delta g(w))'\right] = \Phi(v(w)). \tag{8}$$

Next, set p_1 and p_2 to the functions

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + x_1 z + x_2 z^2 + \ldots,$$

$$p_2(w) = \frac{1+v(w)}{1-v(w)} = 1 + y_1 w + y_2 w^2 + \ldots,$$

with p_1 and p_2 are analytic functions in D (where $p_1(0) = p_2(0) = 1$) as u and v are Schwarz functions, we obtain the equations

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[x_1 z + \left(x_2 - \frac{x_1^2}{2}\right) z^2\right] + \ldots,$$

$$v(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[y_1 w + \left(y_2 - \frac{y_1^2}{2}\right) w^2\right] + \ldots,$$

leading to

$$\Phi(u(z)) = 1 + \frac{\Phi_1 x_1}{2} z + \left[\frac{1}{2}\left(x_2 - \frac{x_1^2}{2}\right)\Phi_1 + \frac{x_1^2}{4}\Phi_2\right] z^2 + \ldots,$$

$$\Phi(v(w)) = 1 + \frac{\Phi_1 y_1}{2} w + \left[\frac{1}{2}\left(y_2 - \frac{y_1^2}{2}\right)\Phi_1 + \frac{y_1^2}{4}\Phi_2\right] w^2 + \dots.$$

Now, by comparing the identical coefficients in (7) and (8), we will acquire:

$$[1+\delta]^n (1+\rho)\, a_2 = \frac{\Phi_1 x_1}{2}, \tag{9}$$

$$[1+2\delta]^n (1+2\rho)\, a_3 = \frac{1}{2}\left(x_2 - \frac{x_1^2}{2}\right)\Phi_1 + \frac{x_1^2}{4}\Phi_2, \tag{10}$$

$$-[1+\delta]^n (1+\rho)\, a_2 = \frac{\Phi_1 y_1}{2}, \tag{11}$$

$$[1+2\delta]^n (1+2\rho)\left(2a_2^2 - a_3\right) = \frac{1}{2}\left(y_2 - \frac{y_1^2}{2}\right)\Phi_1 + \frac{y_1^2}{4}\Phi_2. \tag{12}$$

Clearly, from (9) and (11) we obtain :

$$x_1 = -y_1. \tag{13}$$

From (13), we have

$$2[1+\delta]^{2n} (1+\rho)^2 a_2^2 = \frac{\Phi_1^2}{4}\left(x_1^2 + y_1^2\right). \tag{14}$$

By adding (10) to (12), we get :

$$2(1+2\rho)[1+2\delta]^n a_2^2 = \frac{\Phi_1}{2}(x_2 + y_2) + (\Phi_2 - \Phi_1)\left(\frac{x_1^2}{4} + \frac{y_1^2}{4}\right). \tag{15}$$

By making use of (14) in (15), we have

$$a_2^2 = \frac{\Phi_1^3 (x_2 + y_2)}{4\left([1+2\delta]^n (1+2\rho)\Phi_1^2 - (\Phi_2 - \Phi_1)[1+\delta]^{2n}(1+\rho)^2\right)}, \tag{16}$$

and by utilising (3) and (6) in (16) yields

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{|\Lambda\left([1+2\delta]^n (1+2\rho) - 3[1+\delta]^{2n}(1+\rho)^2\right) + [1+\delta]^{2n}(1+\rho)^2|}}.$$

Next, subtracting (12) from (10) yields

$$2[1+2\delta]^n (1+2\rho)(a_3 - a_2^2) = \frac{\Phi_1}{2}(x_2 - y_2). \tag{17}$$

Then, in view of (14), the equation (17) becomes

$$a_3 = \frac{\Phi_1^2(x_1^2 + y_1^2)}{8[1+\delta]^{2n}(1+\rho)^2} + \frac{\Phi_1(x_2 - y_2)}{4[1+2\delta]^n(1+2\rho)}.$$

Hence, we obtain

$$|a_3| \leq \frac{\Lambda^2}{[1+\delta]^{2n}(1+\rho)^2} + \frac{|\Lambda|}{[1+2\delta]^n(1+2\rho)}.$$

For $\delta = 0$, Theorem 1 affirms Altinkaya's [2] findings in Corollary 3.2.

Corollary 1 *If $f \in D_{\Sigma}^{n,\rho}(\Phi)$, then*

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{|\Lambda\left((1+2\rho) - 3(1+\rho)^2\right) + (1+\rho)^2|}},$$

$$|a_3| \leq \frac{\Lambda^2}{(1+\rho)^2} + \frac{|\Lambda|}{(1+2\rho)}.$$

For $\delta = 0$ and $\rho = 0$, Theorem 1 reduces to Altinkaya ([2], Corollary 3.3).

Corollary 2 *If $f \in D_{\Sigma}^{n}(\Phi)$, then*

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{|-9\Lambda + 4|}},$$

$$|a_3| \leq \frac{\Lambda^2}{4} + \frac{|\Lambda|}{3}.$$

Theorem 2 *Let $f \in F_{\Sigma,\delta}^{n,\alpha}(\Phi)$. Then*

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{\begin{array}{l}|([1+\delta]^n(1+\alpha))^2 - ([1+\delta]^{2n}(1+3\alpha) - 2[1+2\delta]^n(1+2\alpha)\\ +3([1+\delta]^n(1+\alpha))^2)\Lambda|\end{array}}},$$

and

$$|a_3| \leq \frac{\begin{array}{r}|\Lambda|\Big[([1+\delta]^n(1+\alpha))^2 - ([1+\delta]^{2n}(1+3\alpha)\\ - 2[1+2\delta]^n(1+2\alpha) + 3([1+\delta]^n(1+\alpha))^2)\Lambda\Big]\end{array}}{\begin{array}{r}2[1+2\delta]^n(1+2\alpha)\Big[([1+\delta]^n(1+\alpha))^2 - ([1+\delta]^{2n}(1+3\alpha)\\ - 2[1+2\delta]^n(1+2\alpha) + 3([1+\delta]^n(1+\alpha))^2)\Lambda\Big]\end{array}}.$$

Proof. Suppose that $f \in F_{\Sigma,\delta}^{n,\alpha}(\Phi)$. By considering Definition (4), then

$$\alpha\left(1+\frac{zD_\delta^n f''(z)}{D_\delta^n f'(z)}\right)+(1-\alpha)\left(\frac{zD_\delta^n f'(z)}{D_\delta^n f(z)}\right)=\Phi(u(z)),$$

$$\alpha\left(1+\frac{wD_\delta^n g''(w)}{D_\delta^n g'(w)}\right)+(1-\alpha)\left(\frac{wD_\delta^n g'(w)}{D_\delta^n g(w)}\right)=\Phi(v(w)),$$

with $\Lambda=\dfrac{(1-\sqrt{5})}{2}\approx -0.618$ where g is given by (2) and $z,w\in D$. Since

$$\alpha\left(1+\frac{zD_\delta^n f''(z)}{D_\delta^n f'(z)}\right)+(1-\alpha)\left(\frac{zD_\delta^n f'(z)}{D_\delta^n f(z)}\right)=$$
$$1+([1+\delta]^n\alpha+[1+\delta]^n)a_2z+(4[1+2\delta]^n\alpha+2[1+2\delta]^n)a_3z^2-((3\alpha+1)[1+\delta]^{2n})a_2^2z^2+\ldots,$$

and

$$\alpha\left(1+\frac{wD_\delta^n g''(w)}{D_\delta^n g'(w)}\right)+(1-\alpha)\left(\frac{wD_\delta^n g'(w)}{D_\delta^n g(w)}\right)=1-([1+\delta]^n(1+\alpha))a_2w+(4[1+2\delta]^n(1+2\alpha)-[1+\delta]^{2n}(1+3\alpha))a_2^2w^2-2([1+2\delta]^n(1+2\alpha))a_3w^2+\ldots.$$

Thus we have

$$\begin{aligned}1+([1+\delta]^n\alpha+[1+\delta]^n)a_2z+(4[1+2\delta]^n\alpha+2[1+2\delta]^n)a_3z^2\\ -((3\alpha+1)[1+\delta]^{2n})a_2^2z^2+\ldots\\ =1+\frac{\Phi_1x_1z}{2}+\left[\frac{1}{2}\left(x_2-\frac{x_1^2}{2}\right)\Phi_1+\frac{x_1^2}{4}\Phi_2\right]z^2+\ldots,\end{aligned}\tag{18}$$

and

$$\begin{aligned}1-([1+\delta]^n(1+\alpha))a_2w+(4[1+2\delta]^n(1+2\alpha)\\ -[1+\delta]^{2n}(1+3\alpha))a_2^2w^2-2([1+2\delta]^n(1+2\alpha))a_3w^2+\ldots\\ =1+\frac{\Phi_1y_1w}{2}+\left[\frac{1}{2}\left(y_2-\frac{y_1^2}{2}\right)\Phi_1+\frac{y_1^2}{4}\Phi_2\right]w^2+\ldots.\end{aligned}\tag{19}$$

It follows from (18) and (19) that

$$([1+\delta]^n(1+\alpha))a_2=\frac{x_1\Phi_1}{2},\tag{20}$$

$$\begin{aligned}(([1+2\delta]^n2(1+2\alpha))a_3-[1+\delta]^{2n}(1+3\alpha)a_2^2)\\ =\frac{1}{2}\left(x_2-\frac{x_1^2}{2}\right)\Phi_1+\frac{x_1^2}{4}\Phi_2,\end{aligned}\tag{21}$$

and

$$-([1+\delta]^n(1+\alpha))a_2=\frac{\Phi_1y_1}{2},\tag{22}$$

$$(4[1+2\delta]^n(1+2\alpha)-[1+\delta]^{2n}(1+3\alpha))a_2^2-2([1+2\delta]^n(1+2\alpha))a_3 = \frac{1}{2}\left(y_2-\frac{y_1^2}{2}\right)\Phi_1+\frac{y_1^2}{4}\Phi_2. \tag{23}$$

From (20) and (22), we have

$$x_1=-y_1,$$

and

$$2a_2^2=\frac{x_1^2+y_1^2}{4([1+\delta]^n(1+\alpha))}\Phi_1^2. \tag{24}$$

Now, by adding (21) with (23), we obtain

$$(2[1+\delta]^{2n}(1-3\alpha)+4[1+2\delta]^n(1+2\alpha))a_2^2 = \frac{1}{2}(x_2+y_2)\Phi_1-\frac{1}{4}(x_1^2+y_1^2)\Phi_1+\frac{\Phi_2}{4}(x_1^2+y_1^2). \tag{25}$$

By placing (24) in (25) and making use of (6), we have

$$2[(-[1+\delta]^{2n}(1+3\alpha)+2[1+2\delta]^n(1+2\alpha)-3([1+\delta]^n(1+\alpha))^2)\Lambda+([1+\delta]^n(1+\alpha))^2]a_2^2=\frac{1}{2}(x_2+y_2)\Lambda^2.$$

Therefore, using Lemma (1) we obtain

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{\begin{array}{l}|([1+\delta]^n(1+\alpha))^2-([1+\delta]^{2n}(1+3\alpha)-2[1+2\delta]^n(1+2\alpha)\\+3([1+\delta]^n(1+\alpha))^2)\Lambda|\end{array}}}. \tag{26}$$

Thus, to measure the bound on $|a_3|$, subtract from (21) and (23). Thereby, we acquire

$$4[1+2\delta]^n(1+2\alpha)a_3-4(1+2\alpha)[1+2\delta]^n a_2^2=\frac{1}{2}(x_2-y_2)\Lambda.$$

Hence, we get

$$4[1+2\delta]^n(1+2\alpha)|a_3|\leq 2|\Lambda|+4(1+2\alpha)[1+2\delta]^n|a_2|^2.$$

Then, in view of (26), we obtain

$$|a_3|\leq\frac{|\Lambda|[(1+\alpha)^2-(3\alpha^2+\alpha)|\Lambda|]}{2(1+2\alpha)(1+\alpha)[(1+\alpha)-(2+3\alpha)\Lambda]}.$$

Corollary 3 *If* $f\in F_\Sigma^{n,\alpha}(\Phi)$, *then*

$$|a_2|\leq\frac{|\Lambda|}{\sqrt{|(1+\alpha)^2-(1+\alpha)(2+3\alpha)\Lambda|}},$$

$$|a_3| \leq \frac{|\Lambda|[(1+\alpha)^2 - (3\alpha^2+\alpha)|\Lambda|]}{2(1+2\alpha)(1+\alpha)[(1+\alpha) - (2+3\alpha)\Lambda]}.$$

For $\delta = 0$, Theorem 2 affirms Güney et al.'s [14] findings in Theorem 2.

Corollary 4 *If* $f \in F_{\Sigma}^{n}(\Phi)$, *then*

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{|1-2\Lambda|}},$$

$$|a_3| \leq \frac{|\Lambda|}{2(1-2\Lambda)}.$$

For $\delta = 0$ and $\alpha = 0$, Theorem 2 reduces to the one in Güney et al [14] as stated in Corollary 1.

Corollary 5 *If* $f \in F_{\Sigma}^{n,1}(\Phi)$, *then*

$$|a_2| \leq \frac{|\Lambda|}{\sqrt{|4-10\Lambda|}},$$

$$|a_3| \leq \frac{|\Lambda|[1-\Lambda]}{3(2-5\Lambda)}.$$

For $\delta = 0$ and $\alpha = 1$, Theorem 2 affirms Güney et al [14] Corollary 2.

4 Conclusions

Our motivation is to get many interesting and fruitful usages of a wide variety of Fibonacci numbers in Geometric Function Theory. We presented two new subclasses of the bi-univalent functions related to the Fibonacci numbers based on the Al-Oboudi differential operator. From the new subclasses proposed, we attain the initial Taylor-Maclaurin coefficient estimates of the bi-univalent function classes $D_{\Sigma,\delta}^{n,\rho}(\Phi)$ and $F_{\Sigma,\delta}^{n,\alpha}(\Phi)$. Here, we have outlined some implications and generalisations from previous research using corollaries of our main theorems. Therefore, the results that we obtained in this study are the best possible. These results were an improvement on the estimates obtained in the recent studies. Some interesting remarks of the results presented here were also discussed.

References

[1] F. M. Al-Oboudi. On univalent functions defined by a generalized Sălăgean operator. *International Journal of Mathematics and Mathematical Sciences*, 2004(27):1429–1436, 2004.

[2] Ş. Altinkaya. Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers. *Turkish Journal Of Mathematics*, 44:553–560, 2020.

[3] Ş. Altinkaya and S. Yalçin. Some applications of generalized Srivastava-Attiya operator to the bi-concave functions. *Miskolc Mathematical Notes*, 21(1):51, 2020.

[4] Ş. Altınkaya, S. Yalçın, and S. Çakmak. A Subclass of Bi-Univalent Functions Based on the Faber Polynomial Expansions and the Fibonacci Numbers. *Mathematics*, 7(2):160, February 2019.

[5] K. M. Ball. *Strange curves, counting rabbits, and other mathematical explorations*. Princeton University Press, Princeton, N.J, 3rd printing, and first paperback printing edition, 2007.

[6] D.A. Brannan and T.S. Taha. On Some Classes Of Bi-Univalent Functions. In *Mathematical Analysis and its Applications*, pages 53–60. Elsevier, 1988.

[7] S. Bulut. Some Properties For An Integral Operator Defined By Al-Oboudi Differential Operator. *Journal of Inequalities in Pure and Applied Mathematics*, 9(4):115, 5, 2008.

[8] S. Bulut. A New General Integral Operator Defined by Al-Oboudi Differential Operator. *Journal of Inequalities and Applications*, 2009(1):158408, 2009.

[9] P. L. Duren. *Univalent functions*. Number 259 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York, 1983.

[10] J. Dziok, R. K. Raina, and J. Sokół. On alpha-convex functions related to shell-like functions connected with Fibonacci numbers. *Applied Mathematics and Computation*, 218(3):996–1002, 2011.

[11] S. S. Eker and S. Owa. New Applications of classes of analytic functions involving the Salagean operator. *In Proceedings Of The International Symposium On Complex Function Theory And Applications*, pages 21–34, 2006.

[12] S. S. Eker and B. Seker. On a class of multivalent functions defined by Salagean operator. *General Mathematics*, 15:154–163, 2007.

[13] F. W. Gehring. Aspects Of Contemporary Complex Analysis Proceedings of an instructional conference organized by the London Mathematical Society at the University of Durham, July 1979. *Bulletin of the London Mathematical Society*, 14(4):382–383, July 1982.

[14] H. Ö. Güney, G. Murugusundaramoorthy, and J. Sokół. Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers. *Acta Universitatis Sapientiae, Mathematica*, 10(1):70–84, 2018.

[15] H. Ö. Güney, G. Murugusundaramoorthy, and J. Sokół. Certain subclasses of bi-univalent functions related to k-Fibonacci numbers. *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*, 68(2):1909–1921, July 2019.

[16] P. Koebe. Über die Uniformisierung beliebiger analytischer Kurven, (1907), 191-210. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 191–210, 1907.

[17] M Lewin. On a coefficients problem of bi-univalent functions. *Proceedings of the American Mathematical Society*, 18:63–68, 1967.

[18] P. Lokesh and B. S. Keerthi. Fekete–Szego Problem for Sakaguchi kind of Functions Related to Shell–like Curves Connected with Fibonacci Numbers. *International Journal of Innovative Technology and Exploring Engineering*, 8(10S):222–226, 2019.

[19] N. Magesh, C. Abirami, and S. Altinkaya. Initial Bounds For Certain Classes Of Bi-Univalent Functions Defined By The (p,q)-Lucas Polynomials. *Journal of Applied and Engineering Mathematics*, 11(1):282+, 2021.

[20] N. Magesh, J. Nirmala, and J. Yamini. Initial estimates for certain subclasses of bi-univalent functions with k-Fibonacci numbers. *arXiv:2001.08569 [math]*, January 2020. arXiv: 2001.08569.

[21] S. S. Miller and P. Mocanu. *Differential Subordinations: Theory and Applications*. Number 225 in Monographs and textbooks in pure and applied mathematics. M. Dekker, New York, 2000.

[22] G. Murugusundaramoorthy, K. Vijaya, and H. Ö Güney. On λ-Pseudo Bi-Starlike Functions with Respect to Symmetric Points Associated to Shell-Like Curves. *Kragujevac Journal of Mathematics*, 45(01):103–114, 2021.

[23] E. Netanyahu. The Minimal Distance of the Image Boundary from the Origin and the Second Coefficient of a Univalent Function in Iz|<1. *Archive for Rational Mechanics and Analysis*, 32(2):100–112, 1969.

[24] S. O. Olatunji and H. Dutta. Sigmoid Function in the Space of Univalent Lambda-Pseudo-(p,q)-Derivative Operators Related to Shell-Like Curves Connected with Fibonacci Numbers of Sakaguchi Type Functions. *Malaysian Journal Of Mathematical Sciences*, 13(1):95–106, 2019.

[25] H. Orhan, N. Magesh, and C. Abirami. Feketo-Szegö problem for Bi-Bazilevic functions related to shell-like curves. *AIMS Mathematics*, 5(5):4412–4423, 2020.

[26] S. Ruscheweyh. New Criteria For Univalent Functions. *Proceedings of the American Mathematical Society*, 49(1):109–115, May 1975.

[27] F. M. Sakar, H. Ö. Güney, and M. Kamali. A New Generalized Al-Oboudi Differential Operator Defined By Wright's Generalized Hypergeometric Functions. *Acta Universitatis Apulensis*, 31:293–306, 2012.

[28] T. G. Shaba, M. G. Khan, and B. Ahmad. Coefficients Bounds for Certain New Subclasses of Meromorphic Bi-univalent Functions Associated with Al-Oboudi Differential Operator. *Palestine Journal of Mathematics*, 9(2), 2020. Publisher: Unpublished.

[29] T. G. Shaba, A. T. Tiamiyu, I. O. Ibrahim, and A. A. Ibrahim. Subclass of p-valent Function with Negative Coefficients Applying Generalized Al-Oboudi Differential Operator. *Earthline Journal of Mathematical Sciences*, 6(1):87–103, 2021.

[30] G. Singh, G. Singh, and G. Singh. A Subclass of Bi-Univalent Functions Defined by Generalized Sălăgean Operator Related to Shell-Like Curves Connected with Fibonacci Numbers. *International Journal of Mathematics and Mathematical Sciences*, 2019:7 : Article ID 7628083, 2019.

[31] G. Singh, G. Singh, and G. Singh. Certain subclasses of univalent and bi-univalent functions related to shell-like curves connected with Fibonacci numbers. *General Mathematics*, 28(1):125–140, June 2020.

[32] G. S. Sălăgean. Subclasses of univalent functions. In Cabiria Andreian Cazacu, Nicu Boboc, Martin Jurchescu, and Ion Suciu, editors, *Complex Analysis — Fifth Romanian-Finnish Seminar*, volume 1013, pages 362–372. Springer Berlin Heidelberg, Berlin, Heidelberg, 1983. Series Title: Lecture Notes in Mathematics.

[33] J. Sokól. On starlike functions connected with Fibonacci numbers. *Folia scientiarum Universitatis Technicae Resoviensis*, 175:111–116, 1999.

[34] H.M. Srivastava, A.K. Mishra, and P. Gochhayat. Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters*, 23(10):1188–1192, 2010.

[35] P. Sumalatha, R. B. Sharma, and M. H. Priya. The third Hankel determinant for starlike functions with respect to symmetric points subordinate to k-Fibonacci sequence. *AIP Conference Proceedings*, 2112:020069, 2019.

[36] D. M. Thirucheran and T. Stalin. On a New Subclass of Multivalent Functions Defined by Al-Oboudi Differential Operator. *Global Journal of Pure and Applied Mathematics*, 14(5):733–741, 2018.

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