FORMALIZED MATHEMATICS
Vol. **30**, No. **2**, Pages 125–134, **2022**
ISSN: 1426–2630, e-ISSN: 1898–9934
DOI: 10.2478/forma-2022-0010

sciendo
sciendo.com/journal/forma

Definition of Centroid Method as Defuzzification

Takashi Mitsuishi
Faculty of Business and Informatics
Nagano University, Japan

Summary. In this study, using the Mizar system [1], [2], we reuse formalization efforts in fuzzy sets described in [5] and [6]. This time the centroid method which is one of the fuzzy inference processes is formulated [10]. It is the most popular of all defuzzied methods ([11], [13], [7]) – here, defuzzified crisp value is obtained from domain of membership function as weighted average [8]. Since the integral is used in centroid method, the integrability and bounded properties of membership functions are also mentioned to fill the formalization gaps present in the Mizar Mathematical Library, as in the case of another fuzzy operators [4]. In this paper, the properties of piecewise linear functions consisting of two straight lines are mainly described.

MSC: 68V20 93C42

Keywords: defuzzification; centroid; piecewise linear function

MML identifier: `FUZZY_6`, version: `8.1.12 5.71.1431`

From now on A denotes a non empty, closed interval subset of $\mathbb{R}$.

Let A be a non empty, closed interval subset of $\mathbb{R}$ and f be a function from $\mathbb{R}$ into $\mathbb{R}$. The functor centroid(f, A) yielding a real number is defined by the term

(Def. 1) $$\frac{\int_A (\mathrm{id}_{\mathbb{R}} \cdot f)(x)dx}{\int_A f(x)dx}.$$

Now we state the propositions:

(1) Let us consider real numbers a, b, c. Suppose $a < b$ and $c > 0$. Then centroid(AffineMap$(0, c), [a, b]) = \frac{a+b}{2}$.

© 2022 The Author(s) / AMU
(Association of Mizar Users)
under CC BY-SA 3.0 license

PROOF: Set $F = \frac{c}{2} \cdot (\square^2)$. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom}(F'_{\restriction \Omega_{\mathbb{R}}})$ holds $(F'_{\restriction \Omega_{\mathbb{R}}})(x) = (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(0, c)))(x)$ by [12, (2)]. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom}((\mathrm{AffineMap}(c, 0))'_{\restriction \Omega_{\mathbb{R}}})$ holds $((\mathrm{AffineMap}(c, 0))'_{\restriction \Omega_{\mathbb{R}}})(x) = (\mathrm{AffineMap}(0, c))(x)$. $\square$

(2) Let us consider real numbers a, b. Then

(i) $\mathrm{id}_{\mathbb{R}}$ is integrable on $[a, b]$, and

(ii) $\mathrm{id}_{\mathbb{R}} \restriction [a, b]$ is bounded.

(3) (i) $\mathrm{id}_{\mathbb{R}}$ is integrable on A, and

(ii) $\mathrm{id}_{\mathbb{R}} \restriction A$ is bounded.

(4) Let us consider a real number e, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $A \subseteq \operatorname{dom} f$ and for every real number x such that $x \in A$ holds $f(x) = e$. Then

(i) f is integrable on A, and

(ii) $f \restriction A$ is bounded, and

(iii) $\int_{\inf A}^{\sup A} f(x)dx = e \cdot (\sup A - \inf A)$.

Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(5) If for every real number x such that $x \in A$ holds $f(x) = 0$, then $\int_A f(x)dx = 0$. The theorem is a consequence of (4).

(6) Suppose f is integrable on A and $f \restriction A$ is bounded. Then

(i) $\mathrm{id}_{\mathbb{R}} \cdot f$ is integrable on A, and

(ii) $(\mathrm{id}_{\mathbb{R}} \cdot f) \restriction A$ is bounded.

The theorem is a consequence of (3).

(7) Let us consider real numbers a, b, c. Suppose $a < b$. Then

(i) $[a, b] \subseteq \Omega_{\mathbb{R}}$, and

(ii) $\inf[a, b] = a$, and

(iii) $\sup[a, b] = b$.

Let us consider real numbers a, b, c and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(8) Suppose $a < b \leqslant c$ and f is integrable on $[a, c]$ and $f \restriction [a, c]$ is bounded and for every real number x such that $x \in [b, c]$ holds $f(x) = 0$. Then $\operatorname{centroid}(f, [a, c]) = \operatorname{centroid}(f, [a, b])$. The theorem is a consequence of (3).

(9) Suppose $a \leqslant b < c$ and f is integrable on $[a, c]$ and $f\restriction[a, c]$ is bounded and for every real number x such that $x \in [a, b]$ holds $f(x) = 0$. Then $\operatorname{centroid}(f, [a, c]) = \operatorname{centroid}(f, [b, c])$. The theorem is a consequence of (3).

(10) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is integrable on A and $f\restriction A$ is bounded and $\int_A f(x)dx > 0$. Then there exists a real number c such that

(i) $c \in A$, and

(ii) $f(c) > 0$.

PROOF: Set $g = (-1) \cdot f$. There exists a real number r such that for every set y such that $y \in \operatorname{dom}(g\restriction A)$ holds $|(g\restriction A)(y)| < r$. For every real number x such that $x \in A$ holds $0 \leqslant (g\restriction A)(x)$. □

(11) Let us consider a real number r, a fuzzy set f of $\mathbb{R}$, and a function F from $\mathbb{R}$ into $\mathbb{R}$. Suppose $r > 0$ and f is integrable on A and $f\restriction A$ is bounded and for every real number x, $F(x) = \min(r, f(x))$. Then $\int_A F(x)dx \geqslant 0$.

PROOF: There exists a real number r such that for every set y such that $y \in \operatorname{dom}(F\restriction A)$ holds $|(F\restriction A)(y)| < r$. For every real number x such that $x \in A$ holds $0 \leqslant (F\restriction A)(x)$. □

Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(12) $\min(f, g) = \frac{1}{2} \cdot (f + g - |f - g|)$.

PROOF: For every object x such that $x \in \operatorname{dom}(\min(f, g))$ holds $(\min(f, g))(x) = (\frac{1}{2} \cdot (f + g - |f - g|))(x)$. □

(13) Suppose f is integrable on A and $f\restriction A$ is bounded and g is integrable on A and $g\restriction A$ is bounded. Then

(i) $\min(f, g)$ is integrable on A, and

(ii) $\min(f, g)\restriction A$ is bounded, and

(iii) $\int_A (\min(f, g))(x)dx = \frac{1}{2} \cdot (\int_A f(x)dx + \int_A g(x)dx - \int_A |f - g|(x)dx)$.

The theorem is a consequence of (12).

(14) $\max(f, g) = \frac{1}{2} \cdot (f + g + |f - g|)$.

PROOF: For every object x such that $x \in \operatorname{dom}(\max(f, g))$ holds $(\max(f, g))(x) = (\frac{1}{2} \cdot (f + g + |f - g|))(x)$. □

(15) Suppose f is integrable on A and $f\restriction A$ is bounded and g is integrable on A and $g\restriction A$ is bounded. Then

(i) $\max(f, g)$ is integrable on A, and

(ii) $\max(f,g)\restriction A$ is bounded, and

(iii) $\int_A (\max(f,g))(x)dx = \frac{1}{2}\cdot(\int_A f(x)dx + \int_A g(x)dx + \int_A |f-g|(x)dx)$.

The theorem is a consequence of (14).

(16) Let us consider real numbers r_1, r_2, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is integrable on A and $f\restriction A$ is bounded. Then

(i) $\min(\mathrm{AffineMap}(0,r_1), r_2\cdot f)$ is integrable on A, and

(ii) $\min(\mathrm{AffineMap}(0,r_1), r_2\cdot f)\restriction A$ is bounded.

The theorem is a consequence of (13).

(17) Let us consider real numbers r_1, r_2, and functions f, F from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is integrable on A and $f\restriction A$ is bounded and for every real number x, $F(x) = \min(r_1, r_2\cdot f(x))$. Then

(i) F is integrable on A, and

(ii) $F\restriction A$ is bounded.

The theorem is a consequence of (16).

(18) Let us consider a real number s, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Then $f\restriction]-\infty, s[+ \cdot g\restriction [s, +\infty[$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

Let us consider real numbers a, b, c and functions f, g, F from $\mathbb{R}$ into $\mathbb{R}$.

(19) If $a \leqslant b \leqslant c$ and $F = f\restriction [a,b] + \cdot g\restriction [b,c]$, then F is a function from $[a,c]$ into $\mathbb{R}$.

(20) If $a \leqslant b \leqslant c$ and $F = f\restriction [a,b] + \cdot g\restriction [b,c]$, then $F = F\restriction [a,c]$.

Let us consider real numbers a, b, c and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$.

(21) Suppose $a \leqslant b \leqslant c$ and $f\restriction [a,c]$ is bounded and $g\restriction [a,c]$ is bounded and $h = f\restriction [a,b] + \cdot g\restriction [b,c]$ and $f(b) = g(b)$. Then $h\restriction [a,c]$ is bounded.
PROOF: $f\restriction [a,b]$ tolerates $g\restriction [b,c]$. There exists a real number r such that for every set y such that $y \in \mathrm{dom}(h\restriction [a,c])$ holds $|(h\restriction [a,c])(y)| < r$. □

(22) Suppose $a \leqslant b \leqslant c$ and $f\restriction [a,c]$ is bounded and $g\restriction [a,c]$ is bounded and $h\restriction [a,c] = f\restriction [a,b] + \cdot g\restriction [b,c]$ and $f(b) = g(b)$. Then $h\restriction [a,c]$ is bounded.
PROOF: $f\restriction [a,b]$ tolerates $g\restriction [b,c]$. There exists a real number r such that for every set y such that $y \in \mathrm{dom}(h\restriction [a,c])$ holds $|(h\restriction [a,c])(y)| < r$. □

Now we state the propositions:

(23) Let us consider a real number c, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose $f\restriction A$ is bounded and $g\restriction A$ is bounded. Then $(f\restriction]-\infty, c[+ \cdot g\restriction [c, +\infty[)\restriction A$ is bounded.
PROOF: Set $F = f\restriction]-\infty, c[+ \cdot g\restriction [c, +\infty[$. There exists a real number r such that for every set y such that $y \in \mathrm{dom}(F\restriction A)$ holds $|(F\restriction A)(y)| < r$. □

(24) Let us consider real numbers a, b, c, and functions f, g, h, F from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant b \leqslant c$ and f is continuous and g is continuous and $h\restriction[a,c] = f\restriction[a,b]+\cdot g\restriction[b,c]$ and $f(b) = g(b)$ and $F = h\restriction[a,c]$. Then F is continuous.

PROOF: For every real numbers x_0, r such that $x_0 \in [a,c]$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number x_1 such that $x_1 \in [a,c]$ and $|x_1 - x_0| < s$ holds $|h(x_1) - h(x_0)| < r$. □

(25) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is continuous. Then

(i) f is integrable on A, and

(ii) $f\restriction A$ is bounded.

(26) Let us consider a real number c, and functions f, g, F from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is Lipschitzian and g is Lipschitzian and $f(c) = g(c)$ and $F = f\restriction]-\infty,c[+\cdot g\restriction[c,+\infty[$. Then F is Lipschitzian.

PROOF: Consider r_3 being a real number such that $0 < r_3$ and for every real numbers x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r_3 \cdot |x_1 - x_2|$. Consider r_4 being a real number such that $0 < r_4$ and for every real numbers x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} g$ holds $|g(x_1)-g(x_2)| \leqslant r_4\cdot|x_1-x_2|$. There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} F$ holds $|F(x_1) - F(x_2)| \leqslant r \cdot |x_1 - x_2|$. □

(27) Let us consider real numbers a, b. Then AffineMap(a,b) is Lipschitzian.

PROOF: Set $f =$ AffineMap(a,b). There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. □

Let us consider real numbers a, b, p, q and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(28) Suppose $a \neq p$ and $f = (\text{AffineMap}(a,b))\restriction]-\infty,\frac{q-b}{a-p}[+\cdot(\text{AffineMap}(p,q))\restriction[\frac{q-b}{a-p},+\infty[$. Then f is Lipschitzian. The theorem is a consequence of (27) and (26).

(29) Suppose $a \neq p$ and $f = (\text{AffineMap}(a,b))\restriction]-\infty,\frac{q-b}{a-p}[+\cdot(\text{AffineMap}(p,q))\restriction[\frac{q-b}{a-p},+\infty[$. Then

(i) f is integrable on A, and

(ii) $f\restriction A$ is bounded.

The theorem is a consequence of (28).

(30) Let us consider real numbers a, b, p, q. Suppose $a \neq p$.
Then $(\text{AffineMap}(a,b))(\frac{q-b}{a-p}) = (\text{AffineMap}(p,q))(\frac{q-b}{a-p})$.

(31) Every membership function of $\mathbb{R}$ is bounded.

PROOF: There exists a real number r such that for every set x such that $x \in \operatorname{dom} f$ holds $|f(x)| < r$ by [9, (1)]. □

(32) Let us consider a real number r, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $r \neq 0$ and f is integrable on A and $f\restriction A$ is bounded. Then $\operatorname{centroid}(r \cdot f, A) = \operatorname{centroid}(f, A)$. The theorem is a consequence of (6).

Let us consider real numbers a, b, c and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$.

(33) Suppose $a \leqslant b \leqslant c$ and f is integrable on $[a, c]$ and $f\restriction[a, c]$ is bounded and g is integrable on $[a, c]$ and $g\restriction[a, c]$ is bounded and $h\restriction[a, c] = f\restriction[a, b] + \cdot g\restriction[b, c]$ and h is integrable on $[a, c]$ and $f(b) = g(b)$.

Then $\int\limits_{[a,c]} h(x)dx = \int\limits_{[a,b]} f(x)dx + \int\limits_{[b,c]} g(x)dx$.

PROOF: $f\restriction[a, b]$ tolerates $g\restriction[b, c]$. Reconsider $h_1 = h\restriction[a, b]$ as a partial function from $[a, b]$ to $\mathbb{R}$. Reconsider $f_1 = f\restriction[a, b]$ as a partial function from $[a, b]$ to $\mathbb{R}$. Reconsider $H = \operatorname{upper_sum_set} h_1$ as a function from $\operatorname{divs}[a, b]$ into $\mathbb{R}$. Reconsider $F = \operatorname{upper_sum_set} f_1$ as a function from $\operatorname{divs}[a, b]$ into $\mathbb{R}$. $H = F$.

Reconsider $h_2 = h\restriction[b, c]$ as a partial function from $[b, c]$ to $\mathbb{R}$. Reconsider $g_1 = g\restriction[b, c]$ as a partial function from $[b, c]$ to $\mathbb{R}$. Reconsider $H_1 = \operatorname{upper_sum_set} h_2$ as a function from $\operatorname{divs}[b, c]$ into $\mathbb{R}$. Reconsider $G = \operatorname{upper_sum_set} g_1$ as a function from $\operatorname{divs}[b, c]$ into $\mathbb{R}$. $H_1 = G$. $h\restriction[a, c]$ is bounded. □

(34) Suppose $a \leqslant b \leqslant c$ and f is continuous and g is continuous and $h = f\restriction[a, b] + \cdot g\restriction[b, c]$ and $f(b) = g(b)$.

Then $\int\limits_{[a,c]} (\mathrm{id}_{\mathbb{R}} \cdot h)(x)dx = \int\limits_{[a,b]} (\mathrm{id}_{\mathbb{R}} \cdot f)(x)dx + \int\limits_{[b,c]} (\mathrm{id}_{\mathbb{R}} \cdot g)(x)dx$.

PROOF: $\mathrm{id}_{\mathbb{R}} \cdot f$ is integrable on $[a, c]$ and $(\mathrm{id}_{\mathbb{R}} \cdot f)\restriction[a, c]$ is bounded and $\mathrm{id}_{\mathbb{R}} \cdot g$ is integrable on $[a, c]$ and $(\mathrm{id}_{\mathbb{R}} \cdot g)\restriction[a, c]$ is bounded. Set $G = (\mathrm{id}_{\mathbb{R}} \cdot f)\restriction[a, b] + \cdot (\mathrm{id}_{\mathbb{R}} \cdot g)\restriction[b, c]$. For every object x such that $x \in \operatorname{dom} G$ holds $G(x) = (\mathrm{id}_{\mathbb{R}} \cdot h)(x)$. $\mathrm{id}_{\mathbb{R}} \cdot h$ is integrable on $[a, c]$. □

Let us consider a real number c and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(35) $f\restriction]-\infty, c[+ \cdot g\restriction[c, +\infty[= f\restriction]-\infty, c] + \cdot g\restriction[c, +\infty[$.

PROOF: Set $f_1 = f\restriction]-\infty, c[+ \cdot g\restriction[c, +\infty[$. Set $f_2 = f\restriction]-\infty, c] + \cdot g\restriction[c, +\infty[$. For every object x such that $x \in \operatorname{dom} f_1$ holds $f_1(x) = f_2(x)$. □

(36) Suppose $f\restriction A$ is bounded and $g\restriction A$ is bounded.

Then $(f\restriction]-\infty, c] + \cdot g\restriction[c, +\infty[)\restriction A$ is bounded. The theorem is a consequence of (23) and (35).

(37) Let us consider real numbers a, b, c, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant c \leqslant b$. Then $f\restriction[a,c[+ \cdot g\restriction[c,b] = f\restriction[a,c] + \cdot g\restriction[c,b]$.
PROOF: Set $f_1 = f\restriction[a,c[+ \cdot g\restriction[c,b]$. Set $f_2 = f\restriction[a,c] + \cdot g\restriction[c,b]$. For every object x such that $x \in \operatorname{dom} f_1$ holds $f_1(x) = f_2(x)$. □

(38) Let us consider real numbers a, b, c, and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant c$ and $h\restriction[a,c] = f\restriction[a,b] + \cdot g\restriction[b,c]$ and $f(b) = g(b)$. Then

(i) if $b \leqslant a$, then $h\restriction[a,c] = g\restriction[a,c]$, and

(ii) if $c \leqslant b$, then $h\restriction[a,c] = f\restriction[a,c]$.

PROOF: If $b \leqslant a$, then $h\restriction[a,c] = g\restriction[a,c]$. If $c \leqslant b$, then $h\restriction[a,c] = f\restriction[a,c]$. □

(39) Let us consider a real number b, and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$. Suppose $h = f\restriction]-\infty,b[+ \cdot g\restriction[b,+\infty[$ and $f(b) = g(b)$. Then

(i) if $b \leqslant \inf A$, then $h\restriction A = g\restriction A$, and

(ii) if $\sup A \leqslant b$, then $h\restriction A = f\restriction A$.

PROOF: If $b \leqslant \inf A$, then $h\restriction A = g\restriction A$ by [3, (4)]. If $\sup A \leqslant b$, then $h\restriction A = f\restriction A$ by [3, (4)]. □

(40) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $f = (\text{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a-p}[+ \cdot (\text{AffineMap}(p,q))\restriction[\frac{q-b}{a-p}, +\infty[$ and $\frac{q-b}{a-p} \in A$.
Then $f\restriction A = (\text{AffineMap}(a,b))\restriction[\inf A, \frac{q-b}{a-p}] + \cdot (\text{AffineMap}(p,q))\restriction[\frac{q-b}{a-p}, \sup A]$.
PROOF: Set $F = (\text{AffineMap}(a,b))\restriction[\inf A, \frac{q-b}{a-p}] + \cdot (\text{AffineMap}(p,q))\restriction[\frac{q-b}{a-p}, \sup A]$. For every object x such that $x \in \operatorname{dom} F$ holds $F(x) = (f\restriction A)(x)$. □

(41) Let us consider real numbers a, b. Then

(i) $(\text{AffineMap}(a,b))\restriction A$ is bounded, and

(ii) $\text{AffineMap}(a,b)$ is integrable on A.

Let us consider real numbers a, b, p, q and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(42) Suppose $a \neq p$ and $f = (\text{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a-p}[+ \cdot (\text{AffineMap}(p,q))\restriction[\frac{q-b}{a-p}, +\infty[$. Then

(i) if $\frac{q-b}{a-p} \in A$, then $\int\limits_A f(x)dx = \int\limits_{[\inf A, \frac{q-b}{a-p}]} (\text{AffineMap}(a,b))(x)dx + \int\limits_{[\frac{q-b}{a-p}, \sup A]} (\text{AffineMap}(p,q))(x)dx$, and

(ii) if $\frac{q-b}{a-p} \leqslant \inf A$, then $\int\limits_A f(x)dx = \int\limits_A (\mathrm{AffineMap}(p,q))(x)dx$, and

(iii) if $\frac{q-b}{a-p} \geqslant \sup A$, then $\int\limits_A f(x)dx = \int\limits_A (\mathrm{AffineMap}(a,b))(x)dx$.

PROOF: $(\mathrm{AffineMap}(a,b))(\frac{q-b}{a-p}) = (\mathrm{AffineMap}(p,q))(\frac{q-b}{a-p})$. $\mathrm{AffineMap}(a,b)$ is integrable on $[\inf A, \sup A]$ and $(\mathrm{AffineMap}(a,b))\restriction[\inf A, \sup A]$ is bounded. $\mathrm{AffineMap}(p,q)$ is integrable on $[\inf A, \sup A]$. $\mathrm{AffineMap}(p,q)\restriction[\inf A, \sup A]$ is bounded. f is integrable on $[\inf A, \sup A]$. If $\frac{q-b}{a-p} \in A$, then $\int\limits_A f(x)dx = \int\limits_{[\inf A, \frac{q-b}{a-p}]} (\mathrm{AffineMap}(a,b))(x)dx + \int\limits_{[\frac{q-b}{a-p}, \sup A]} (\mathrm{AffineMap}(p,q))(x)dx$. If $\frac{q-b}{a-p} \leqslant \inf A$, then $\int\limits_A f(x)dx = \int\limits_A (\mathrm{AffineMap}(p,q))(x)dx$. If $\frac{q-b}{a-p} \geqslant \sup A$, then $\int\limits_A f(x)dx = \int\limits_A (\mathrm{AffineMap}(a,b))(x)dx$. □

(43) Suppose $a \neq p$ and $f\restriction A = \mathrm{AffineMap}(a,b)\restriction[\inf A, \frac{q-b}{a-p}] + \cdot\, \mathrm{AffineMap}(p,q)\restriction[\frac{q-b}{a-p}, \sup A]$ and $\frac{q-b}{a-p} \in A$. Then $\int\limits_A (\mathrm{id}_{\mathbb{R}} \cdot f)(x)dx = \int\limits_{[\inf A, \frac{q-b}{a-p}]} (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(a,b)))(x)dx + \int\limits_{[\frac{q-b}{a-p}, \sup A]} (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(p,q)))(x)dx$.

PROOF: $(\mathrm{id}_{\mathbb{R}} \cdot f)\restriction[\inf A, \sup A] = (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(a,b)))\restriction[\inf A, \frac{q-b}{a-p}] + \cdot (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(p,q)))\restriction[\frac{q-b}{a-p}, \sup A]$. Set $F = (\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a-p}[+ \cdot \mathrm{AffineMap}(p,q)\restriction[\frac{q-b}{a-p}, +\infty[$. $F\restriction[\inf A, \sup A]$ is integrable. $F\restriction[\inf A, \sup A] = f\restriction A$. f is integrable on $[\inf A, \sup A]$ and $f\restriction[\inf A, \sup A]$ is bounded. $\mathrm{id}_{\mathbb{R}} \cdot f$ is integrable on $[\inf A, \sup A]$. □

(44) Let us consider real numbers a, b. Then $\mathrm{id}_{\mathbb{R}} \cdot \mathrm{AffineMap}(a,b) = a \cdot \square^2 + b \cdot \square^1$.

PROOF: For every object x such that $x \in \mathbb{R}$ holds $\mathrm{id}_{\mathbb{R}} \cdot \mathrm{AffineMap}(a,b)(x) = a \cdot (\square^2 + b \cdot \square^1)(x)$. □

(45) Let us consider real numbers a, b, c, d. Suppose $c \leqslant d$.

Then $\int\limits_c^d (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(a,b)))(x)dx = \frac{1}{3} \cdot a \cdot (d \cdot d \cdot d - c \cdot c \cdot c) + \frac{1}{2} \cdot b \cdot (d \cdot d - c \cdot c)$.

The theorem is a consequence of (44).

(46) Let us consider real numbers a, b. Then $\mathrm{AffineMap}(a,b) = a\cdot\square^1 + b\cdot\square^0$.
PROOF: For every object x such that $x \in \mathbb{R}$ holds $\mathrm{AffineMap}(a,b)(x) = (a\cdot\square^1 + b\cdot\square^0)(x)$. $\square$

(47) Let us consider real numbers a, b, c, d. Suppose $c \leqslant d$.
Then $\int_c^d (\mathrm{AffineMap}(a,b))(x)dx = \frac{1}{2}\cdot a\cdot(d\cdot d - c\cdot c) + b\cdot(d-c)$. The theorem is a consequence of (46).

(48) Let us consider real numbers a, b, p, q, c, d, e, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \neq p$ and $f\restriction A = \mathrm{AffineMap}(a,b)\restriction[\inf A, \frac{q-b}{a-p}] +\cdot \mathrm{AffineMap}(p,q)\restriction[\frac{q-b}{a-p}, \sup A]$ and $\frac{q-b}{a-p} \in A$. Then $\mathrm{centroid}(f,A) = \frac{\frac{1}{3}\cdot a\cdot((\frac{q-b}{a-p})^{\mathbf{3}} - (\inf A)^{\mathbf{3}}) + \frac{1}{2}\cdot b\cdot((\frac{q-b}{a-p})^{\mathbf{2}} - (\inf A)^{\mathbf{2}}) + \frac{1}{3}\cdot p\cdot((\sup A)^{\mathbf{3}} - (\frac{q-b}{a-p})^{\mathbf{3}}) + \frac{1}{2}\cdot q\cdot((\sup A)^{\mathbf{2}} - (\frac{q-b}{a-p})^{\mathbf{2}})}{\frac{1}{2}\cdot a\cdot((\frac{q-b}{a-p})^{\mathbf{2}} - (\inf A)^{\mathbf{2}}) + b\cdot(\frac{q-b}{a-p} - \inf A) + \frac{1}{2}\cdot p\cdot((\sup A)^{\mathbf{2}} - (\frac{q-b}{a-p})^{\mathbf{2}}) + q\cdot(\sup A - \frac{q-b}{a-p})}$.
The theorem is a consequence of (18), (40), (42), (43), (45), and (47).

(49) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$.
Then $\max_+(f) = \max(\mathrm{AffineMap}(0,0), f)$.

REFERENCES

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[3] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(**1**):93–102, 1999.

[4] Adam Grabowski. Fuzzy implications in the Mizar system. In *30th IEEE International Conference on Fuzzy Systems, FUZZ-IEEE 2021, Luxembourg, July 11–14, 2021*, pages 1–6. IEEE, 2021. doi:10.1109/FUZZ45933.2021.9494593.

[5] Adam Grabowski and Takashi Mitsuishi. *Extending Formal Fuzzy Sets with Triangular Norms and Conorms*, volume 642: *Advances in Intelligent Systems and Computing*, pages 176–187. Springer International Publishing, Cham, 2018. doi:10.1007/978-3-319-66824-6_16.

[6] Adam Grabowski and Takashi Mitsuishi. Initial comparison of formal approaches to fuzzy and rough sets. In Leszek Rutkowski, Marcin Korytkowski, Rafal Scherer, Ryszard Tadeusiewicz, Lotfi A. Zadeh, and Jacek M. Zurada, editors, *Artificial Intelligence and Soft Computing – 14th International Conference, ICAISC 2015, Zakopane, Poland, June 14-18, 2015, Proceedings, Part I*, volume 9119 of *Lecture Notes in Computer Science*, pages 160–171. Springer, 2015. doi:10.1007/978-3-319-19324-3_15.

[7] Tetsuro Katafuchi, Kiyoji Asai, and Hiroshi Fujita. Investigation of deffuzification in fuzzy inference: Proposal of a new defuzzification method (in Japanese). *Medical Imaging and Information Sciences*, 18(1):19–30, 2001. doi:10.11318/mii1984.18.19.

[8] Ebrahim H. Mamdani. Application of fuzzy algorithms for control of simple dynamic plant. *IEE Proceedings*, 121:1585–1588, 1974.

[9] Takashi Mitsuishi, Katsumi Wasaki, and Yasunari Shidama. Basic properties of fuzzy set operation and membership function. *Formalized Mathematics*, 9(**2**):357–362, 2001.

[10] Masaharu Mizumoto. Improvement of fuzzy control (IV)-case by product-sum-gravity method. In *Proc. 6th Fuzzy System Symposium, 1990*, pages 9–13, 1990.

[11] Timothy J. Ross. *Fuzzy Logic with Engineering Applications*. John Wiley and Sons Ltd, 2010.

[12] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.

[13] Werner Van Leekwijck and Etienne E. Kerre. Defuzzification: Criteria and classification. *Fuzzy Sets and Systems*, 108(2):159–178, 1999.

Accepted July 23, 2022