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Approximating intuitionistic fuzzy fractals by densifiability techniques [1]

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Abstract

We present a sequence of sets converging, under suitable conditions and respect to the Hausdorff intuitionistic fuzzy metric, to the attractor set of certain intuitionistic fuzzy iterated function systems. For this goal, we will introduce a *fuzzy version* of the so called α-dense curves which have been used by the author to approximate, with arbitrarily small and controlled error, the attractor set of certain (metric) iterated function systems. In this way, we relate the above mentioned concepts of the intuitionistic fuzzy metric spaces with the α-density theory.

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1 Introduction

Since the introduction of the Fractal Analysis in 1975 by the seminal work of Mandelbrot [21], this theory has been studied and popularized by several authors, among whose we can stress the works of Barnsley [5] and Hutchinson [18]. Meaning by a *fractal*, in our context, a compact set which is the fixed point of an iterated function system (see Section 2), as we will discuss in Section 3, there are some procedures to approximate such set. This will be the main goal of this paper: approximate certain class of fractals using a novel technique.

On the other hand, the definition of *fuzzy set* was introduced in 1965 by Zadeh [39] and since then, a theory for the so called *fuzzy metric spaces* has been widely

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developed by many authors (see, for instance, [13, 17, 19, 20, 22, 27, 28, 29, 37, 38] and references therein). In particular, such works contain many interesting results related to the *intuitionistic fuzzy metric spaces*, defined in 2004 by Park [27]. We will review briefly these concepts in Section 2.

In Section 3 we recall the concepts of α-dense curves and densifiable sets in metric spaces and show a procedure to approximate under suitable conditions and with arbitrarily small and controlled error, the attractor set of an iterated function system (called *intuitionistic fuzzy fractal*) in a special class of fuzzy metric spaces, namely, the *standard intuitionistic fuzzy metric spaces* (see Theorem 3.1). The main tool in such result will be the α-dense curves.

To end the paper, in Section 4 (see Theorem 4.1), we will show a sequence of finite sets converging, under suitable conditions, to the fuzzy fractal of an iterated function system in a general intuitionistic fuzzy metric space. Such sequence of sets will be constructed using a new type of curves, the *fuzzy α-dense curves* which are defined in a fuzzy metric space and will be more suitable for this goal than the α-dense curves.

We conclude this section with the following observation. The concept of fuzzy metric space, as expected, has been generalized in many ways; see, for instance, [10, 36] and references therein. However, in this paper we focus in the intuitionistic fuzzy metric space because these spaces are more suitable for our goals: such spaces can be generated from an usual metric space (X, d), where there is a robust theory of the main tool that we use in our developments, namely, the α-dense curves. Therefore, the new concepts introduced here, and related with such curves, can be defined in a more natural way than in other generalized fuzzy metric spaces. The same for the corresponding *Hausdorff intuitionistic fuzzy metric* (see Definition 2.9).

2 Iterated function systems and intuitionistic fuzzy iterated function systems

For a better comprehension of the manuscript, we need to recall some basic definitions and well known facts related to the iterated function systems. In the following definitions, (X, d) will be a metric space and $\mathcal{K}(X)$ the class of all non-empty compact subsets of X.

Definition 2.1 (Hausdorff metric, [6, 32])**.** *The Hausdorff metric $H_d : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow [0, +\infty)$ is defined by*

$$H_d(A, B) := \max\big\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\big\},$$

for each $A, B \in \mathcal{K}(X)$.

It is a well known fact (see, for instance, [6, 30]) that $(\mathcal{K}(X), H_d)$ is a complete metric space. Bearing in mind that $f : (X, d) \longrightarrow (X, d)$ is called a r-contraction, for some $0 < r < 1$, if $d(f(x), f(y)) \leq rd(x, y)$, for all $x, y \in X$, we have the following key definition for our goals:

Definition 2.2 (Iterated function system, [32]). *Let $f_i : X \longrightarrow X$ be a r_i-contraction for some $0 < r_i < 1$, for $i = 1, \ldots, n$. Then, $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ is called an iterated function system, in short IFS.*

Another key concept to understand the theory of fractals in metric spaces is the following:

Definition 2.3 (Hutchinson-Barnsley operator, [5]). *Let $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ be an IFS. The set valued map $F : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$ given by*

$$F(K) := \bigcup_{i=1}^{n} f_i(K) \quad \text{for all } B \in \mathcal{K}(X),$$

is called the Hutchinson-Barnsley operator, in short the HB operator.

As the HB operator F is a r-contraction, $r := \max\{r_i : i = 1, \ldots, n\}$, in $(\mathcal{K}(X), H_d)$ (see, for instance, [18, 30]) the celebrated Banach contraction principle states that F has a unique fixed point, called the *attractor set* (or the *fractal*) of the IFS $\mathcal{F}$. Formally:

Theorem 2.1 (Collage Theorem for IFS, [5]). *Let $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ be an IFS and F the HB operator. Then, F is a r-contraction, with $r := \max\{r_i : i = 1, \ldots, n\}$, its (unique) fixed point $\mathcal{A}$ is called the attractor set (or the fractal) of the IFS $\mathcal{F}$ and*

$$\lim_k F^k(B) = \mathcal{A},$$

for every $B \in \mathcal{K}(X)$, where the limit means respect to the Hausdorff metric and the exponent for the composition. Moreover, for each $B \in \mathcal{K}(X)$ we have

$$H_d(\mathcal{A}, B) \leq \frac{H_d(B, F(B))}{1 - r}.$$

As expected, there are many generalizations of the IFS (see, for instance, [6, 8, 12, 32, 33, 34, 35] and references therein.

On the other hand, and as we have pointed out in Section 1, we are interested in the so called *intuitionistic fuzzy fractals*. Before, we need to recall the basic concepts related to the fuzzy metric spaces. We follow the notation and terminology used in [22, 27, 37, 38].

The concept of *fuzzy set* was defined in 1965 by Zadeh [39]:

Definition 2.4 (Fuzzy set). *A fuzzy set M in X is a function with domain X and values in $I := [0, 1]$.*

Informally, a fuzzy set is a non-empty set X with a "membership function" $M : X \longrightarrow I$. As an extension of the fuzzy sets above defined, Atanassov [2] introduced in 1986 the concept of *intuitionistic fuzzy set*:

Definition 2.5 (Intuitionistic fuzzy set)**.** *An intuitionistic fuzzy set is a 3-tuple* (X, M, N) *where*$N, M : X \longrightarrow I$ *and* $0 \leq M(x) + N(x) \leq 1$*, for every* $x \in X$*.*

Roughly speaking, in an intuitionistic fuzzy set for each $x \in X$, in addition to the "degree of membership" function $M(x)$, we can also quantify (with the function N) the "degree of non-membership" of x respect to M. For a more detailed of the above ideas, we refer to the Atanassov works [2, 3, 4].

A very important concept to understand the concepts of *fuzzy* and *intuitionistic fuzzy metric space* is that of *continuous t-norm* (see, for instance, [27, 31]):

Definition 2.6 (Continuous t-norm)**.** *A continuous t-norm is a binary operation* $* : I^2 \longrightarrow I$ *satisfying the following conditions:*

(i) $*$ *is continuous, commutative and associative.*

(ii) $a * 1 = a$ *for all* $a \in I$*.*

(iii) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$*, for all* $a, b, c, d \in I$*.*

From the above definition, we can give the following (see, for instance, [17, 29]):

Definition 2.7 (Fuzzy metric space)**.** *A 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* $*$ *is a continuous t-norm and* M *is a fuzzy set defined on* $X^2 \times (0, +\infty)$ *satisfying the following conditions:*

(i) $M(x, y, t) > 0$*.*

(ii) $M(x, y, t) = 1$ *if, and only if,* $x = y$*.*

(iii) $M(x, y, t) = M(y, x, t)$*.*

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*.*

(v) $M(x, y, \cdot) : (0, +\infty) \longrightarrow (0, 1]$ *is continuous.*

for every $x, y, z \in X$ *and* $s, t > 0$*. The fuzzy set* M *is often called a metric on* X*.*

Next, we show a class of spaces introduced in 2004 by Park [27] (see also [37, 38]) as a natural generalization of the fuzzy metric spaces:

Definition 2.8 (Intuitionistic fuzzy metric space)**.** *A 5-tuple* $(X, M, N, *, \Diamond)$ *is said to be an intuitionistic fuzzy metric space if* $*, \Diamond$ *are continuous t-norms and* M, N *are fuzzy sets defined on* $X^2 \times (0, +\infty)$ *satisfying the following conditions:*

(i) $M(x, y, t) > 0$*,* $N(x, y, t) < 1$ *and* $0 \leq M(x, y, t) + N(x, y, t) \leq 1$*.*

(ii) $M(x, y, t) = 1$ *if, and only if,* $x = y$*.*

(iii) $N(x, y, t) = 0$ *if, and only if,* $x = y$*.*

(iv) $M(x,y,t) = M(y,x,t)$ *and* $N(x,y,t) = N(y,x,t)$.

(v) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$.

(vi) $N(x,y,t) \Diamond N(y,z,s) \geq N(x,z,t+s)$.

(vii) $M(x,y,\cdot), N(x,y,\cdot) : (0,+\infty) \longrightarrow (0,1]$ *are continuous.*

for every $x,y,z \in X$ *and* $s,t > 0$. *The pair* (M,N) *is called an intuitionistic fuzzy metric on* X.

For instance, if $(X,M,*)$ is a fuzzy metric space, putting $N(x,y,t) := 1 - M(X,y,t)$, for each $x,y \in X$ and $t > 0$, and $a \Diamond b := 1 - ((1-a) * (1-b))$, for each $a,b \in I$, we find that $(X,M,N,*,\Diamond)$ is an intuitionistic fuzzy metric space. In the following example, we show how we can construct an intuitionistic fuzzy metric space from a metric space:

Example 2.1. *Let* (X,d) *be a metric space,* $*, \Diamond$ *two t-norms and* $M, N : X^2 \longrightarrow (0,+\infty)$ *the fuzzy sets defined by*

$$M_d(x,y,t) := \frac{t}{t+d(x,y)} \quad \textit{and} \quad N_d(x,y,t) := 1 - M_d(x,y,t) = \frac{d(x,y)}{t+d(x,y)},$$

for every $x,y \in X$ *and* $t > 0$. *Then,* $(X,M_d,*)$ *(resp.* $(X,M_d,N_d,*,\Diamond)$*) is known as the standard fuzzy metric space (resp. the standard intuitionistic fuzzy metric space) induced by d, and* M_d *(resp.* (M_d,N_d)*) the standard fuzzy metric (resp. the standard intuitionistic fuzzy metric); see, for instance, [27].*

Hereinafter $(X,M,N,*,\Diamond)$ will be an intuitionistic fuzzy metric space, $\mathcal{T}_{(M,N)}$ the topology induced by the intuitionistic fuzzy metric (M,N) (see also Remark 4.1) and $\mathcal{K}_0(X)$ the class of all non-empty compact subsets of $(X,\mathcal{T}_{(M,N)})$.

Remark 2.1. *It can be proved (see [37]) that a non-empty set* $B \subset X$ *is compact in* (X,d) *if, and only if,* B *is compact in* $(X,\mathcal{T}_{(M_d,N_d)})$*, where* $\mathcal{T}_{(M_d,N_d)}$ *is the topology induced by the standard intuitionistic fuzzy metric* (M_d,N_d).

Definition 2.9 (Hausdorff intuitionistic fuzzy metric [37])**.** *For each* $A,B \in \mathcal{K}_0(X)$ *and* $t > 0$ *let*

$$M(A,B,t) := \inf_{x\in A} \sup_{y\in B} M(x,y,t) \quad \textit{and} \quad N(A,B,t) := \sup_{x\in A} \inf_{y\in B} N(x,y,t).$$

Then, the Hausdorff intuitionistic fuzzy metric (H_M,H_N) *is defined as*

$$H_M(A,B,t) := \min\big\{M(A,B,t), M(B,A,t)\big\} \quad \textit{and}$$

$$H_N(A,B,t) := \max\big\{N(A,B,t), N(B,A,t)\big\}.$$

Of course, (H_M, H_N) is an intuitionistic fuzzy metric on $\mathcal{K}_0(X)$ and hence $(\mathcal{K}_0(X), H_M, H_N, *, \Diamond)$ is an intuitionistic fuzzy metric space, called the *Hausdorff intuitionistic fuzzy metric space*. Recall (see, for instance, [17, 38]) that a fuzzy (or intuitionistic fuzzy) metric space is complete if each Cauchy sequence is convergent, that is to say, if $(x_n)_{n\geq 1} \subset X$ satisfies that $\lim_n M(x_n, x_{n+p}, t) = 1$ (and therefore $\lim_n N(x_n, x_{n+p}, t) = 0$) for every integer $p \geq 1$ and $t > 0$, then there is $x \in X$ such that $\lim_n M(x_n, x, t) = 1$ (and therefore $N(x_n, x, t) = 0$) for every $t > 0$. Then, for the completeness of $(\mathcal{K}_0(X), H_{M_d}, H_{N_d}, *, \Diamond)$, we have the following well known result (see, for instance, [37, Proposition III.1] and the proof of [37, Theorem V.1]):

Proposition 2.1. *The following conditions are equivalent:*

(1) (X, d) *is complete.*

(2) The standard intuitionistic fuzzy metric space induced by d, $(X, M_d, N_d, *, \Diamond)$, *is complete.*

(3) The intuitionistic fuzzy metric space $(\mathcal{K}_0(X), H_{M_d}, H_{N_d}, *, \Diamond)$ *is complete.*

In the above result, the fuzzy metrics must be those induced by d, as it is evidenced in the following example.

Example 2.2. *Let* $X := (0, 1]$, $*$ *and* $\Diamond$ *two t-norms and* $d := |\cdot|$. *As* (X, d) *is not complete, by the above result* $(X, M_d, N_d, *, \Diamond)$ *is not complete. However, if we define*

$$M(x, y, t) := \frac{t}{t + |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right|} \quad \text{and} \quad N(x, y, t) := 1 - M(x, y, t),$$

for each $x, y \in X$ *and* $t > 0$, *it is straightforward to show that* $(X, M, N, *, \Diamond)$ *is complete (see also [29]).*

Another interesting property of the Hausdorff intuitionistic fuzzy metric is the following (see, for instance, [13, Proposition 1]):

Proposition 2.2. *The Hausdorff intuitionistic fuzzy metric* (H_{M_d}, H_{N_d}) *of the standard intuitionistic fuzzy metric* (M_d, N_d) *coincides with the standard intuitionistic fuzzy metric* (M_{H_d}, N_{H_d}) *of the Hausdorff metric on* $\mathcal{K}_0(X)$, *that is to say,*

$$H_{M_d}(A, B, t) = M_{H_d}(A, B, t) = \tfrac{t}{t + H_d(A,B)} \quad \text{and}$$

$$H_{N_d}(A, B, t) = N_{H_d}(A, B, t) = \frac{H_d(A, B)}{t + H_d(A, B)},$$

for every $A, B \in \mathcal{K}_0(X)$ *and* $t > 0$.

Now, we can define the concept of IFS in intuitionistic fuzzy metric spaces.

Definition 2.10 (Intuitionistic fuzzy iterated function system, [37]). *Given $f : X \longrightarrow X$ and $0 < r < 1$, f is called intuitionistic fuzzy r-contraction in $(X, M, N, *, \Diamond)$ if*

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq r\Big(\frac{1}{M(x, y, t)} - 1\Big) \quad \textit{and}$$

$$\frac{1}{N(f(x), f(y), t)} - 1 \geq \frac{1}{r}\Big(\frac{1}{N(x, y, t)} - 1\Big),$$

for each $x, y \in X$ and $t > 0$. If $f_i : X \longrightarrow X$, for $i = 1, \ldots, n$, is an intuitionistic fuzzy r_i-contraction, for some $0 < r_i < 1$, then $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ is called an intuitionistic fuzzy iterated function system, shortly IF-IFS, in $(X, M, N, *, \Diamond)$.

A large class of IF-IFS in the standard intuitionistic fuzzy metric space induced by d, $(X, M_d, N_d, *, \Diamond)$, is obtained from the following result (see [22]):

Proposition 2.3. *$\mathcal{F} := \{X; f_1, \ldots, f_n\}$ is an IFS if, and only if, $\mathcal{F}$ is an IF-IFS in $(X, M_d, N_d, *, \Diamond)$.*

At this point, we can define the HB operator in the Hausdorff intuitionistic fuzzy metric space $(\mathcal{K}_0(X), H_M, H_N, *, \Diamond)$.

Definition 2.11 (The intuitionistic fuzzy HB operator, [37]). *Let $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ be an IF-IFS in $(X, M, N, *, \Diamond)$, Then, the intuitionistic fuzzy HB operator, in short the IF-HB operator, is the set valued map $F : \mathcal{K}_0(X) \longrightarrow \mathcal{K}_0(X)$ defined as:*

$$F(K) := \bigcup_{i=1}^{n} f_i(K) \quad \textit{for all } K \in \mathcal{K}_0(X).$$

The following collage theorem for IF-IFS holds (see [37, Theorem IV.3]):

Theorem 2.2 (Collage Theorem for IF-IFS in standard intuitionistic fuzzy metric spaces). *Let (X, d) be a complete metric space, $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ an IF-IFS in $(X, M_d, N_d, *, \Diamond)$ and the IF-HB operator. Then, F is a r-contraction, with $r := \max\{r_i : i = 1, \ldots, n\}$, its (unique) fixed point $\mathcal{A}$ is called the attractor (or the intuitionistic fuzzy fractal) of the IF-IFS $\mathcal{F}$, and*

$$\lim_k F^k(B) = \mathcal{A},$$

for every $B \in \mathcal{K}(X)$. Moreover, for each $B \in \mathcal{K}_0(X)$:

$$\begin{aligned} &H_{M_d}(\mathcal{A}, B, t) \geq H_{M_d}(B, F(B), t(1-r)) \quad \textit{and} \\ &H_{N_d}(\mathcal{A}, B, t) \leq H_{N_d}(B, F(B), t(1-r)). \end{aligned} \tag{2.1}$$

Although the above result gives us a lower and an upper bound for $H_{M_d}(\mathcal{A}, B, t)$ and $H_{N_d}(\mathcal{A}, B, t)$, i.e. an approximation error for the intuitionistic fuzzy fractal, we will show in Theorem 3.1 a sequence of sets which provides us *more fine* bounds for the approximation error.

For $(X, M, N, *, \lozenge)$, we have the following result:

Theorem 2.3. *Assume that $(X, M, N, *, \lozenge)$ is complete, $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ is an IF-IFS in such intuitionistic fuzzy metric space. Then, the IF-HB operator F is a r-contraction, with $r := \max\{r_i : i = 1, \ldots, n\}$ and its (unique) fixed point $\mathcal{A}$ is called the attractor (or the intuitionistic fuzzy fractal) of the IF-IFS $\mathcal{F}$.*

Proof. As $(X, M, N, *, \lozenge)$ is complete, $(\mathcal{K}_0(X), H_M, H_N, *, \lozenge)$ is complete (see [13, Theorem 5]). It is not very hard to prove that the intuitionistic fuzzy HB operator F (see Definition 2.11) is an intuitionistic fuzzy r-contraction in $(\mathcal{K}_0(X), H_M, H_N, *, \lozenge)$ (see also the proof of Theorem 4.1), and therefore the result follows from [28, Theorem 3.6].

□

3 α-dense curves and densifiable sets in metric spaces

The concepts of α-dense curves and densifiable sets in a metric space (X, d) were introduced in 1997 by Mora and Cherruault [25]:

Definition 3.1. *Given $\alpha \geq 0$ and $B \subseteq X$ non-empty and bounded, a continuous map $\gamma : I \longrightarrow (X, d)$ is said to be an α-dense curve in B if the following conditions hold:*

(i) $\gamma(I) \subset B$.

(ii) For any $x \in B$, there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$.

If for every $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Clearly, given a non-empty and bounded set $B \subseteq X$, there is always an α-dense curve in B for any $\alpha \geq \mathrm{Diam}(B)$, the diameter of B. Indeed, fixed $x_0 \in B$, the map $\gamma(\tau) := x_0$, for all $\tau \in I$, is an α-dense curve in B whenever $\alpha \geq \mathrm{Diam}(B)$.

Example 3.1. *For each positive integer m, the map $\gamma_m : I \longrightarrow \mathbb{R}^d$, $d > 1$, given by*

$$\gamma_m(\tau) := \Big(\tau, \frac{1}{2}\big(1 - \cos(m\pi\tau)\big), \ldots, \frac{1}{2}\big(1 - \cos(m^{d-1}\pi\tau)\big)\Big) \quad \textit{for all } \tau \in I,$$

is a $\frac{\sqrt{d-1}}{m}$-dense curve in I^d, called the cosines curve (see [7, Proposition 9.5.4, p. 144]).

Note that the parameter α of Definition 3.1 coincides with the Hausdorff distance from B to $\gamma(I)$. So, we can say that the α-dense curves are a generalization of the so called *space-filling curves* (see [30]). In fact, when $B := I^d$ for some integer

$d > 1$, a 0-dense curve in B is, precisely, a space-filling curve in B. Furthermore, it can be proved that the class of densifiable sets is strictly between the class of Peano Continua (i.e. those sets that are the continuous image of I) and the class of connected and precompact sets (see [24, 26]).

We have the following result (see [26]):

Proposition 3.1. *Let $B \subseteq X$ non-empty, arc-connected and precompact. Then, B is densifiable.*

Of course, not every bounded (even compact) subset of X is densifiable:

Example 3.2. *In the Euclidean plane $X := \mathbb{R}^2$ the set*

$$B := \big\{\big(x, \sin(x^{-1})\big) : x \in [1,0) \cup (0,1]\big\} \cup \{(0,y) : y \in [-1,1]\}.$$

is a compact and connected set that is not densifiable. Indeed, since B is not arc-connected, it is immediate to check that there is no an α-dense curve in B for any $0 < \alpha < 1$.

For a detailed exposition of the α-dense curves and their applications, see [7, 14, 23, 25] and references therein.

On the other hand, if $\mathcal{F} := \{X; f_1, \ldots, f_n\}$, $n > 1$, is an IFS in the metric space (X, d), there are some procedures to approximate the attractor set of $\mathcal{F}$; see, for instance, [9, 11]. Moreover, the author has proved in [15] (see also [16]) that the α-dense curves can be used to approximate under suitable conditions, with arbitrarily small and controlled error, the attractor set of the so called *countable iterated functions systems* (see, for instance, [33]) and, consequently, the attractor set of certain IFS. We expose this result in the following lines.

Let $r := \max\{r_i : i = 1, \ldots, n\}$ where r_i is the ratio contraction of f_i, for $i = 1, \ldots, n$, F the HB operator and $\mathcal{A}$ its fixed point (i.e., the attractor set of $\mathcal{F}$). Assume X is densifiable, and for each $k \geq 1$ let $\gamma_k : I \longrightarrow X$ be an α_k-dense curve in X, with $\alpha_k \to 0$. Define $B_1 := \{f_1(x_1)\}$, and for each $k > 1$ let

$$\begin{aligned} B_k := \big\{ & f_{j_1 \cdots j_k}(y_k) : d(x_{i_1 \cdots i_{k-1}}, y_k) \leq \alpha_k, \text{ for each } x_{i_1 \cdots i_{k-1}} \in B_{k-1} \\ & \text{and some } y_k \in \gamma_k(I), \text{ where } j_1, \ldots, j_k \in N_k \text{ and } i_1, \ldots, i_{k-1} \in N_{k-1}\big\}. \end{aligned} \tag{3.1}$$

where $f_{j_1 \cdots j_k} := f_{j_1} \circ \cdots \circ f_{j_k}$, $N_k := \{1, \ldots, k\}$ for $1 < k < n$ and $N_k := \{1, \ldots, n\}$ for $k \geq n$. Then, the sequence $(B_k)_{k \geq 1} \subset X$ converges, with respect to the metric H_d, to the attractor set $\mathcal{A}$. More specifically:

Proposition 3.2. *[15, Corollary 3.1]. Under the above conditions, we have:*

$$H_d(\mathcal{A}, B_k) \leq r^k \Big(\frac{\mathrm{Diam}(X)}{1 - r^k} + \alpha_k \Big) \quad \textit{for all } k \geq n.$$

As, by virtue of Proposition 2.3, $\mathcal{F}$ is an IF-IFS in the standard intuitionistic fuzzy metric space $(X, M_d, N_d, *, \lozenge)$ induced by d, if we consider the IF-HB operator F and its fixed point $\mathcal{A}$ (i.e., the intuitionistic fuzzy fractal of $\mathcal{F}$), we have the following result:

Theorem 3.1. *Assume that (X, d) is densifiable, and let $(B_k)_{k\geq 1} \subset X$ be the sequence defined in (3.1). Then, for every $t > 0$, the inequalities*

$$H_{M_d}(\mathcal{A}, B_k, t) \geq \frac{t(1-r^k)}{t(1-r^k)+\rho_k} \quad \textit{and} \quad H_{N_d}(\mathcal{A}, B_k, t) \leq \frac{\rho_k}{t(1-r^k)+\rho_k},$$

hold for every $k \geq n$, where $\rho_k := r^k(\mathrm{Diam}(X) + (1-r^k)\alpha_k)$.

Proof. Fixed $k \geq n$, by Proposition 3.2, we have

$$H_d(\mathcal{A}, B_k) \leq r^k\Big(\frac{\mathrm{Diam}(X)}{1-r^k} + \alpha_k\Big) \quad \text{for all } k \geq n. \tag{3.2}$$

Next, given $t > 0$, by Proposition 2.2 and noticing (3.2) we have

$$H_{M_d}(\mathcal{A}, B_k, t) = M_{H_d}(\mathcal{A}, B_k, t) = \tfrac{t}{t+H_d(\mathcal{A},B_k)} \geq$$

$$\frac{t(1-r^k)}{t(1-r^k)+r^k\big(\mathrm{Diam}(X)+(1-r^k)\alpha_k\big)} = \frac{t(1-r^k)}{t(1-r^k)+\rho_k},$$

and, as $0 \leq H_{M_d}(\mathcal{A}, B_k, t) + H_{N_d}(\mathcal{A}, B_k, t) \leq 1$, we find

$$H_{N_d}(\mathcal{A}, B_k, t) \leq 1 - H_{M_d}(\mathcal{A}, B_k, t) \leq 1 - \frac{t(1-r^k)}{t(1-r^k)+\rho_k} = \frac{\rho_k}{t(1-r^k)+\rho_k},$$

and the result follows.

□

Clearly, the above result provide us a sequence to approximate (with controlled and arbitrarily small error) the intuitionistic fuzzy fractal of $\mathcal{F}$ as, roughly speaking, the degree of *nearness* between $\mathcal{A}$ and B_k, $H_{M_d}(\mathcal{A}, B_k, t)$, increases with k while the degree of *non-nearness* $H_{N_d}(\mathcal{A}, B_k, t)$ decreases, for every $t > 0$. Formally:

$$\lim_k H_{M_d}(\mathcal{A}, B_k, t) = 1 \quad \text{and} \quad \lim_k H_{N_d}(\mathcal{A}, B_k, t) = 0, \quad \text{for all } t > 0. \tag{3.3}$$

Also, let us note that each B_k is a finite set, and therefore it can be easily draw with a suitable software. We conclude this section with an example.

Example 3.3. *Let T be the Sierpińsky triangle, which is the attractor set of the IFS $\{I^2; f_1, f_2, f_3\}$ where*

$$f_1(x,y) := \Big(\frac{1}{2}x, \frac{1}{2}y\Big), \quad f_2(x,y) := \Big(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\Big), \quad f_3(x,y) := \Big(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{1}{2}\Big),$$

for each $x, y \in I$ (see, for instance, [5, 30]). Then, in Figure 3.1 we can see an approximation of T by using the so called chaos game (see [5]) starting at $x_1 := (0, 0)$, and some points generated by the procedure of Proposition 3.2 using the cosines curve of Example 2.1. In this case $\mathrm{Diam}(I^2) = \sqrt{2}$, *$r = 1/2$ and $\alpha_k = 1/k$, for each $k \geq 1$.*

Of course, the inequalities of Theorem 3.1 hold for the IF-IFS $\{I^2; f_1, f_2, f_3\}$ in the standard intuitionistic fuzzy metric space $(I^2, M_d, N_d, *, \Diamond)$ induced by d, d being the Euclidean distance.

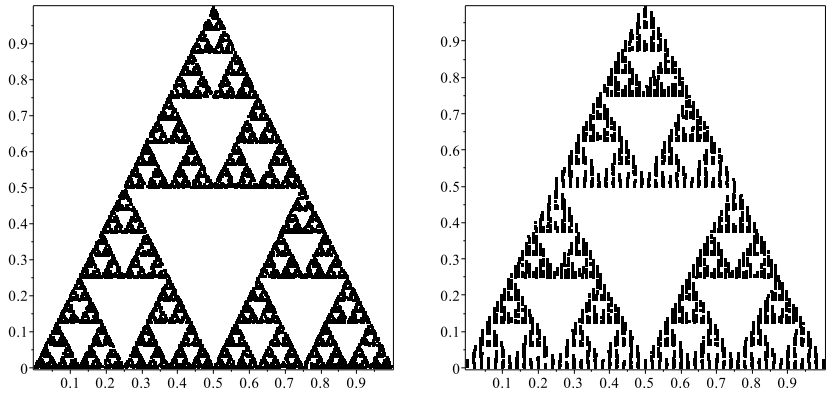


Figure 3.1: Some points in T (left) and in the set B_{50} (right).

4 The main result

To state a similar result to Theorem 3.1 in an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, first we need to introduce the concept of α-dense curve and densifiable set in a fuzzy metric space $(X, M, *)$.

Definition 4.1. *A continuous function $\varphi : (0, +\infty) \times [0, +\infty) \longrightarrow I$ is said to be a fuzzy comparison function if satisfies the following conditions for every $t > 0$:*

(i) $\varphi(t, \cdot)$ is decreasing.

(ii) $\varphi(t, \alpha) = 1$ if, and only if, $\alpha = 0$.

(iii) $\lim_{\alpha \to +\infty} \varphi(t, \alpha) = 0$.

The suggested name for φ in the above definition is justified by the following one:

Definition 4.2. *Let $(X, M, *)$ be a fuzzy metric space, $B \subseteq X$ non-empty, $\alpha \geq 0$ and φ a fuzzy comparison function. A continuous map $\gamma : I \longrightarrow X$ is said to be a fuzzy α-dense curve in B respect to φ if the following conditions hold:*

(i) $\gamma(I) \subset B$.

(ii) For each $x \in B$ *there is* $y \in \gamma(I)$ *such that* $M(x, y, t) \geq \varphi(t, \alpha)$ *for every* $t > 0$.

If the above conditions are satisfied for every $\alpha > 0$*, then* B *is said to be fuzzy densifiable respect to* φ.

Observe that in the above definition we do not need the boundedness of the set B, as the case of the α-dense curves. Furthermore, for every non-empty $B \subset (X, M, *)$ we can find a fuzzy α-dense curve in B for each $\alpha \geq 1$ and certain fuzzy comparison function φ. Indeed, taking the function

$$\varphi(t, \alpha) := \begin{cases} 1 - \alpha, & \text{if} \quad 0 \leq \alpha < 1 \\ 0, & \text{if} \quad \alpha \geq 1 \end{cases} \qquad \text{for all } t > 0, \alpha \geq 0, \tag{4.1}$$

is clear that fixed any $x_0 \in B$, the map $\gamma(\tau) := x_0$, for each $\tau \in I$, is a fuzzy α-dense curve in B respect to φ, for every $\alpha \geq 1$. Also, the fuzzy α-dense curves shares the following property with the α-dense curves: if γ is a fuzzy α-dense curve in B respect to certain fuzzy comparison function φ and some $\alpha > 0$, then γ is a fuzzy α'-dense curve in B respect to φ, for every $\alpha' > \alpha$.

The class of fuzzy α-dense curves in the standard fuzzy metric space induced by d, $(X, M_d, *)$ (see Example 2.1), respect to certain comparison function φ is large:

Proposition 4.1. *Let* $(X, M_d, *)$ *the standard fuzzy metric space induced by* d*,* $B \subseteq X$ *non-empty and bounded, and* $\gamma : I \longrightarrow X$ *an* α*-dense curve in* B*, for certain* $\alpha > 0$*. Then,* γ *is a fuzzy* α*-dense curve in* B *respect to* $\varphi(t, \alpha) := t/(t + \alpha)$*. In particular, if* B *is densifiable then is fuzzy densifiable respect to* φ.

Proof. Given any $x \in B$ there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$. Thus, for every $t > 0$ we have

$$M(x, y, t) = \frac{t}{t + d(x, y)} \geq \frac{t}{t + \alpha} = \varphi(t, \alpha),$$

and therefore γ is a fuzzy α-dense in B respect to φ. Clearly, if B is densifiable then the above inequality holds for every $\alpha > 0$ and so B is fuzzy densifiable respect to φ. □

A similar result to Proposition 3.1 is the following. For a general background on the topology of the fuzzy and the intuitionistic fuzzy metric spaces, see [1, 19, 20, 29].

Proposition 4.2. *Let* $(X, M, *)$ *a fuzzy metric space, and* $B \subseteq X$ *non-empty, arc-connected and precompact in the topology* $\mathcal{T}_M$ *induced by the fuzzy metric* M*. Then,* B *is fuzzy densifiable respect to the fuzzy comparison function* φ *defined in (4.1).*

Proof. First at all, let us note that the family of closed balls with center $y \in X$ and radius $0 < r < 1$ with respect to $t > 0$ defined as

$$B_M(y, r, t) := \{x \in X : M(x, y, t) \geq 1 - r\},$$

generates a base of open neighborhoods for the topology $\mathcal{T}_M$ on $(X, M, *)$. So, if B is precompact in the topology $\mathcal{T}_M$, given any $0 < \alpha < 1$ and $t > 0$ there is a finite set $\{y_1, \ldots, y_m\} \subset B$ such that

$$B \subset \bigcup_{i=1}^{m} B_M(y_i, \alpha, t).$$

Therefore, taking a continuous map $\gamma : I \longrightarrow X$ joining the points $y_1, \ldots, y_m$ (it is possible as B is arc-connected) the result follows.

□

Remark 4.1. *The family of open balls in the above proof also generates a Hausdorff and first countable topology, $\mathcal{T}_{(M,N)}$, for the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. In this case, one can give the following more precise definition of such family:*

$$B_{(M,N)}(y, r, t) := \{x \in X : M(x, y, t) \geq 1 - r, N(x, y, t) \leq r\},$$

for every $t > 0$ (see, for instance, [29]).

Remark 4.2. *Of course, in view of Proposition 4.2, a non-empty, arc-connected and precompact set $B \subset (X, M, *)$ can be fuzzy densifiable respect to a fuzzy comparison function φ not necessarily equal to the given one in (4.1).*

As expected, not every non-empty subset of a fuzzy metric space is fuzzy densifiable:

Example 4.1. *Let $X := (0, +\infty)$ and*

$$M(x, y, t) := \begin{cases} \dfrac{x}{y}, & \text{if } \ x \leq y, \\ \\ \dfrac{y}{x}, & \text{if } \ x > y, \end{cases} \qquad \text{for all } x, y \in X, t > 0.$$

Given $B := (a, b) \cup (a', b') \subset X$ with $0 < a < b < a' < b' < +\infty$, assume that $\gamma : I \longrightarrow X$ is a fuzzy α-dense curve in B, for some $\alpha > 0$ and certain fuzzy comparison function φ. If $\gamma(\tau) = x_0 \in X$ for all $\tau \in I$, with $x_0 \in (a, b)$ (the case $x_0 \in (a', b')$ is treated in the same way), taking any $y \in (a', b')$ we have

$$1 > \frac{x_0}{a'} > \frac{x_0}{y} = M(x_0, y, t) \geq \varphi(t, \alpha).$$

So, if $\alpha > 0$ is such that $\varphi(t, \alpha) \geq x_0/a'$ (such α there exists because, according to Definitions 4.1, $\varphi(t, \cdot)$ is continuous and $\varphi(t, \alpha) \to 1$ as $\alpha \to 0$, for every $t > 0$), γ can not be a fuzzy α-dense curve in B respect to φ.

Next, assume that $[\underline{\gamma}, \overline{\gamma}] \subset B$, with $\underline{\gamma} < \overline{\gamma}$, is the image of I under γ and $[\underline{\gamma}, \overline{\gamma}] \subset (a, b)$ (the case $[\underline{\gamma}, \overline{\gamma}] \subset (a', b')$ is completely analogous). So, given any $t > 0$ and $x \in (a', b')$, for each $y \in [\underline{\gamma}, \overline{\gamma}]$ we have

$$1 > \frac{\overline{\gamma}}{a'} > \frac{y}{x} = M(x, y, t) \geq \varphi(t, \alpha).$$

Therefore, reasoning as above, γ can not be a fuzzy α-dense curve in B.

In what follows, $\mathcal{F} := \{X; f_1, \ldots, f_n\}$ will be an IF-IFS in the complete intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ (and therefore $(X, M, *)$ will be a fuzzy metric space), and $\mathcal{A}$ its intuitionistic fuzzy fractal. The existence of $\mathcal{A}$ holds by Theorem 2.3.

The next result has been proved in [9, Lemma 2] for an IFS, although it remains true for an IF-IFS:

Lemma 1. *For each $k \geq n$, we have*

$$\mathcal{A} = F^k(\mathcal{A}) = \bigcup_{j_1,\ldots,j_k} f_{j_1 \cdots j_k}(\mathcal{A}),$$

the union being performed over all possible $j_1, \ldots, j_k \in N_n := \{1, 2, \ldots, n\}$ and $f_{j_1 \cdots j_k} := f_{j_1} \circ \cdots \circ f_{j_k}$

Also, we will need the following lemmas, which can be easily proved only using the involved definitions and the inequalities

$$0 \leq H_M(A, C, t) + H_N(A, C, t) \leq 1 \quad \text{and} \quad 0 \leq M(x, y, t) + N(x, y, t) \leq 1,$$

for every $t > 0$, $A, B \in \mathcal{K}_0(X)$ and $x, y \in X$.

Lemma 2. *For each $A, B, C \in \mathcal{K}_0(X)$ and $t > 0$ we have the inequalities*

$$H_M(A, C, 2t) \geq H_M(A, B, t) * H_M(B, C, t) \quad \textit{and}$$

$$H_N(A, C, 2t) \leq H_N(A, B, t) \Diamond H_N(B, C, t).$$

Lemma 3. *Let $(X, M, *)$ be a fuzzy metric space and γ a fuzzy α-dense curve in X, for some $\alpha > 0$, respect to certain fuzzy comparison function φ. Then, in the space $(X, M, N, *, \Diamond)$, for each $x \in X$ there is $y \in \gamma(I)$ obeying*

$$M(x, y, t) \geq \varphi(t, \alpha) \quad \textit{and} \quad N(x, y, t) \leq 1 - \varphi(t, \alpha),$$

for every $t > 0$.

As we have pointed out in Section 3, the α-dense curves have been used, in the framework of the IFS and under suitable conditions, to approximate the attractor set of certain IFS. In this section, we have introduced and analyzed the concept of fuzzy α-dense curves, respect to a given fuzzy comparison function φ, and we will use this novel tool to approximate, under suitable conditions, the intuitionistic fuzzy fractal $\mathcal{A}$ of a given an IF-IFS. More specifically, by using fuzzy α-dense curves in the next result we construct a sequence of finite sets which approximates (in the specified sense) $\mathcal{A}$ with controlled and arbitrarily small error.

Now, we are ready to state and prove our main result:

Theorem 4.1. *Assume that $(X, M, *)$ is fuzzy densifiable respect to certain fuzzy comparison function φ, and let γ_k be a fuzzy α_k-dense curve in X, for each $k \geq 1$, respect to φ, with $\alpha_k \to 0$ and $0 < \varphi(t, \alpha_k) < 1$ for each $t > 0$ and $k \geq 1$. Fixed $x_1 \in X$, let $B_1 := \{f_1(x_1)\}$ and for $k > 1$ define*

$$B_k := \big\{f_{j_1 \cdots j_k}(y_k) : M(x_{i_1 \cdots i_{k-1}}, y_k, t) \geq \varphi(t, \alpha_k), \text{ for all } t > 0, \text{ for each}$$

$$x_{i_1 \cdots i_{k-1}} \in B_{k-1} \text{ and some } y_k \in \gamma_k(I), \text{ where } j_1, \ldots, j_k \in N_k \text{ and } i_1, \ldots, i_{k-1} \in N_{k-1}\big\},$$

where $N_k := \{1, \ldots, k\}$ for $1 < k < n$ and $N_k := \{1, \ldots, n\}$ for $k \geq n$. Then, we have

$$H_M(\mathcal{A}, B_k, 2t) \geq \tag{4.2}$$
$$\frac{H_M(\mathcal{A}, B_{k-1}, t)}{H_M(\mathcal{A}, B_{k-1}, t) + r^k(1 - H_M(\mathcal{A}, B_{k-1}, t))} * \frac{\varphi(t, \alpha_k)}{\varphi(t, \alpha_k) + r^k(1 - \varphi(t, \alpha_k))},$$

and

$$H_N(\mathcal{A}, B_k, 2t) \leq \frac{r^k H_N(\mathcal{A}, B_{k-1}, t)}{r^k H_N(\mathcal{A}, B_{k-1}, t) + 1 - H_N(\mathcal{A}, B_{k-1}, t)} \Diamond \frac{r^k(1 - \varphi(t, \alpha_k))}{\varphi(t, \alpha_k)}, \tag{4.3}$$

for every $k \geq n$ and $t > 0$.

Proof. Let any $t > 0$ and $k \geq n$. As $\mathcal{A}$ and B_k belong to $\mathcal{K}_0(X)$, by Lemmas 1 and 2 we have

$$H_M(\mathcal{A}, B_k, 2t) \geq H_M(\mathcal{A}, F^k(B_{k-1}), t) * H_M(F^k(B_{k-1}), B_k, t) =$$
$$H_M(F^k(\mathcal{A}), F^k(B_{k-1}), t) * H_M(F^k(B_{k-1}), B_k, t). \tag{4.4}$$

Also, in view of Definition 2.10, as

$$\frac{1}{M(f_{j_1 \ldots j_k}(x), f_{j_1 \ldots j_k}(y), t)} - 1 \leq r^k\Big(\frac{1}{M(x, y, t)} - 1\Big),$$

for every $j_1, \ldots, j_k \in N_k$ and $x \in F^k(\mathcal{A})$, $y \in F^k(B_{k-1})$, we infer that

$$\frac{1}{H_M(F^k(\mathcal{A}), F^k(B_{k-1}), t)} - 1 \leq r^k\Big(\frac{1}{H_M(\mathcal{A}, B_{k-1}, t)} - 1\Big),$$

and therefore

$$H_M(F^k(\mathcal{A}), F^k(B_{k-1}), t) \geq \frac{H_M(\mathcal{A}, B_{k-1}, t)}{H_M(\mathcal{A}, B_{k-1}, t) + r^k(1 - H_M(\mathcal{A}, B_{k-1}, t))}. \tag{4.5}$$

So, replacing (4.5) in (4.4), we find

$$H_M(\mathcal{A}, B_k, 2t) \geq \tag{4.6}$$
$$\frac{H_M(\mathcal{A}, B_{k-1}, t)}{H_M(\mathcal{A}, B_{k-1}, t) + r^k(1 - H_M(\mathcal{A}, B_{k-1}, t))} * H_M(F^k(B_{k-1}), B_k, t).$$

Next, we need to estimate $H_M(F^k(B_{k-1}), B_k, t)$. Let $x \in F^k(B_{k-1})$, put $x := f_{j_1\dots j_k}(x_0)$ for some $j_1, \dots, j_k \in N_k$ and $x_0 \in B_{k-1}$, and take (by virtue of Lemma 3) $y := f_{j_1\dots j_k}(y_0)$ with $y_0 \in \gamma_k(I)$ such that $M(x, y, t) \geq \varphi(t, \alpha_k)$. Then, from this and noticing Definition 2.10 we have

$$M(x, y, t) = M(f_{j_1\dots j_k}(x_0), f_{j_1\dots j_k}(y_0)) \geq \frac{\varphi(t, \alpha_k)}{\varphi(t, \alpha_k) + r^k(1 - \varphi(t, \alpha_k))},$$

and so

$$\sup_{y \in B_k} M(x, y, t) \geq \frac{\varphi(t, \alpha_k)}{\varphi(t, \alpha_k) + r^k(1 - \varphi(t, \alpha_k))}.$$

From the arbitrariness of $x \in F^k(B_{k-1})$, we conclude

$$\inf_{x \in F^k(B_{k-1})} \sup_{y \in B_k} M(x, y, t) \geq \frac{\varphi(t, \alpha_k)}{\varphi(t, \alpha_k) + r^k(1 - \varphi(t, \alpha_k))}.$$

Likewise, we can prove that

$$\inf_{x \in B_k} \sup_{y \in F^k(B_{k-1})} M(x, y, t) \geq \frac{\varphi(t, \alpha_k)}{\varphi(t, \alpha_k) + r^k(1 - \varphi(t, \alpha_k))},$$

and therefore

$$H_M(F^k(B_{k-1}), B_k, t) \geq \frac{\varphi(t, \alpha_k)}{\varphi(t, \alpha_k) + r^k(1 - \varphi(t, \alpha_k))}, \tag{4.7}$$

So, replacing (4.7) in (4.6) the inequality (4.2) follows.

On the other hand, by Lemmas 1 and 2, we have

$$\begin{aligned} &H_N(\mathcal{A}, B_k, 2t) \leq H_N(\mathcal{A}, F^k(B_{k-1}), t) \Diamond H_N(F^k(B_{k-1}), B_k, t) = \\ &H_N(F^k(\mathcal{A}), F^k(B_{k-1}), t) \Diamond H_N(F^k(B_{k-1}), B_k, t), \end{aligned} \tag{4.8}$$

and, reasoning as above

$$H_N(F^k(\mathcal{A}), F^k(B_{k-1}), t) \leq \frac{r^k H_N(\mathcal{A}, B_{k-1}, t)}{r^k H_N(\mathcal{A}, B_{k-1}, t) + 1 - H_N(\mathcal{A}, B_{k-1}, t)}. \tag{4.9}$$

Then, replacing (4.9) in (4.8)

$$H_N(\mathcal{A}, B_k, 2t) \leq \frac{r^k H_N(\mathcal{A}, B_{k-1}, t)}{r^k H_N(\mathcal{A}, B_{k-1}, t) + 1 - H_N(\mathcal{A}, B_{k-1}, t)} \Diamond H_N(F^k(B_{k-1}), B_k, t). \tag{4.10}$$

To estimate $H_N(F^k(B_{k-1}), B_k, t)$, noticing Lemma 3 and reasoning in the same way that for the inequality (4.7), we can conclude straightforwardly that

$$H_N(F^k(B_{k-1}), B_k, t) \leq \frac{r^k(1 - \varphi(t, \alpha_k))}{\varphi(t, \alpha_k)}, \tag{4.11}$$

and joining (4.11) and (4.10), the inequality (4.3) follows. The proof is now complete. □

Clearly, as in Theorem 3.1, the limits shown in (3.3) also hold in the above result as

$$\lim_k \frac{H_M(\mathcal{A},B_{k-1},t)}{H_M(\mathcal{A},B_{k-1},t)+r^k(1-H_M(\mathcal{A},B_{k-1},t))} =$$

$$\lim_k \frac{\varphi(t,\alpha_k)}{\varphi(t,\alpha_k)+r^k(1-\varphi(t,\alpha_k))} = 1,$$

and

$$\lim_k \frac{r^k H_N(\mathcal{A},B_{k-1},t)}{r^k H_N(\mathcal{A},B_{k-1},t)+1-H_N(\mathcal{A},B_{k-1},t)} = \lim_k \frac{r^k(1-\varphi(t,\alpha_k))}{\varphi(t,\alpha_k)} = 0,$$

by the continuity of the t-norms $*$ and $\Diamond$, we deduce

$$\lim_k H_M(\mathcal{A},B_k,2t) = 1 \quad \text{and} \quad \lim_k H_N(\mathcal{A},B_k,2t) = 0, \quad \text{for all } t > 0.$$

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