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Near Approximations in Modules

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Abstract. Rough set theory is a mathematical approach to imperfect knowledge. The near set approach leads to partitions of ensembles of sample objects with measurable information content and an approach to feature selection. In this paper, we apply the previous results of Bagirmaz [Appl. Algebra Engrg. Comm. Comput., 30(4) (2019) 285-29] and [Davvaz et al., Near approximations in rings. AAECC (2020). https://doi.org/10.1007/s00200-020-00421-3] to module theory. We introduce the notion of near approximations in a module over a ring, which is an extended notion of a rough approximations in a module presented in [B. Davvaz and M. Mahdavipour, Roughness in modules, Information Sciences, 176 (2006) 3658-3674]. Then we define the lower and upper near submodules and investigate their properties.

Keywords: Near set, lower and upper approximations, rough set, module, submodule

1. Introduction

In computer science, a rough set is a formal approximation of a crisp set in terms of a pair of sets which give the lower and the upper approximation of the original set. Rough set theory was introduced by Pawlak in [10] to handle imprecision, vagueness, and uncertainty in data analysis. The relations between rough sets and algebraic systems have been already considered by many mathematicians. A key concept in Pawlak rough set model is an equivalence relation. The equivalence classes are the

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building blocks for the construction of the lower and upper approximations. Indeed, the concept is mainly concerned with the approximation of objects using an equivalence relation on the universe of approximation space. The notion of nearness between sets (proximity relation between sets) was firstly investigated from a spatial meaning of distance by Efremovič [6]. Later, it would seem that the notion of proximity between sets was more abstract, i.e., proximity not limited to a spatial interpretation (see, also [7, 12]). Peters studied near sets in [11], where objects with affinities were considered perceptually near to each other. We know that any rough set has a non-empty boundary region, but the near set may also be a null set. In fact, this is the main difference between a rough set and a near set. Peters and Wasilewski [13] introduced an approach to the foundations of information science considered in the context of near sets.

In applied mathematics, applied sciences and our daily life we encounter many examples of mathematical objects that can be added to each other and multiplied by scalar numbers. First of all, the real numbers themselves are such objects. Other examples are real valued functions, the complex numbers, infinite series, vectors in n-dimensional space, and vector-valued functions. All of these examples and many others are special cases of a general mathematical concept, called a module over a ring. Indeed, module over a ring R is an abelian group M together with an external product $R \times M$ into M (the image of (r, x) being denoted by rx) such that for all $r, s \in R$ and $x, y \in M$: $r(x + y) = rx + ry$, $(r + s)x = rx + sy$ and $r(sx) = (rs)x$. A non-empty subset A is a submodule of M provided that A is an additive subgroup of M and $ra \in A$, for all $r \in R$ and $a \in A$. Modules over a ring are useful because they generalize the concept of vector spaces; instead of restricting the scalars to lie in a field we allow them to be element of an arbitrary ring. Any use you know of the theory of vector spaces (e.g. all of linear algebra), is a use of a particularly nice and well-understood part of the theory modules. Much of the motivation of modules comes from wanting to apply the ideas of linear algebra in settings where nonzero "scalars" might not be invertible (so one cannot just use linear algebra). In [5], Davvaz and Mahdavipour considered a module as a universal set and introduced the notion of rough submodule with respect to a submodule of a module, which is an extended notion of a submodule in a module.

İnan and Öztürk [8, 9] investigated the basic concepts of the algebraic structures of the near set theory. Bağirmaz [2] introduced the notion of near subsemigroups, near ideals and homomorphisms of near semigroups on near approximation spaces. Also, Bağirmaz [1] introduced the notion of a near approximations in a group. Recently, Davvaz et al. [3] investigated the upper and lower near subrings and ideals and studied their properties.

The paper is organized as follows: after the introduction and preliminaries, we apply the previous results of Bagirmaz [1] and Davvaz et al. [3] to module theory. We introduce the notion of near approximations in a module over a ring, which is an extended notion of a rough approximations in a module presented in [5]. Then we define lower and upper near submodules and investigate their properties.

2. Preliminaries

Here, we present the main definitions and properties of near sets as in [13].

Symbol	Interpretation
$\mathbb{R}$	Set of real numbers
$\mathcal{O}$	Set of perceptual objects
A	$A \subseteq \mathcal{O}$, set of sample objects
x	$x \in \mathcal{O}$, sample object
$\mathbb{F}$	A set of functions representing object features
$\mathcal{B}$	$\mathcal{B} \subseteq \mathbb{F}$
Φ	$\Phi : \mathcal{O} \to \mathbb{R}^n$, object description
n	n is a description length
φ_i	$\varphi_i \in \mathcal{B}$, where $\varphi_i : \mathcal{O} \to \mathbb{R}$
$\Phi(x)$	$\Phi(x) = \big(\varphi_1(x), \ldots, \varphi_n(x)\big)$, description

Table 1. Description symbols

Definition 2.1. [13] A perceptual information system $\langle \mathcal{O}, \mathbb{F} \rangle$ or, more concisely, perceptual system, is a real-valued total deterministic information system where $\mathcal{O}$ is a non-empty set of perceptual objects, while $\mathbb{F}$ is a countable set of probe functions.

An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in A$. The important thing to notice is the choice of functions $\varphi_i \in \mathcal{B}$ used to describe an object of interest. Assume that $\mathcal{B} \subseteq \mathbb{F}$ (see Table 1) is a given set of functions representing features of sample objects $A \subseteq \mathcal{O}$.

Definition 2.2. [13] Let $\langle \mathcal{O}, \mathbb{F} \rangle$ be a perceptual system. For every $\mathcal{B} \subseteq \mathbb{F}$ the indiscernibility relation $\sim_{\mathcal{B}}$ is defined as follows:

$$\sim_{\mathcal{B}} = \{(x, y) \in \mathcal{O} \times \mathcal{O} \ : \ \forall \varphi_i \in \mathcal{B}, \varphi_i(x) = \varphi_i(y)\}.$$

If $\mathcal{B} = \{\varphi\}$, for some $\varphi \in \mathbb{F}$, instead of $\sim_{\{\varphi\}}$ we write $\sim_{\varphi}$.

This relation defines a partition of $\mathcal{O}$ into non-empty, pairwise disjoint subsets, where they are equivalence classes, and denoted by $[x]_{\mathcal{B}}$, where $[x]_{\mathcal{B}} = \{y \in \mathcal{O} \ : \ x \sim_{\mathcal{B}} y\}$. These classes form the quotient set $\mathcal{O}/\sim_{\mathcal{B}}$, where $\mathcal{O}/\sim_{\mathcal{B}} = \{[x]_{\mathcal{B}} \ : \ x \in \mathcal{O}\}$.

The basic idea in the near set approach to object recognition is to compare object descriptions. Sample perceptual objects $x, y \in \mathcal{O}$ $(x \neq y)$ are near each other if and only if x and y have similar descriptions. Let A and A' are two subsets of $\mathcal{O}$ and $\mathcal{B} \subseteq \mathbb{F}$. We say that A is near A' if there exist $x \in A$, $y \in A'$ and $\varphi \in \mathcal{B}$ such that $x \sim_{\varphi} y$.

Definition 2.3. The family of neighborhood with respect to one object feature is defined as:

$$N_1(\mathcal{B}) = \cup\{[x]_{\varphi_i} \ : \ x \in \mathcal{O}, \ \varphi_i \in \mathcal{B}\}.$$

The family of neighborhoods with respect to two object features is defined as

$$N_2(\mathcal{B}) = \cup\{[x]_{\{\varphi_i,\varphi_j\}} \ : \ x \in \mathcal{O},\ \varphi_i, \varphi_j \in \mathcal{B}, i \neq j\}.$$

Finally, the family of neighborhoods with respect to all object features is defined as

$$N_{|\mathcal{B}|}(\mathcal{B}) = \cup\{[x]_{\{\varphi_1,\ldots,\varphi_{|\mathcal{B}|}\}} \ : \ x \in \mathcal{O},\ \varphi_i \in \mathcal{B}\}.$$

Definition 2.4. [11] A nearness approximation space (NAS) is a tuple $NAS = (\mathcal{O}, \mathbb{F}, \sim_{\mathcal{B}_r}, \mathcal{B}_r, N_r)$ where the approximation space NAS is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathbb{F}$ representing object features, indiscernibility relation $\sim_{\mathcal{B}_r}$ defined relative to $\mathcal{B}_r \subseteq \mathcal{B} \subseteq \mathbb{F}$, collection of partitions (families of neighborhoods) $N_r(\mathcal{B})$.

Definition 2.5. [11] Let $(\mathcal{O}, \mathbb{F}, \sim_{\mathcal{B}_r}, \mathcal{B}_r, N_r)$ be a nearness approximation space and let A be an arbitrary subset of $\mathcal{O}$. We define

$$\underline{N_r(\mathcal{B})}(A) = \{x \in \mathcal{O} \ : \ [x]_{\mathcal{B}_r} \subseteq A\}$$

and

$$\overline{N_r(\mathcal{B})}(A) = \{x \in \mathcal{O} \ : \ [x]_{\mathcal{B}_r} \cap A \neq \emptyset\}.$$

$\underline{N_r(\mathcal{B})}(A)$ is called $N_r(\mathcal{B})$-lower approximation and $\overline{N_r(\mathcal{B})}(A)$ is called $N_r(\mathcal{B})$-upper approximation of A in $(\mathcal{O}, \mathbb{F}, \sim_{\mathcal{B}_r}, \mathcal{B}_r, N_r)$.

3. Near approximation in a module

3.1. Approximations in rings and modules

Let I be an ideal of R and A be a non-empty subset of R. According to [4], the set

$$\underline{Apr_I}(A) = \{x \in R \ : \ x + I \subseteq A\}$$

and

$$\overline{Apr_I}(A) = \{x \in R \ : \ x + I \cap A \neq \emptyset\}$$

are called lower and upper approximations of the set A with respect to the ideal I of R respectively.

Let M be an R-module. If A is a non-empty subset of R and S is a non-empty subset of M, then AS is defined by

$$AS = \left\{ \sum_{i=1}^{n} a_i s_i \ : \ a_i \in A, s_i \in S \text{ and } n \in \mathbb{N} \right\}.$$

Let S be a submodule of M and A be a non-empty subset of M. According to [5], the sets

$$\underline{Apr_S}(A) = \{x \in M \ : \ x + S \subseteq A\}$$

and

$$\overline{Apr_S}(A) = \{x \in M \ : \ x + S \cap A \neq \emptyset\}$$

are called lower and upper approximations of set A with respect to the submodule S of M, respectively.

EXAMPLE 1. Let $M = \mathbb{Z}_{18}$ and $R = \mathbb{Z}_{18}$ so M is a module over itself. Let $S = \langle 6\rangle$ be a submodule of M and suppose that $A = \{0,\ 1,\ 2,\ 4,\ 6,\ 7,\ 8,\ 10,\ 12,\ 13,\ 14,\ 16,\ 17\}$. Then, we have $\underline{Apr_S}(A) = S \cup (1+S) \cup (2+S) \cup (4+S) = A - \{17\}$ and $\overline{Apr_S}(A) = S \cup (1+S) \cup (\overline{2+S}) \cup (4+S) \cup (5+S) = \mathbb{Z}_{18} - (3+S) = \mathbb{Z}_{18} - \{3, 9, 15\}$.

3.2. Near approximations of a set with respect to submodules

We denote $\mathbb{F}$ the set of all submodules in a module M and $\mathcal{B} \subseteq \mathbb{F}$. In this case, we use the pair $(M, \mathbb{F})$ instead of the nearness approximation space $(\mathcal{O}, \mathbb{F}, \sim_{\mathcal{B}_r}, \mathcal{B}_r, N_r)$.

If $(M, \mathbb{F})$ is a nearness approximation space, then

$$[x]_{\mathcal{B}_r} = x + \mathcal{B}_r = x + \Big(\bigcap_{S \in \mathcal{B}_r} S\Big).$$

Definition 3.1. Let $(M, \mathbb{F})$ be a nearness approximation space and A be a non-empty subset of M. Then, the sets

$$\underline{N_r(\mathcal{B})}(A) = \{x \in M \ : \ x + \mathcal{B}_r \subseteq A\}$$

and

$$\overline{N_r(\mathcal{B})}(A) = \{x \in M \ : \ x + \mathcal{B}_r \cap A \neq \emptyset\}$$

are called near lower and near upper approximations of A with respect to the submodules $\mathcal{B}$ of M, respectively. Clearly, we observe

$$\underline{N_r(\mathcal{B})}(A) = \bigcup_{\mathcal{B}_r \subseteq \mathcal{B}} \underline{Apr_{\mathcal{B}_r}}(A) \text{ and } \overline{N_r(\mathcal{B})}(A) = \bigcup_{\mathcal{B}_r \subseteq \mathcal{B}} \overline{Apr_{\mathcal{B}_r}}(A).$$

EXAMPLE 2. Let $M = \mathbb{Z}_{18}$ and $R = \mathbb{Z}_{18}$ so M is a module over it self. The set $\mathcal{B} = \{S_1,\ S_2,\ S_3\}$ is a family of submodules of M, where $S_1 = \langle 2\rangle$, $S_2 = \langle 3\rangle$ and $S_3 = \langle 9\rangle$. If $A = \{0,\ 1,\ 4,\ 6,\ 7,\ 8,\ 10,\ 12,\ 13,\ 14,\ 16,\ 17\}$, then

(1) We observe that $\underline{Apr_{S_1}}(A) = \emptyset$, $\underline{Apr_{S_2}}(A) = (1+S_2) = \{1,\ 4,\ 7,\ 10,\ 13,\ 16\}$, $\underline{Apr_{S_3}}(A) = (1+\overline{S_3}) \cup (\overline{4+S_3}) \cup (\overline{7+S_3}) \cup (8+S_3) = \{1,\ 4,\ 7,\ 8,\ 10,\ 13,\ 16,\ 17\}$. So, $\underline{N_1(\mathcal{B})}(A) = \underline{Apr_{S_1}}(A) \cup \underline{Apr_{S_2}}(A) \cup \underline{Apr_{S_3}}(A) = \{1,\ 4,\ 7,\ 8,\ 10,\ 13,\ 16,\ 17\}$;

(2) We have $\overline{Apr_{S_1}}(A) = S_1 \cup (1+S_1) = \mathbb{Z}_{18}$, $\overline{Apr_{S_2}}(A) = S_2 \cup (1+S_2) \cup (2+S_2) = \mathbb{Z}_{18}$, $\overline{Apr_{S_3}}(A) = \mathbb{Z}_{18} - \{2+S_3\} = Z_{18} - \{2, 11\}$. Hence, $\overline{N_1(\mathcal{B})}(A) = \overline{Apr_{S_1}}(A) \cup \overline{Apr_{S_2}}(A) \cup \overline{Apr_{S_3}}(A) = \mathbb{Z}_{18}$;

(3) We obtain $S_1 \cap S_2 = \langle 6\rangle$, $S_1 \cap S_3 = \langle 0\rangle$ and $S_2 \cap S_3 = \langle 9\rangle$. Thus, $\underline{Apr_{S_1 \cap S_2}}(A) = (S_1 \cap S_2) \cup (1 + S_1 \cap S_2)) \cup (4 + S_1 \cap S_2) = \{0,\ 1,\ 4,\ 6,\ 7,\ 10,\ 12,\ 13,\ 16\}$, $\underline{Apr_{S_1 \cap S_3}}(A) = A$, $\underline{Apr_{S_2 \cap S_3}}(A) = \underline{Apr_{S_3}}(A) = \{1, 4, 7, 8, 10, 13, 16, 17\}$. So, we conclude that $\underline{N_2(\mathcal{B})}(A) = \underline{Apr_{S_1 \cap S_2}}(A) \cup \underline{Apr_{S_1 \cap S_3}}(A) \cup \underline{Apr_{S_2 \cap S_3}}(A) = A$;

(4) We find that $S_1 \cap S_2 = \langle 6\rangle$, $S_1 \cap S_3 = \langle 0\rangle$ and $S_2 \cap S_3 = \langle 9\rangle$. Then, $\overline{Apr_{S_1\cap S_2}}(A) = \mathbb{Z}_{18} - (3 + S_1 \cap S_2) = \mathbb{Z}_{18} - \{3, 9, 15\}$, $\overline{Apr_{S_1\cap S_3}}(A) = A$ and $\overline{Apr_{S_2\cap S_3}}(A) = \overline{Apr_{S_3}}(A) = \mathbb{Z}_{18} - \{2, 11\}$. Thus, $\overline{N_2(\mathcal{B})}(A) = \overline{Apr_{S_1\cap S_2}}(A) \cup \overline{Apr_{S_1\cap S_3}}(A) \cup \overline{Apr_{S_2\cap S_3}}(A) = \mathbb{Z}_{18}$;

(5) We get $S_1 \cap S_2 \cap S_3 = \langle 0\rangle$. Then, $\underline{N_3(\mathcal{B})}(A) = \underline{Apr_{S_1\cap S_2\cap S_3}}(A) = A$ and $\overline{N_3(\mathcal{B})}(A) = \overline{Apr_{S_1\cap S_2\cap S_3}}(A) = A$.

Proposition 3.2. *For every approximation space $(M, \mathbb{F})$ and every subsets A and B of M, we have*

(1) $\underline{N_r(\mathcal{B})}(A) \subseteq A \subseteq \overline{N_r(\mathcal{B})}(A)$;

(2) $\underline{N_r(\mathcal{B})}(\emptyset) = \emptyset = \overline{N_r(\mathcal{B})}(\emptyset)$;

(3) $\underline{N_r(\mathcal{B})}(M) = M = \overline{N_r(\mathcal{B})}(M)$;

(4) *If* $A \subseteq B$, then $\underline{N_r(\mathcal{B})}(A) \subseteq \underline{N_r(\mathcal{B})}(B)$ and $\overline{N_r(\mathcal{B})}(A) \subseteq \overline{N_r(\mathcal{B})}(B)$;

(5) $\underline{N_r(\mathcal{B})}(\underline{N_r(\mathcal{B})}(A)) = \underline{N_r(\mathcal{B})}(A)$;

(6) $\overline{N_r(\mathcal{B})}(\overline{N_r(\mathcal{B})}(A)) \supseteq \overline{N_r(\mathcal{B})}(A)$;

(7) $\overline{N_r(\mathcal{B})}(\underline{N_r(\mathcal{B})}(A)) \supseteq \underline{N_r(\mathcal{B})}(A)$;

(8) $\underline{N_r(\mathcal{B})}(\overline{N_r(\mathcal{B})}(A)) = \overline{N_r(\mathcal{B})}(A)$;

(9) $\underline{N_r(\mathcal{B})}(A) \supseteq \left(\overline{N_r(\mathcal{B})}(A^C)\right)^C$;

(10) $\overline{N_r(\mathcal{B})}(A) \supseteq \left(\underline{N_r(\mathcal{B})}(A^C)\right)^C$;

(11) $\underline{N_r(\mathcal{B})}(A \cap B) = \underline{N_r(\mathcal{B})}(A) \cap \underline{N_r(\mathcal{B})}(B)$;

(12) $\overline{N_r(\mathcal{B})}(A \cap B) \subseteq \overline{N_r(\mathcal{B})}(A) \cap \overline{N_r(\mathcal{B})}(B)$;

(13) $\underline{N_r(\mathcal{B})}(A \cup B) \supseteq \underline{N_r(\mathcal{B})}(A) \cup \underline{N_r(\mathcal{B})}(B)$;

(14) $\overline{N_r(\mathcal{B})}(A \cup B) = \overline{N_r(\mathcal{B})}(A) \cup \overline{N_r(\mathcal{B})}(B)$.

Proof. The proofs of parts (1), (4), (11), (12), (13) and (14) follow from Proposition 3.1 in [1] as submodules are also normal subgroups.
For numbers 2 and 3 the proofs follow from Definition 3.1.

(5) By Proposition 3.2 (1), we have $\underline{N_r(\mathcal{B})}(\underline{N_r(\mathcal{B})}(A)) \subseteq \underline{N_r(\mathcal{B})}(A)$. For the converse, suppose that x is any element of $\underline{N_r(\mathcal{B})}(A)$. Then, $x + \overline{\mathcal{B}_r} \subseteq A$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ so that $x + \mathcal{B}_r \subseteq \underline{N_r(\mathcal{B})}(A)$ for some $\overline{\mathcal{B}_r} \subseteq \mathcal{B}$ which implies $x \in \underline{N_r(\mathcal{B})}(\underline{N_r(\mathcal{B})}(A))$. Therefore, $\underline{N_r(\mathcal{B})}(A) \subseteq \underline{N_r(\mathcal{B})}(\underline{N_r(\mathcal{B})}(A))$.

(6) By Proposition 3.2(1), we have $\overline{N_r(\mathcal{B})}(A) \subseteq \overline{N_r(\mathcal{B})}(\overline{N_r(\mathcal{B})}(A))$.

(7) By Proposition 3.2 (1), we have $\underline{N_r(\mathcal{B})}(A) \subseteq \overline{N_r(\mathcal{B})}(\underline{N_r(\mathcal{B})}(A))$.

(8) By Proposition 3.2 (1), we have $\underline{N_r(\mathcal{B})}(\overline{N_r(\mathcal{B})}(A)) \subseteq \overline{N_r(\mathcal{B})}(A)$. Conversely, let x be any element of $\overline{N_r(\mathcal{B})}(A)$, then $x + \mathcal{B}_r \cap A \neq \emptyset$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ so that $x + \mathcal{B}_r \subseteq \overline{N_r(\mathcal{B})}(A)$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ which implies $x \in \underline{N_r(\mathcal{B})}(\overline{N_r(\mathcal{B})}(A))$. Therefore, $\overline{N_r(\mathcal{B})}(A) \subseteq \underline{N_r(\mathcal{B})}(\overline{N_r(\mathcal{B})}(A))$.

(9) Suppose that x is any element of $\left(\overline{N_r(\mathcal{B})}(A^C)\right)^C$. Then, we obtain $x \notin \overline{N_r(\mathcal{B})}(A^C)$ so that $x + \mathcal{B}_r \cap A^C = \emptyset$ for every $\mathcal{B}_r \subseteq \mathcal{B}$. Hence, we have $x + \mathcal{B}_r \subseteq A$ for every $\mathcal{B}_r \subseteq \mathcal{B}$, which implies $x \in \underline{N_r(\mathcal{B})}(A)$, Therefore, we get $\left(\overline{N_r(\mathcal{B})}(A^C)\right)^C \subseteq \underline{N_r(\mathcal{B})}(A)$.

(10) Suppose that x is any element of $\left(\underline{N_r(\mathcal{B})}(A^C)\right)^C$. Then, we have $x \notin \underline{N_r(\mathcal{B})}(A^C)$ so that $x + \mathcal{B}_r \not\subseteq A^C$ for every $\mathcal{B}_r \subseteq \mathcal{B}$. Hence, we have $x + \mathcal{B}_r \cap A \neq \emptyset$ for every $\mathcal{B}_r \subseteq \mathcal{B}$, which implies $x \in \overline{N_r(\mathcal{B})}(A)$. Therefore, we deduce that $\left(\underline{N_r(\mathcal{B})}(A^C)\right)^C \subseteq \overline{N_r(\mathcal{B})}(A)$. □

3.3. Near approximations of a set with respect to special submodules

We begin with the following definition:

Definition 3.3. Let $(M, \mathbb{F})$ be a nearness approximation space, and suppose that I is an ideal of R and A is a non-empty subset of M. Then, the sets

$$\underline{N_r(I\mathcal{B})}(A) = \{x \in M \ : \ x + I\mathcal{B}_r \subseteq A\}$$

and

$$\overline{N_r(I\mathcal{B})}(A) = \{x \in M \ : \ x + I\mathcal{B}_r \cap A \neq \emptyset\}$$

are called near lower and near upper approximations of A with respect to the submodule $I\mathcal{B}$ of M, respectively.

Proposition 3.4. *Let $(M, \mathbb{F})$ be a nearness approximation space and I be an ideal of R. If A is a non-empty subset of R and B is a non-empty subset of M, then*

$$\overline{N_r(I\mathcal{B})}(AB) \subseteq \overline{Apr_I}(A)\overline{N_r(\mathcal{B})}(B).$$

Proof. Suppose that x is any element of $\overline{N_r(I\mathcal{B})}(AB)$. Then, $x + I\mathcal{B}_r \cap AB \neq \emptyset$ for some $\mathcal{B}_r \subseteq \mathcal{B}$. Hence, there exists an element $y \in M$ such that $y \in x + I\mathcal{B}_r$ and

$y \in AB$. Thus, we can write $y = \sum_{i=1}^{n} a_i b_i$ for some $a_i \in A$ and $b_i \in B$. Now, we have

$$\begin{aligned} x \in y + I\mathcal{B}_r &= (\sum_{i=1}^{n} a_i b_i) + I\mathcal{B}_r \\ &= \sum_{i=1}^{n}(a_i b_i + I\mathcal{B}_r) = \sum_{i=1}^{n}(a_i b_i + a_i 0 + 0 b_i + I\mathcal{B}_r) \\ &\subseteq \sum_{i=1}^{n}(a_i b_i + a_i\mathcal{B}_r + I b_i + I\mathcal{B}_r) = \sum_{i=1}^{n}(a_i + I)(b_i + \mathcal{B}_r). \end{aligned}$$

Hence, there exist $x_i \in a_i + I$ and $y_i \in b_i + \mathcal{B}_r$ such that $x = \sum_{i=1}^{n} x_i y_i$. Therefore, $a_i \in x_i + I \cap A$ and similarly $b_i \in y_i + \mathcal{B}_r \cap B$. Now, we obtain $x_i \in \overline{Apr_I}(A)$ and $y_i \in \underline{N_r(\mathcal{B})}(B)$, which yields that $x \in \overline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B)$. Therefore, $\overline{N_r(I\mathcal{B})}(AB) \subseteq \overline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B)$. □

Example 3. Let $M = \mathbb{Z}_{12}$ and $R = \mathbb{Z}_{18}$. It is easy to see that M is an R-module in the usual manner. The set $\mathcal{B} = \{S_1,\ S_2,\ S_3\}$ is a family of submodules of M, where $S_1 = \langle 2\rangle = \{0,\ 2,\ 4,\ 6,\ 8,\ 10\}$, $S_2 = \langle 3\rangle = \{0,\ 3,\ 6,\ 9\}$, $S_3 = \langle 6\rangle = \{0,\ 6\}$ and $I = \langle 3\rangle = \{0,\ 3,\ 6,\ 9,\ 12,\ 15\}$ is an ideal of R. Let $A = \{0, 6\}$ be a subset of R and $B = \{0, 1, 6, 9\}$ be a subset of M. Then,

- $\overline{Apr_I}(A) = \langle 3\rangle = \{0,\ 3,\ 6,\ 9,\ 12,\ 15\}$;
- $\overline{N_1(\mathcal{B})}(B) = \mathbb{Z}_{12}$, $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = S_1 \cap S_2 \cap S_3 = S_3 = \langle 6\rangle = \{0,\ 6\}$ so that $\overline{N_2(\mathcal{B})}(B) = \overline{N_3(\mathcal{B})}(B) = \{0,\ 1,\ 3,\ 6,\ 7,\ 9\}$;
- $AB \subseteq M, AB = \langle 6\rangle = \{0,\ 6\}$, $IS_1 = IS_3 = \langle 6\rangle = \{0,\ 6\}$ and $IS_2 = \langle 3\rangle = \{0,\ 3,\ 6,\ 9\}$ so that $\overline{N_1(I\mathcal{B})}(AB) = \langle 3\rangle = \{0,\ 3,\ 6,\ 9\}$ and $\overline{N_2(I\mathcal{B})}(AB) = \overline{N_3(I\mathcal{B})}(AB) = \langle 6\rangle = \{0,\ 6\}$.

Consequently, we have

$$\overline{N_1(I\mathcal{B})}(AB) = \overline{Apr_I}(A)\overline{N_1(\mathcal{B})}(B), \overline{N_2(I\mathcal{B})}(AB) \subset \overline{Apr_I}(A)\overline{N_2(\mathcal{B})}(B),$$
$$\overline{N_3(I\mathcal{B})}(AB) \subset \overline{Apr_I}(A)\overline{N_3(\mathcal{B})}(B).$$

This yields that $\overline{N_r(I\mathcal{B})}(AB) = \overline{Apr_I}(A)\overline{N_r(\mathcal{B})}(B)$ is not true in general.

Let $\mathcal{B}^* = \{I_1,\ I_2,\ I_3, \cdots,\ I_n\}$, where $I_1,\ I_2,\ I_3, \cdots,\ I_n$ are ideals of R and A is a non-empty subset of R. Similar to Definition 3.1, $\underline{N_r(\mathcal{B}^*)}(A)$ and $\overline{N_r(\mathcal{B}^*)}(A)$ are defined by

$$\underline{N_r(\mathcal{B}^*)}(A) = \{x \in R\ :\ x + {\mathcal{B}_r}^* \subseteq A\}$$

and

$$\overline{N_r(\mathcal{B}^*)}(A) = \{x \in R\ :\ x + {\mathcal{B}_r}^* \cap A \neq \emptyset\}.$$

Definition 3.5. Let $(M, \mathbb{F})$ be a nearness approximation space and B be a non-empty subset of M. Then, the sets

$$\underline{N_r(\mathcal{B}^*\mathcal{B})}(AB) = \{x \in M \ : \ x + \mathcal{B}_r{}^*\mathcal{B}_r \subseteq AB\}$$

and

$$\overline{N_r(\mathcal{B}^*\mathcal{B})}(AB) = \{x \in M \ : \ x + \mathcal{B}_r{}^*\mathcal{B}_r \cap AB \neq \emptyset\}$$

are called near lower and near upper approximations of set AB with respect to the submodules $\mathcal{B}^*\mathcal{B}$ of M, respectively.

Proposition 3.6. *Let $(M, \mathbb{F})$ be a nearness approximation space and let*

$$\mathcal{B}^* = \{I_1, \ I_2, \ I_3, \ \cdots, \ I_n\},$$

where $I_1, \ I_2, \ I_3, \ \cdots, \ I_n$ are ideals of R. Let A be a non-empty subset of R and B be a non-empty subset of M. Then,

$$\overline{N_r(\mathcal{B}^*\mathcal{B})}(AB) \subseteq \overline{N_r(\mathcal{B}^*)}(A)\overline{N_r(\mathcal{B})}(B).$$

Proof. Suppose that x is any element of $\overline{N_r(\mathcal{B}^*\mathcal{B})}(AB)$. Then, $x + \mathcal{B}_r{}^*\mathcal{B}_r \cap AB \neq \emptyset$ for some $\mathcal{B}_r^* \subseteq \mathcal{B}^*$ and $\mathcal{B}_r \subseteq \mathcal{B}$. Hence, there exists an element $y \in M$ such that $y \in x + \mathcal{B}_r{}^*\mathcal{B}_r$ and $y \in AB$. Thus, we can write $y = \sum_{i=1}^{n} a_i b_i$ for some $a_i \in A$ and $b_i \in B$. Now, we have

$$\begin{aligned} x \in y + \mathcal{B}_r{}^*\mathcal{B}_r \quad &= (\sum_{i=1}^{n} a_i b_i) + \mathcal{B}_r{}^*\mathcal{B}_r = \sum_{i=1}^{n}(a_i b_i + \mathcal{B}_r{}^*\mathcal{B}_r) \\ &= \sum_{i=1}^{n}(a_i b_i + a_i 0 + 0 b_i + \mathcal{B}_r{}^*\mathcal{B}_r) \\ &\subseteq \sum_{i=1}^{n}(a_i b_i + a_i\mathcal{B}_r + \mathcal{B}_r^* b_i + \mathcal{B}_r{}^*\mathcal{B}_r) \\ &= \sum_{i=1}^{n}(a_i + \mathcal{B}_r^*)(b_i + \mathcal{B}_r). \end{aligned}$$

Hence, there exist $x_i \in a_i + \mathcal{B}_r^*$ and $y_i \in b_i + \mathcal{B}_r$ such that $x = \sum_{i=1}^{n} x_i y_i$. This yields that $a_i \in x_i + \mathcal{B}_r^* \cap A$ and similarly we have $b_i \in y_i + \mathcal{B}_r \cap B$. Now, we obtain $x_i \in \overline{N_r(\mathcal{B}^*)}(A)$ and $y_i \in \overline{N_r(\mathcal{B})}(B)$ which yields that $x \in \overline{N_r(\mathcal{B}^*)}(A)\overline{N_r(\mathcal{B})}(B)$. Consequently, we have $\overline{N_r(\mathcal{B}^*\mathcal{B})}(AB) \subseteq \overline{N_r(\mathcal{B}^*)}(A)\overline{N_r(\mathcal{B})}(B)$. □

Proposition 3.7. *Let $(M, \mathbb{F})$ be a nearness approximation space and I be an ideal of R. Let A be a non-empty subsets of R and B be a non-empty subset of M, then*

$$\underline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B) \subseteq \underline{N_r(I\mathcal{B})}(AB).$$

Proof. Suppose that x is any element of $\underline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B)$. Then we can write $x = \sum_{i=1}^{n} a_i b_i$ for some $a_i \in \underline{Apr_I}(A)$ and $b_i \in \overline{N_r(\mathcal{B})}(B)$. Thus $a_i + I \subseteq A$ and $b_i + \mathcal{B}_{r_i} \subseteq B$ for some $\mathcal{B}_{r_i} \subseteq B$. Now, we have

$$\begin{aligned} x + I\mathcal{B}_{r_i} &= (\sum_{i=1}^{n} a_i b_i) + I\mathcal{B}_{r_i} = \sum_{i=1}^{n}(a_i b_i + I\mathcal{B}_{r_i}) \\ &= \sum_{i=1}^{n}(a_i b_i + a_i 0 + 0 b_i + I\mathcal{B}_{r_i}) \subseteq \sum_{i=1}^{n}(a_i b_i + a_i \mathcal{B}_{r_i} + I b_i + I\mathcal{B}_{r_i}) \\ &= \sum_{i=1}^{n}(a_i + I)(b_i + \mathcal{B}_{r_i}) \subseteq AB, \end{aligned}$$

which yields that $x \in \underline{N_r(I\mathcal{B})}(AB)$. Therefore, we conclude that $\underline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B) \subseteq \underline{N_r(I\mathcal{B})}(AB)$. □

Example 4. Let $M = \mathbb{Z}_{12}$ and $R = \mathbb{Z}_{18}$. It is easy to see that M is an R-module in the usual manner. The set $\mathcal{B} = \{S_1,\ S_2,\ S_3\}$ is a family of submodules of M, where $S_1 = \langle 2\rangle = \{0,\ 2,\ 4,\ 6,\ 8,\ 10\}$, $S_2 = \langle 4\rangle = \{0,\ 4,\ 8\}$, $S_3 = \langle 6\rangle = \{0,\ 6\}$ and $I = \langle 6\rangle = \{0,\ 6,\ 12\}$ is an ideal of R. Let $A = \{0,\ 3,\ 6,\ 12\}$ be a subset of R and $B = \{0,\ 1,\ 6,\ 9\}$ be a subset of M. Then

- $\underline{Apr_I}(A) = \langle 6\rangle = \{0,\ 6,\ 12\}$,
- $\underline{N_1(\mathcal{B})}(B) = \langle 6\rangle = \{0, 6\}$, $S_1 \cap S_2 = S_2 = \langle 4\rangle = \{0,\ 4,\ 8\}$, $S_1 \cap S_3 = S_3 = \langle 6\rangle = \{0,\ 6\}$, $S_2 \cap S_3 = \{0\}$, $S_1 \cap S_2 \cap S_3 = \{0\}$. Hence, $\underline{N_2(\mathcal{B})}(B) = \underline{N_3(\mathcal{B})}(B) = B = \{0, 1, 6, 9\}$,
- $AB \subseteq M, AB = \langle 3\rangle = \{0,\ 3,\ 6,\ 9\}, IS_1 = IS_2 = IS_3 = I(S_1 \cap S_2) = I(S_1 \cap S_3) = I(S_2 \cap S_3) = I(S_1 \cap S_2 \cap S_3) = \{0\}$. So, $\underline{N_1(I\mathcal{B})}(AB) = \underline{N_2(I\mathcal{B})}(AB) = \underline{N_3(I\mathcal{B})}(AB) = AB = \langle 3\rangle = \{0,\ 3,\ 6,\ 9\}$.

Therefore, we conclude that

$$\underline{Apr_I}(A)\underline{N_1(\mathcal{B})}(B) = \{0\} \subset \underline{N_1(I\mathcal{B})}(AB),$$

$$\underline{Apr_I}(A)\underline{N_2(\mathcal{B})}(B) = \{0,\ 6\} \subset \underline{N_2(I\mathcal{B})}(AB),$$

$$\underline{Apr_I}(A)\underline{N_3(\mathcal{B})}(B) = \{0,\ 6\} \subset \underline{N_3(I\mathcal{B})}(AB).$$

This shows that $\underline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B) = \underline{N_r(I\mathcal{B})}(AB)$ is not true in general.

Proposition 3.8. *Let $(M, \mathbb{F})$ be a nearness approximation space and let*

$$\mathcal{B}^* = \{I_1,\ I_2,\ I_3, \cdots, I_n\},$$

where $I_1,\ I_2,\ I_3,\ \cdots,\ I_n$ are ideals of R. If A is a non-empty subset of R and B is a non-empty subset of M, then

$$\underline{N_r(\mathcal{B}^*)}(A)\underline{N_r(\mathcal{B})}(B) \subseteq \underline{N_r(\mathcal{B}^*\mathcal{B})}(AB).$$

Proof. Suppose that x is any element of $\underline{N_r(\mathcal{B}^*)}(A)\underline{N_r(\mathcal{B})}(B)$. Then, we can write $x = \sum_{i=1}^{n} a_i b_i$, for some $a_i \in \underline{N_r(\mathcal{B}^*)}(A)$ and $b_i \in \underline{N_r(\mathcal{B})}(B)$. Thus, we have $a_i + \mathcal{B}^*_{r_i} \subseteq A$, for some $\mathcal{B}^*_{r_i} \subseteq \mathcal{B}^*$ and $b_i + \mathcal{B}_{r_i} \subseteq \mathcal{B}$, for some $\mathcal{B}_{r_i} \subseteq B$. Now, we have

$$\begin{aligned} x + \mathcal{B}^*_{r_i}\mathcal{B}_{r_i} &= (\sum_{i=1}^{n} a_i b_i) + \mathcal{B}^*_{r_i}\mathcal{B}_{r_i} = \sum_{i=1}^{n}(a_i b_i + \mathcal{B}^*_{r_i}\mathcal{B}_{r_i}) \\ &= \sum_{i=1}^{n}(a_i b_i + a_i 0 + 0 b_i + \mathcal{B}^*_{r_i}\mathcal{B}_{r_i}) \subseteq \sum_{i=1}^{n}(a_i b_i + a_i \mathcal{B}_{r_i} + \mathcal{B}^*_{r_i} b_i + \mathcal{B}^*_{r_i}\mathcal{B}_{r_i}) \\ &= \sum_{i=1}^{n}(a_i + \mathcal{B}^*_{r_i})(b_i + \mathcal{B}_{r_i}) \subseteq AB, \end{aligned}$$

which yields that $x_i \in \underline{N_r(\mathcal{B}^*\mathcal{B})}(AB)$. This shows that $\underline{N_r(\mathcal{B}^*)}(A)\underline{N_r(\mathcal{B})}(B) \subseteq \underline{N_r(\mathcal{B}^*\mathcal{B})}(AB)$. □

Proposition 3.9. *If I and A are ideals of R and $I \subseteq A$, then $\underline{Apr_I}(A) = A$.*

Proof. Clearly, we have $\underline{Apr_I}(A) \subseteq A$. Suppose that $a \in A$. Since A is an ideal of R and $I \subseteq A$, it follows that $a + I \subseteq A$ which implies $a \in \underline{Apr_I}(A)$. Therefore $A \subseteq \underline{Apr_I}(A)$. □

The following corollary follows directly from Proposition 3.9.

Corollary 3.10. Let $\mathcal{B}^* = \{I_1,\ I_2,\ \cdots,\ I_n\}$, where $I_1,\ I_2,\ I_3,\ \cdots,\ I_n$ are ideals of R. Let A be an ideal of R such that $\mathcal{B}^*_r \subseteq A$, for some $\mathcal{B}^*_r \subseteq \mathcal{B}^*$, then

$$\underline{N_r(\mathcal{B}^*)}(A) = A.$$

Proof. By Proposition 3.9, since $\mathcal{B}^*_r \subseteq A$, it follows $\underline{Apr_{B^*_r}}(A) = A$. Hence, $A = \underline{Apr_{B^*_r}}(A) \subseteq \underline{N_r(\mathcal{B}^*)}(A) \subseteq A$. Therefore, we obtain $\underline{N_r(\mathcal{B}^*)}(A) = A$. □

Proposition 3.11. *Let $(M, \mathbb{F})$ be a nearness approximation space. If A is a submodule of M such that $\underline{Apr_{B_r}}(A) \neq \emptyset$, for some $\mathcal{B}_r \subseteq \mathcal{B}$, then*

$$\underline{N_r(\mathcal{B})}(A) = A.$$

Proof. By Proposition 3.17 in [5], we have $\underline{Apr_{B_r}}(A) = A$. Since $A = \underline{Apr_{B_r}}(A) \subseteq \underline{N_r(\mathcal{B})}(A) \subseteq A$, it follows that $\underline{N_r(\mathcal{B})}(A) = A$. □

The following corollary follows immediately from Proposition 3.11.

Corollary 3.12. Let $(M, \mathbb{F})$ be a nearness approximation space. Let A be a submodule of M such that $\mathcal{B}_r \subseteq A$ for some $\mathcal{B}_r \subseteq \mathcal{B}$, then

$$\underline{N_r(\mathcal{B})}(A) = A.$$

Proof. Since $0 + \mathcal{B}_r \subseteq A$, it follows that $0 \in \underline{Apr_{B_r}}(A)$. By Proposition 3.11 it yields $\underline{N_r(\mathcal{B})}(A) = A$. □

Proposition 3.13. *Let $(M, \mathbb{F})$ be a nearness approximation space and I is an ideal of R. If A is an ideal of R and B is a submodule of M such that $I \subseteq A$, for some $\mathcal{B}_r \subseteq B$, then*

$$\underline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B) = \underline{N_r(I\mathcal{B})}(AB).$$

Proof. By Proposition 3.9 and Corollary 3.12, we have $\underline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B) = (AB)$. Since $I \subseteq A$ and $\mathcal{B}_r \subseteq \mathcal{B}$, it follows that $I\mathcal{B}_r \subseteq AB$, for some $\overline{I\mathcal{B}_r} \subseteq I\mathcal{B}$, and so $\underline{N_r(I\mathcal{B})}(AB) = AB$. Therefore, we have $\underline{Apr_I}(A)\underline{N_r(\mathcal{B})}(B) = \underline{N_r(I\mathcal{B})}(AB)$. □

Proposition 3.14. *Let $(M, \mathbb{F})$ be a nearness approximation space and let $\mathcal{B}^* = \{I_1, I_2, I_3, \cdots, I_n\}$, where $I_1, I_2, I_3, \cdots, I_n$ are ideals of R. If A is an ideal of R and B is a submodule of M such that $\mathcal{B}_r^* \subseteq A$, for some $\mathcal{B}_r^* \subseteq \mathcal{B}^*$, and $\mathcal{B}_r \subseteq B$, for some $\mathcal{B}_r \subseteq \mathcal{B}$, then*

$$\underline{N_r(\mathcal{B}^*)}(A)\underline{N_r(\mathcal{B})}(B) = \underline{N_r(\mathcal{B}^*\mathcal{B})}(AB).$$

Proof. By Corollary 3.10 and Corollary 3.12, we have $\underline{N_r(\mathcal{B}^*)}(A)\underline{N_r(\mathcal{B})}(B) = AB$. Since $\mathcal{B}_r^* \subseteq A$ and $\mathcal{B}_r \subseteq B$, it follows that $\mathcal{B}_r^*\mathcal{B}_r \subseteq AB$, for some $\overline{\mathcal{B}_r^*\mathcal{B}_r} \subseteq \mathcal{B}^*\mathcal{B}$. This yields that $\underline{N_r(\mathcal{B}^*\mathcal{B})}(AB) = AB$. Therefore, we conclude that $\underline{N_r(\mathcal{B}^*)}(A)\underline{N_r(\mathcal{B})}(B) = \underline{N_r(\mathcal{B}^*\mathcal{B})}(AB)$. □

By Corollary 3.12 and Proposition 3.2 (11), we have the following proposition.

Proposition 3.15. *Let $(M, \mathbb{F})$ be a nearness approximation space and Let A and B be two submodule of M such that $\mathcal{B}_r \subseteq A$ and $\mathcal{B}'_r \subseteq B$ for some $\mathcal{B}_r, \mathcal{B}'_r \subseteq \mathcal{B}$,*

$$\underline{N_r(\mathcal{B})}(A) \cup \underline{N_r(\mathcal{B})}(B) = \underline{N_r(\mathcal{B})}(A \cup B).$$

Proof. By Corollary 3.12, we have $\underline{N_r(\mathcal{B})}(A) \cup \underline{N_r(\mathcal{B})}(B) = A \cup B$. Since $\mathcal{B}_r \subseteq A$ it follows $\mathcal{B}_r \subseteq A \cup B$. Hence, $\overline{\underline{N_r(\mathcal{B})}}(A \cup B = A \cup B$. Therefore, we have $\underline{N_r(\mathcal{B})}(A) \cup \underline{N_r(\mathcal{B})}(B) = \underline{N_r(\mathcal{B})}(A \cup B)$. □

Definition 3.16. Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$. Let A be a non-empty subset of M. Then, the sets

$$\underline{N_r(\mathcal{B} + \mathcal{B}')}(A) = \{x \in M \ : \ x + (\mathcal{B}_r + \mathcal{B}'_r) \subseteq A\}$$

and

$$\overline{N_r(\mathcal{B} + \mathcal{B}')}(A) = \{x \in M \ : \ x + (\mathcal{B}_r + \mathcal{B}'_r) \cap A \neq \emptyset\}$$

are called near lower and near upper approximations of A with respect to the submodules $\mathcal{B} + \mathcal{B}'$ of M, respectively.

Proposition 3.17. *Let $(M, \mathbb{F})$ be a nearness approximation space. If A and B are two non-empty subsets of M, then*

(1) $\overline{N_r(\mathcal{B})}(A + B) \subseteq \overline{N_r(\mathcal{B})}(A) + \overline{N_r(\mathcal{B})}(B)$;

(2) $\overline{N_r(\mathcal{B})}(A) + \overline{N_r(\mathcal{B})}(B) \subseteq \overline{N_r(\mathcal{B}+\mathcal{B})}(A+B)$.

Proof. As submodules in a module M are normal subgroups in a group M so the proof can be obtained as Proposition 4 and Proposition 5 in [1]. □

The following corollary follows from Proposition 3.17.

Corollary 3.18. Let $(M, \mathbb{F})$ be a nearness approximation space. If A and B are two non-empty subsets of M such that $\overline{N_r(\mathcal{B})}(A+B) = \overline{N_r(\mathcal{B}+\mathcal{B})}(A+B)$, then

$$\overline{N_r(\mathcal{B})}(A) + \overline{N_r(\mathcal{B})}(B) = \overline{N_r(\mathcal{B})}(A+B).$$

Proposition 3.19. *Let $(M, \mathbb{F})$ be a nearness approximation space. If A and B are two non-empty subsets of M then $\underline{N_r(\mathcal{B})}(A) + \underline{N_r(\mathcal{B})}(B) \subseteq \underline{N_r(\mathcal{B}+\mathcal{B})}(A+B)$.*

Proof. As submodules of a module M are normal subgroups in a group M, so the proof can be obtained as Proposition 6 in [1]. □

Proposition 3.20. *Let $(M, \mathbb{F})$ be a nearness approximation space. If A and B are submodules of M such that $\mathcal{B}_r \subseteq A, B$ for some $\mathcal{B}_r \subseteq \mathcal{B}$, then*

$$\underline{N_r(\mathcal{B})}(A+B) = \underline{N_r(\mathcal{B})}(A) + \underline{N_r(\mathcal{B})}(B).$$

Proof. By Corollary 3.12 we have $\underline{N_r(\mathcal{B})}(A) + \underline{N_r(\mathcal{B})}(B) = A + B$. Since $\mathcal{B}_r \subseteq A, B$ and B is a submodule of M, it follows that $\mathcal{B}_r \subseteq A + 0 \subseteq A + B$. Therefore, we have $\underline{N_r(\mathcal{B})}(A+B) = A + B$. □

Proposition 3.21. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathbb{F}$ and let A be a non-empty subset of M. Then*

(1) $\overline{N_r(\mathcal{B})}(A) \subseteq \overline{N_r(\mathcal{B}')}(A)$,

(2) $\underline{N_r(\mathcal{B})}(A) \subseteq \underline{N_r(\mathcal{B}')}(A)$.

Proof. (1) Suppose that $x \in \overline{N_r(\mathcal{B})}(A)$. Then, we have $x + \mathcal{B}_r \cap A \neq \emptyset$ for some $\mathcal{B}_r \subseteq \mathcal{B} \subseteq \mathcal{B}'$ so that $x \in \overline{N_r(\mathcal{B}')}(A)$.

(2) Suppose that $x \in \underline{N_r(\mathcal{B})}(A)$. Then, we have $x + \mathcal{B}_r \subseteq A$ for some $\mathcal{B}_r \subseteq \mathcal{B} \subseteq \mathcal{B}'$. Hence, $x \in \underline{N_r(\mathcal{B}')}(A)$. □

Proposition 3.22. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathbb{F}$. If A is a submodule of M such that $\mathcal{B}_r \subseteq A$ for some $\mathcal{B}_r \subseteq \mathcal{B}$, then $\underline{N_r(\mathcal{B})}(A) = \underline{N_r(\mathcal{B}')}(A) = A$.*

Proof. By Corollary 3.12 and Proposition 3.21 we have $\underline{N_r(\mathcal{B})}(A) = A$. Then, we obtain $A = \underline{N_r(\mathcal{B})}(A) \subseteq \underline{N_r(\mathcal{B}')}(A) \subseteq A$. This yields that $\underline{N_r(\mathcal{B})}(A) = \underline{N_r(\mathcal{B}')}(A) = A$. □

EXAMPLE 5. Let $M = \mathbb{Z}_{20}$ and $R = \mathbb{Z}_{20}$ so M is a module over itself. Let $\mathcal{B} = \{S_1,\ S_2\}$ and $\mathcal{B}' = \{S_1,\ S_2,\ S_3\}$ be two families of submodules of M, where $S_1 = \langle 4 \rangle$, $S_2 = \langle 2 \rangle$, $S_3 = \langle 10 \rangle$. Let $A = \{0,\ 1,\ 4,\ 10\}$ be a subset of M. Then,

- $\underline{N_1(\mathcal{B})}(A) = \emptyset$, $\underline{N_1(\mathcal{B}')}(A) = \{0,\ 10\}$, $\overline{N_1(\mathcal{B})}(A) = \mathbb{Z}_{20}$, $\overline{N_1(\mathcal{B}')}(A) = \mathbb{Z}_{20}$.
- $S_1 \cap S_2 = S_1 = \langle 4 \rangle$, $S_1 \cap S_3 = \{0\}$, $S_2 \cap S_3 = \langle 10 \rangle$. So, we obtain
  $\underline{N_2(\mathcal{B})}(A) = \emptyset$, $\underline{N_2(\mathcal{B}')}(A) = A$,
  $\overline{N_2(\mathcal{B})}(A) = \{0,\ 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 14,\ 16,\ 17,\ 18\}$ and

$$\begin{aligned}\overline{N_2(\mathcal{B}')}(A) &= \overline{N_2(\mathcal{B})}(A) \cup \{11\} \\ &= \{0,\ 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 16,\ 17,\ 18\}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\underline{N_1(\mathcal{B})}(A) &\subset \underline{N_1(\mathcal{B}')}(A), \\ \underline{N_2(\mathcal{B})}(A) &\subset \underline{N_2(\mathcal{B}')}(A), \\ \overline{N_1(\mathcal{B})}(A) &= \overline{N_1(\mathcal{B}')}(A), \\ \overline{N_2(\mathcal{B})}(A) &\subset \overline{N_2(\mathcal{B}')}(A).\end{aligned}$$

Definition 3.23. Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$. Let A be a non-empty subset of M. Then, the sets

$$\underline{N_r(\mathcal{B} \cap \mathcal{B}')}(A) = \{x \in M \ :\ x + (\mathcal{B}_r \cap \mathcal{B}'_r) \subseteq A\}$$

and

$$\overline{N_r(\mathcal{B} \cap \mathcal{B}')}(A) = \{x \in M \ :\ x + (\mathcal{B}_r \cap \mathcal{B}'_r) \cap A \neq \emptyset\}$$

are called near lower and near upper approximations of set A with respect to the submodules $\mathcal{B} \cap \mathcal{B}'$ of M, respectively.

Proposition 3.24. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$. If A is a non-empty subset of M, then*

$$\underline{N_r(\mathcal{B})}(A) \cap \underline{N_r(\mathcal{B}')}(A) \subseteq \underline{N_r(\mathcal{B} \cap \mathcal{B}')}(A).$$

Proof. Suppose that $x \in \underline{N_r(\mathcal{B})}(A) \cap \underline{N_r(\mathcal{B}')}(A)$. Then $x \in \underline{N_r(\mathcal{B})}(A)$ and $x \in \underline{N_r(\mathcal{B}')}(A)$. Hence, $x + \mathcal{B}_r \subseteq A$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ and $x + \mathcal{B}'_r \subseteq A$ for some $\mathcal{B}'_r \subseteq \mathcal{B}'$. Therefore, we obtain $x + (\mathcal{B}_r \cap \mathcal{B}'_r) \subseteq A$ for some $\mathcal{B}'_r \subseteq \mathcal{B} \cap \mathcal{B}'$ which yields that $x \in \underline{N_r(\mathcal{B} \cap \mathcal{B}')}(A)$. □

Proposition 3.25. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$. If A is a submodule of M such that $\mathcal{B}_r, \mathcal{B}'_r \subseteq A$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ and $\mathcal{B}'_r \subseteq \mathcal{B}'$, then*

$$\underline{N_r(\mathcal{B})}(A) \cap \underline{N_r(\mathcal{B}')}(A) = \underline{N_r(\mathcal{B} \cap \mathcal{B}')}(A).$$

Proof. By Corollary 3.12, we have $\underline{N_r(\mathcal{B})}(A) \cap \underline{N_r(\mathcal{B}')}(A) = A \cap A = A$. Since $\mathcal{B}_r, \mathcal{B}'_r \subseteq A$, it follows that $\mathcal{B}_r \cap \mathcal{B}'_r \subseteq A$ for some $\mathcal{B}_r \cap \mathcal{B}'_r \subseteq \mathcal{B} \cap \mathcal{B}'$. Hence, $\underline{N_r(\mathcal{B} \cap \mathcal{B}')}(A) = A$. Consequently, we have $\underline{N_r(\mathcal{B})}(A) \cap \underline{N_r(\mathcal{B}')}(A) = \underline{N_r(\mathcal{B} \cap \mathcal{B}')}(A)$. □

Proposition 3.26. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$. If A is a non-empty set of M, then*

$$\overline{N_r(\mathcal{B} \cap \mathcal{B}')}(A) \subseteq \overline{N_r(\mathcal{B})}(A) \cap \overline{N_r(\mathcal{B}')}(A).$$

Proof. Suppose that $x \in \overline{N_r(\mathcal{B} \cap \mathcal{B}')}(A)$. Then, $x + (\mathcal{B}_r \cap \mathcal{B}'_r) \cap A \neq \emptyset$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ and $\mathcal{B}'_r \subseteq \mathcal{B}'$. Hence, $x + \mathcal{B}_r \cap A \neq \emptyset$ and $x + \mathcal{B}'_r \cap A \neq \emptyset$, which implies that $x \in \overline{N_r(\mathcal{B})}(A)$ and $x \in \overline{N_r(\mathcal{B}')}(A)$. Therefore, we have $\overline{N_r(\mathcal{B} \cap \mathcal{B}')}(A) \subseteq \overline{N_r(\mathcal{B})}(A) \cap \overline{N_r(\mathcal{B}')}(A)$. □

Proposition 3.27. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$. If A is a non-empty set of M such that $\overline{N_r(\mathcal{B})}(A) = A$ or $\overline{N_r(\mathcal{B}')}(A) = A$, then*

$$\overline{N_r(\mathcal{B} \cap \mathcal{B}')}(A) = \overline{N_r(\mathcal{B})}(A) \cap \overline{N_r(\mathcal{B}')}(A).$$

Proof. By Proposition 3.26 we have $\overline{N_r(\mathcal{B} \cap \mathcal{B}')}(A) \subseteq \overline{N_r(\mathcal{B})}(A) \cap \overline{N_r(\mathcal{B}')}(A)$.

Conversely, suppose that $\overline{N_r(\mathcal{B})}(A) = A$, then $\overline{N_r(\mathcal{B})}(A) \cap \overline{N_r(\mathcal{B}')}(A) = A \cap \overline{N_r(\mathcal{B}')}(A) = A \subseteq \overline{N_r(\mathcal{B} \cap \mathcal{B}')}(A)$. □

Proposition 3.28. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$ and let A be a non-empty subset of M, then*

(1) $\overline{N_r(\mathcal{B})}(A) \subseteq \overline{N_r(\mathcal{B} + \mathcal{B}')}(A)$;

(2) $\underline{N_r(\mathcal{B} + \mathcal{B}')}(A) \subseteq \underline{N_r(\mathcal{B})}(A)$.

Proof. (1) Suppose that $x \in \overline{N_r(\mathcal{B})}(A)$. Then, $x + \mathcal{B}_r \cap A \neq \emptyset$, for some $\mathcal{B}_r \subseteq \mathcal{B}$. Since $\mathcal{B}_r = \mathcal{B}_r + 0 \subseteq \mathcal{B}_r + \mathcal{B}'_r$, it follows that $x + \mathcal{B}_r + \mathcal{B}'_r \cap A \neq \emptyset$ for some $\mathcal{B}'_r \subseteq \mathcal{B}'$ which implies that $x \in \overline{N_r(\mathcal{B} + \mathcal{B}')}(A)$.

(2) Suppose that $x \in \underline{N_r(\mathcal{B} + \mathcal{B}')}(A)$. Then, $x + (\mathcal{B}_r + \mathcal{B}'_r) \subseteq A$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ and $\mathcal{B}'_r \subseteq \mathcal{B}'$. Since $x + \mathcal{B}_r = x + (\mathcal{B}_r + 0) \subseteq x + (\mathcal{B}_r + \mathcal{B}'_r) \subseteq A$, it follows that $x \in \underline{N_r(\mathcal{B})}(A)$. □

Proposition 3.29. *Let $(M, \mathbb{F})$ be a nearness approximation space and $\mathcal{B}, \mathcal{B}' \subseteq \mathbb{F}$. Let A be a submodule of M, then*

$$\overline{N_r(\mathcal{B})}(A) + \overline{N_r(\mathcal{B}')}(A) = \overline{N_r(\mathcal{B} + \mathcal{B}')}(A).$$

Proof. Suppose that $x \in \overline{N_r(\mathcal{B})}(A) + \overline{N_r(\mathcal{B}')}(A)$ then we can write $x = y + z$ for some $y \in \overline{N_r(\mathcal{B})}(A)$ and $z \in \overline{N_r(\mathcal{B}')}(A)$ so that there exist $a_1 \in y + \mathcal{B}_r \cap A$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ and $a_2 \in z + \mathcal{B}'_r \cap A$ for some $\mathcal{B}'_r \subseteq \mathcal{B}'$ which implies that $a_1 \in y + \mathcal{B}_r$, $a_2 \in z + \mathcal{B}'_r$ and $a_1, a_2 \in A$. Since A is a submodule of M, and $a_i, a_2 \in A$ it follows that $a_1 + a_2 \in A$. Thus, $a_1 + a_2 \in y + \mathcal{B}_r + z + \mathcal{B}'_r \cap A = y + z + \mathcal{B}_r + \mathcal{B}'_r \cap A = x + \mathcal{B}_r + \mathcal{B}'_r \cap A$. Therefore $x \in \overline{N_r(\mathcal{B} + \mathcal{B}')}(A)$.

Conversely, suppose that $x \in \overline{N_r(\mathcal{B} + \mathcal{B}')}(A)$, then $x + (\mathcal{B}_r + \mathcal{B}'_r) \cap A \neq \emptyset$ for some $\mathcal{B}_r \subseteq \mathcal{B}$ and $\mathcal{B}'_r \subseteq \mathcal{B}'$. So there exist $y \in \mathcal{B}_r$ such that $x + y + \mathcal{B}'_r \cap A \neq \emptyset$ which yields $x + y \in \overline{N_r(\mathcal{B}')}(A)$. On the other hand, since $-y \in \mathcal{B}_r$ and A is a submodule of M it follows that $(-y + \mathcal{B}_r) \cap A = \mathcal{B}_r \cap A \neq \emptyset$ which yields $-y \in \overline{N_r(\mathcal{B})}(A)$. Now, we have $x = -y + x + y \in \overline{N_r(\mathcal{B})}(A) + \overline{N_r(\mathcal{B}')}(A)$. □

Proposition 3.30. *Let $(M,\mathbb{F})$ be a nearness approximation space and $\mathcal{B},\mathcal{B}'\subseteq\mathbb{F}$. If A is a submodule of M such that $\mathcal{B}_r,\mathcal{B}'_r\subseteq A$ for some $\mathcal{B}_r\subseteq\mathcal{B}$ and $\mathcal{B}'_r\subseteq\mathcal{B}'$, then*

$$\underline{N_r(\mathcal{B})}(A)+\underline{N_r(\mathcal{B}')}(A)=\underline{N_r(\mathcal{B}+\mathcal{B}')}(A).$$

Proof. By Corollary 3.12, we have $\underline{N_r(\mathcal{B})}(A)+\underline{N_r(\mathcal{B}')}(A)=A+A=A$. Since $\mathcal{B}_r,\mathcal{B}'_r\subseteq A$ and A is a submodule of M, it follows $\mathcal{B}_r+\mathcal{B}'_r\subseteq A$ for some $\mathcal{B}_r+\mathcal{B}'_r\subseteq\mathcal{B}+\mathcal{B}'$. This implies that $\underline{N_r(\mathcal{B}+\mathcal{B}')}(A)=A$. Therefore, we conclude that $\underline{N_r(\mathcal{B})}(A)+\underline{N_r(\mathcal{B}')}(A)=\underline{N_r(\mathcal{B}+\mathcal{B}')}(A)$. □

3.4. Upper and lower near submodules

Let $(M,\mathbb{F})$ be a nearness approximation space. Let A be a non-empty subset of M, then A is called an upper near submodule of M if $\overline{N_r(\mathcal{B})}(A)$ is a submodule of M. Similarly, A is called a lower near submodule of M if $\underline{N_r(\mathcal{B})}(A)$ is a submodule of M.

Proposition 3.31. *Let $(M,\mathbb{F})$ be a nearness approximation space and A be a submodule of M such that $\overline{N_r(\mathcal{B})}(A)=\overline{N_r(\mathcal{B}+\mathcal{B})}(A)$. Then, A is an upper near submodule of M.*

Proof. By Proposition 8 in [1], A is an upper near subgroup of M. Now, assume that $r\in R$ and $x\in\overline{N_r(\mathcal{B})}(A)$, then $x+\mathcal{B}_r\cap A\neq\emptyset$ for some $\mathcal{B}_r\subseteq\mathcal{B}$ so there exist $y\in x+\mathcal{B}_r\cap A$. Thus $y\in x+\mathcal{B}_r$ and $y\in A$. Since A is a submodule of M it follows $ry\in A$ so that $ry\in r(x+\mathcal{B}_r)\cap A=rx+r\mathcal{B}_r\cap A\subseteq rx+\mathcal{B}_r\cap A$. Therefore, $rx\in\overline{N_r(\mathcal{B})}(A)$. □

Proposition 3.32. *Let $(M,\mathbb{F})$ be a nearness approximation space and A be a submodule of M such that $\mathcal{B}_r\subseteq A$ for some $\mathcal{B}_r\subseteq\mathcal{B}$. Then A is a lower near submodule of M.*

Proof. By Corollary 3.12, we have $\underline{N_r(\mathcal{B})}(A)=A$ then $\underline{N_r(\mathcal{B})}(A)$ is a submodule of M which mean A is a lower near submodule of M. □

Let X be a non-empty set of M. We mean the set $-X=\{-x\ :\ x\in X\}$.

Proposition 3.33. *Let $(M,\mathbb{F})$ be a nearness approximation space. If A is a non-empty subset of M, then*

(1) $\underline{N_r(\mathcal{B})}(-A)=-\underline{N_r(\mathcal{B})}(A)$;

(2) $\overline{N_r(\mathcal{B})}(-A)=-\overline{N_r(\mathcal{B})}(A)$.

Proof. (1) We can write

$$\begin{aligned}x\in\underline{N_r(\mathcal{B})}(-A)\ &\Leftrightarrow x+\mathcal{B}_r\subseteq -A,\ \text{for some } \mathcal{B}_r\subseteq\mathcal{B}\\ &\Leftrightarrow -x-\mathcal{B}_r\subseteq A\Leftrightarrow -x+\mathcal{B}_r\subseteq A\\ &\Leftrightarrow -x\in\underline{N_r(\mathcal{B})}(A)\Leftrightarrow x\in-\underline{N_r(\mathcal{B})}(A).\end{aligned}$$

(2) We have

$$\begin{aligned} x \in \overline{N_r(\mathcal{B})}(-A) &\Leftrightarrow x + \mathcal{B}_r \cap -A \neq \emptyset \text{ for some } \mathcal{B}_r \subseteq \mathcal{B} \\ &\Leftrightarrow -x - \mathcal{B}_r \cap A \neq \emptyset \Leftrightarrow -x + \mathcal{B}_r \cap A \neq \emptyset \\ &\Leftrightarrow -x \in \overline{N_r(\mathcal{B})}(A) \Leftrightarrow x \in -\overline{N_r(\mathcal{B})}(A). \end{aligned}$$

□

Proposition 3.34. *Let $(M, \mathbb{F})$ be a nearness approximation space. Let A be a non-empty subset of M, then*

$$\underline{N_i(\mathcal{B})}(A) \subseteq \underline{N_j(\mathcal{B})}(A) \subseteq A \subseteq \overline{N_j(\mathcal{B})}(A) \subseteq \overline{N_i(\mathcal{B})}(A) \text{ where } 1 \leq i \leq j \leq r = |\mathcal{B}|.$$

Proof. The proof can be obtained as Proposition 17 in [1]. □

Example 6. If we consider Example 5, we have

$$\underline{N_1(\mathcal{B})}(A) = \emptyset,$$

$$\underline{N_2(\mathcal{B})}(A) = \emptyset,$$

$$\overline{N_1(\mathcal{B})}(A) = \mathbb{Z}_{20},$$

$$\overline{N_2(\mathcal{B})}(A) = \{0,\ 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 14,\ 16,\ 17,\ 18\},$$

$$\underline{N_1(\mathcal{B}')}(A) = \{0,\ 10\},$$

$$\underline{N_2(\mathcal{B}')}(A) = A, \overline{N_1(\mathcal{B}')}(A) = \mathbb{Z}_{20},$$

and

$$\overline{N_2(\mathcal{B}')}(A) = \overline{N_2(\mathcal{B})}(A) \cup \{11\} = \{\{0,\ 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11, 12,\ 13,\ 14,\ 16,\ 17,\ 18\}\},$$

Then, we have

- $\underline{N_1(\mathcal{B})}(A) = \underline{N_2(\mathcal{B})}(A) \subset A \subset \overline{N_2(\mathcal{B})}(A) \subset \overline{N_1(\mathcal{B})}(A)$,
- $\underline{N_1(\mathcal{B}')}(A) \subseteq \underline{N_2(\mathcal{B}')}(A) = A \subset \overline{N_2(\mathcal{B}')}(A) \subset \overline{N_1(\mathcal{B}')}(A)$

The following corollary follows directly from Proposition 3.34.

Corollary 3.35. Let $(M, \mathbb{F})$ be a nearness approximation space and A be a submodule of M such that $\mathcal{B}_i \subseteq A$ for some $\mathcal{B}_i \subseteq \mathcal{B}$. Then $\underline{N_i(\mathcal{B})}(A) = \underline{N_j(\mathcal{B})}(A) = A$, where $1 \leq i \leq j \leq r = |\mathcal{B}|$.

Proof. By Corollary 3.12, $\underline{N_i(\mathcal{B})}(A) = A$. By Proposition 3.34, $A = \underline{N_i(\mathcal{B})}(A) \subseteq \underline{N_j(\mathcal{B})}(A) \subseteq A$, where $1 \leq \overline{i \leq j} \leq r = |\mathcal{B}|$, then $\underline{N_j(\mathcal{B})}(A) = A$, where $\overline{1 \leq i} \leq j \leq r = |\mathcal{B}|$. Therefore, we have $\underline{N_i(\mathcal{B})}(A) = \underline{N_j(\mathcal{B})}\overline{(A)} = A$, where $1 \leq i \leq j \leq r = |\mathcal{B}|$. □

EXAMPLE 7. Let $M = \mathbb{Z}_{30}$ and $R = \mathbb{Z}_{30}$. Then, M is a module over itself. The set $\mathcal{B} = \{S_1,\ S_2,\ S_3\}$ is a family of submodules of M, where $S_1 = \langle 2\rangle$, $S_2 = \langle 3\rangle$, $S_3 = \langle 5\rangle$. Let $A = \langle 6\rangle$ be a submodule of M. Then,

- $\underline{N_1(\mathcal{B})}(A) = \emptyset$,
- $S_1 \cap S_2 = \langle 6\rangle, S_1 \cap S_3 = \langle 10\rangle$, and $S_2 \cap S_3 = \langle 15\rangle$. Since $\underline{Apr_{S_1\cap S_2}}(A) = A$, it follows that $A = \underline{Apr_{S_1\cap S_2}}(A) \subseteq \underline{N_2(\mathcal{B})}(A) \subseteq A$ which yields $\underline{N_2(\mathcal{B})}(A) = A$.
- $S_1 \cap S_2 \cap S_3 = \langle 0\rangle$, therefore $\underline{N_3(\mathcal{B})}(A) = A$.

According to Corollary 3.35, since $S_1 \cap S_2 \subseteq A$, it follows that $\underline{N_2(\mathcal{B})}(A) = A$ which yields that $\underline{N_2(\mathcal{B})}(A) = \underline{N_3(\mathcal{B})}(A) = A$.

4. Conclusion

In applied mathematics and real life we encounter many examples of mathematical objects that can be added to each other and multiplied by scalar numbers. Based on this, we introduced an approach of module theory considered in the context of near set theory. We demonstrated the concepts of near approximations in a module are generalizations of the concepts of rough approximations in a module [5]. We investigated some related properties about submodules. The suggested methods of near approximations open a way for constructing new types of lower and upper approximations.

An information system is a set of interrelated components that collect or retrieve, process, store, and distribute information to support decision making and control in an organization. For future work, it will be interesting to introduce near approximations in information systems.

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