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SOME IMPLICATIONS OF THE MULTIFRACTAL INFORMATIONAL ENTROPY

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Abstract. A new formulation of the informational entropy on multifractal manifolds is proposed. In such a context, some connections with known results from the dynamics of complex systems are presented: correlations of SL(2ℝ) groups with Hamiltonian formalism, simultaneous invariances between two groups (coordinate group and parametric group), linear harmonic oscillator/oscillator ensemble dynamics from the perspective of SL(2ℝ) groups, uncertainty relations on multifractal manifolds etc. Moreover, the existence of a multifractal perspective in the complex dynamics analysis allows direct correspondences with deep learning algorithms.

Keywords: SL(2ℝ) group, Hamiltonian formalism, harmonic oscillator, uncertainty relation, multifractal manifolds.

1. Introduction

It is well-known the role of informational entropy in the study of complex system dynamics. Examples of such systems include: biological, engineering, economic, social, medical, environmental etc. In such a context, various definitions of the informational entropy were given both structurally and functionally (ex. Shannon entropy, Fisher entropy, Fractal entropy etc.) [1,2]. In the present paper, a new definition of informational entropy is given and some of its implications are presented.

2. Multifractal informational entropy

Let the following definition of the informational entropy on multifractal manifolds (multifractal informational entropy) be:

$$H(\rho,\mu)=\int \rho(q,p,\mu)\ln\left[\rho(q,p,\mu)\right]dqdp, \tag{1}$$

where $\rho(q,p,\mu)$ is the multifractal probability density, $q(\mu)$ is the multifractal positional coordinate, $p(\mu)$ is the multifractal impulse coordinate and μ is the resolution scale,

$$\mu=\lambda(dt)^{\left(\frac{2}{f(\alpha)}\right)-1}. \tag{2}$$

In relation (2), λ is the coefficient associated to the multifractal-non-multifractal transition, dt este scara de rezoluție is the resolution scale, $f(\alpha)$ is the singularity spectrum of order $\alpha=\alpha(D_F)$ and D_F is the fractal dimension of the motion curves, through which it is possible to describe the dynamics of any physical system.

3. Variational principle

It is possible to observe that the $H(\rho,\mu)$ facilitates the use, in equal measure, of the specific given by both the theory of non-additive sets as well as the theory of set multifunctions, the multifractal character being dictated by (2). More precisely $H(\rho,\mu)$ simultaneously operates on two multifractal manifolds, one being the position-impulse coordinates and the other being the resolution scales. In such a context, μ corresponds to the set of resolution scales.

Now, if the multifractal informational entropy (1) is subjected to the following set of multifractal restraints:

$$\int \rho\, \rho(q,p,\mu)dqdp = q_0(\mu) \tag{3}$$

$$\int p\, \rho(q,p,\mu)dqdp = p_0(\mu) \tag{4}$$

$$\int (q-q_0)^2\, \rho(q,p,\mu)dqdp = [\delta q(\mu)]^2 \tag{5}$$

$$\int (p-p_0)^2\, \rho(q,p,\mu)dqdp = [\delta p(\mu)]^2 \tag{6}$$

$$\int (q-q_0)(p-p_0)\, \rho(q,p,\mu)dqdp = cov(q,p,\mu), \tag{7}$$

its maximization through the admittance of the functionality of the following multifractal variational principle:

$$\delta H(q,\mu) = 0 \tag{8}$$

with the constraints (3)-(7), has, as a result, the normal multifractal distribution

$$\rho(x,y,\mu) = \frac{\sqrt{ac-b^2}}{2\pi}\exp\left[-\frac{1}{2}H(x,y,\mu)\right], \tag{9}$$

where the variables x, y are of the form

$$x = (q-q_0), y = (p-p_0), \tag{10}$$

and $H(x,y,\mu)$ is the multifractal quadratic form

$$H(x,y,\mu) = ax^2 + 2by^2 + cy^2 \tag{11}$$

of multifractal coefficients a, b, c, correlated with the multifractal constraints (3)-(7) through the relations:

$$a(\mu) = \frac{[\delta q(\mu)]^2}{D(\mu)}, b = -\frac{cov(q,p,\mu)}{D(\mu)}, c = \frac{[\delta p(\mu)]^2}{D(\mu)}, \tag{12}$$

where the following notations were used:

$$D(\mu) = [\delta q(\mu)]^2[\delta p(\mu)]^2 - cov^2(q,p,\mu). \tag{13}$$

In relations (3)-(7), we use the average value of the multifractal position coordinate, $q_0(\mu)$ $p_0(\mu)$ being the average value of the multifractal impulse coordinate, $[\delta q(\mu)]^2$ is the standard deviation of the multifractal position coordinate, $[\delta p(\mu)]^2$ is the standard deviation of the multifractal impulse coordinate and $cov(q,p,\mu)$ is the multifractal covariance of the two multifractal variables.

4. Correspondences of the Lie groups SL(2ℝ) with Hamiltonian formalism

The eq. (11) is invariant with respect to the Lie group SL(2ℝ) [3]. It is a homogenous transformation group of the affine plane; it invariates the areas that are represented through unimodular real matrices

$$x' = \alpha x + \beta y, y' = \gamma x + \delta y, \alpha\delta - \beta\gamma = 1. \tag{14}$$

This group contains two variables with three parameters and only three of $\alpha, \beta, \gamma, \delta$, are independent given the fact that the determinant of the matrix is 1. The following parameterization is adopted for infinitesimal transformations corresponding to the relation [3]:

$$y^i = x^i + \xi_k^i(x)a^k, \tag{15}$$

$$\alpha = 1 + \frac{1}{2}a^2 = a^1, \beta = a^1, \gamma = -a^3, \delta = 1 - \frac{1}{2}a^2, \tag{16}$$

case in which (14) become:

$$x' = x + ya^1 + \frac{x}{2}a^2, y' = y - \frac{y}{2}a^2 - xa^3. \tag{17}$$

It is possible to notice that the action of the group is transitive: there exists at least a group transformation which links two coordinate points (x, y) and respectively, (x', y'). This results from the fact that the equations considered in the unknowns a^1, a^2, a^3 form a compatible system. Now, it is possible to directly read on (17) the matrix (ξ) which characterizes the infinitesimal transformations, necessary in Deltheil equations [3].

$$\xi_1^1 = y, \xi_2^1 = \frac{x}{2}, \xi_3^1 = 0, \xi_1^2 = 0, \xi_2^2 = -\frac{y}{2}, \xi_3^2 = -x. \tag{18}$$

Before writing the Deltheil equations, it is possible to observe that eq. (18) is actually the base of Lie algebra associated with SL(2ℝ), group, it being specified by the infinitesimal generators

$$X_k = \xi_k^i \frac{\partial}{\partial x^i}, \tag{19}$$

meaning

$$X_1 = y\frac{\partial}{\partial x}, X_2 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right), X_3 = -x\frac{\partial}{\partial y}. \tag{20}$$

Using the associated Poisson brackets (or operator switches) of these inifinitesimal operators, we can obtain a group structure:

$$[X_1, X_2] = X_1, [X_2, X_3] = X_3, [X_3, X_1] = -2X_2, \tag{21}$$

where the switches of the respective operators can be found in square brackets. It is possible to reveal the structural constants of the algebra found in the SL(2ℝ)group:

$$C_{12}^1 = C_{23}^3 = 1, C_{31}^2 = -2, \tag{22}$$

all the others terms being zero.

Taking into account (18), the Deltheil equations characteristic of the group are:

$$y \cdot \frac{\partial F}{\partial x} = 0, \frac{\partial}{\partial x}(xF) - \frac{\partial}{\partial y}(yF) = 0, x \cdot \frac{\partial F}{\partial F} = 0 \tag{23}$$

and they adopt the solution $F =$ const. In particular, it is possible to choose $F = 1$. Consequently, the SL(2ℝ) being a measurable group, it has the elementary measure

$$d\mu(x, y) = dx \Lambda dy, \tag{24}$$

In eq. (24), "Λ" is the external product of the differential form. If we take into account the observations of Jaynes [4], and admit the existence of some unspecified circumstances that admit the respective invariance group, then the a priori situations, equally probable, accept the uniform distribution of the elementary measure given by eq. (24).

If we consider the canonical formalism, we can directly obtain such an unspecified circumstance. Indeed, eq. (24) shows that this is a symplectic group, equally, since the group SL(2ℝ) invariates the form 2.

Using eq. (20), the corresponding Hamiltonian dynamic can be generated, in the "tangent" space through the vectors X_1, X_2, X_3, which satisfy the commutation relations (21).

We can define a general "tangent" vector as a linear combination of the form:

$$T = cX_1 + 2b + aX_3 \tag{25}$$

The problem arises of finding the invariant functions along the tangent trajectories of the respective vector, i.e. to find the solutions of the equation:

$$\hat{T}H(x, y) = 0 \tag{26}$$

Equation (26) can be explicitly written in the form of eq. (27) if we taking into account the eq. (20):

$$(bx + cy)\frac{\partial H}{\partial x} - (ax + by)\frac{\partial H}{\partial y} = 0. \tag{27}$$

The characteristic differential system of the equation is:

$$\frac{dx}{bx+cy}=-\frac{dy}{ax+by} \tag{28}$$

therefore

$$H(x,y)=\frac{1}{2}(ax^2+2bxy+cy^2). \tag{29}$$

Consequently, the solution of eq. (27) is a random function of the mentioned format playing a special role in the field of application of the previously discussed theory, i.e. of a Hamiltonian, which is considered a motion generator. The differential system, (28), is the Hamilton equation system associated to eq. (29), being written in the form:

$$\frac{dx}{dt}=\frac{\partial H}{\partial y},\frac{dy}{dt}=-\frac{\partial H}{\partial x} \tag{30}$$

where x is a coordinate, and y is the impulse of the considered physical system. Here, dt means the time derivative of the two differentials in eq. (28), which is the differential function of the parameter on the whole curves of the vector given by eq. (25).

In principle, amongst the solutions of eq. (26) the Gaussian repartition density can be found, expressed in the form:

$$\Gamma(x,y)=\exp\{-H\}, \tag{31}$$

in which case the parameters a,b,c can acquire statistical significance, as previously shown. This fact mathematically legitimates the idea from statistics that the probability density should be a motion integer.

From a dynamic point of view, the Hamiltonian given by eq. (29) characterizes a well-established system, the parameters a,b,c are fixed and eq. (26) specifies a certain class of trajectories belonging to the SL(2$\mathbb{R}$) group, specified by the values of the specified parameters previous. It is not complicated to obtain a class of trajectories using the system of eqs. (28).

Assuming that the parameters a and c are positive and the quadratic form given by eq. (29) is positive, eq. (28) gives us the solution:

$$x'=\left(\cos wt+\frac{b}{w}\sin wt\right)x+\frac{a}{w}(\sin wt)y, y'=-\frac{c}{w}(\sin wt)x+(\cos wt-\frac{b}{w}\sin wt)\,y, \tag{32}$$

where $w=\sqrt{ac-b^2}$. In other working hypothesis regarding the quadratic form (29) another evolution matrix can be obtained, but it will still be unimodular.

According to the principle of maximum multifractal informational entropy, things get even more complicated if eq. (29) is obtained by statistical inference. Indeed, as previously shown, two

types of hypotheses come into play: some acting upon the averages, others upon the statistical variances; from a mathematical point they are intransitive.

But, the variety of hypotheses is not linear, as in the case when averages are specified. An important case is, for example, the case of Gaussians that refer to the same average.

In this situation, x and y represent the deviations of these quantities from the average, i.e. (10).

By a suitable particular choice of the coefficients (a, b, c) from the quadratic form (29) any statistical hypothesis can be specified. All these statistical hypotheses are characterized by the invariance of the mentioned quadratic form, because, in this case, the Gaussian distribution density (non-normed) is written in the form:

$$\Gamma'(x', y') = -\exp\{-H'(x', y')\}, \tag{33}$$

where

$$H'(x', y') = \frac{1}{2}\left(a'x'^2 + 2b'x'y' + c'y'^2\right), \tag{34}$$

with the condition

$$H'(x', y') = H(x, y). \tag{35}$$

However, if x', y' are linked to x, y through a unimodular transformation of type (14), it can obviously be accepted, because the inference is done in constant averages, therefore the condition (35) imposes for a, b, c the following group with three parameters:

$$a' = \delta^2 a - 2\gamma\delta b + \gamma^2 c,\ b' = -\beta\delta a - (\beta\gamma + \alpha\delta)b - \alpha\gamma c,\ c' = \beta^2 a - 2\alpha\beta b + \alpha^2 c. \tag{36}$$

This group is of course isomorph with respect to (14), but it has the property that its action is intrasitive on the space of (a, b, c) variables. In order to show the meaning of the previous statement, for the group (14) the parametrization (16) will be adopted, so, the infinitesimal transformations corresponding are:

$$a' = a - aa^2 + 2ba^3, b' = b - aa^1 + ca^3, c' = c - 2ba^1 + ca^2. \tag{37}$$

Considered as a linear system with the unknowns a^1, a^2, a^3, this system is incompatible. Therefore, no transformation exists which can link (a, b, c) and (a', b', c'), which means that the group of these parameters is intransitive.

This means that the group exists only under the condition of a certain link between the parameters a, b, c, which should be invariantly kept, i.e., on the so-called group transitivity manifolds. We can find invariant functions, knowing that the function must fulfill a necessary and sufficient

condition for the group variance, that is, the respective function must satisfy the system of equations with partial derivatives [3]:

$$\xi_h^i \frac{\partial F}{\partial x^i} = 0. \tag{38}$$

This can easily be deducted from the generated constraint [3]:

$$F(y^1, \dots, y^n) = F(x^1, \dots, x^n) + \frac{\partial F}{\partial x^i} \xi_k^i(x) a^k, \tag{39}$$

with the invariance condition $F(x) = F(y)$, in the hypothesis of the arbitrariness of the a^k parameter values.

In the current case, it can be seen in (37) that the matrix (ξ) is given by the relations:

$$\xi_1^1 = 0, \xi_2^1 = -a, \xi_3^1 = 2b, \xi_1^2 = -a, \xi_2^2 = 0, \xi_3^2 = c, \xi_1^3 = -2b, \xi_2^3 = c, \xi_3^3 = 0, \tag{40}$$

situation in which (38) becomes:

$$-a\frac{\partial F}{\partial b} - 2b\frac{\partial F}{\partial c} = 0, -a\frac{\partial F}{\partial a} + c\frac{\partial F}{\partial c} = 0, 2b\frac{\partial F}{\partial b} + c\frac{\partial F}{\partial b} = 0. \tag{41}$$

The solution of the system (41) is in the form of an arbitrary function:

$$\Delta = ac - b^2, \tag{42}$$

Equation (43) gives us the transitivity varieties of the group (36)

$$ac - b^2 = const. \tag{43}$$

Thus, the property that the discriminant of quadratic forms (29) is constant is specific to the class of statistical hypotheses specific to Gaussians of the same average. The interpretation of this fact is discussed in [5, 6].

According to the principle of maximum multifractal information entropy, another connection between the theory of continuous, measurable groups and Gaussian construction by statistical inference should be noted. The infinitesimal generators of the action for group (36) can be found using eq. (40). These are:

$$A_1 = -a\frac{\partial}{\partial b} - 2b\frac{\partial}{\partial a}, A_2 = -a\frac{\partial}{\partial a} + c\frac{\partial}{\partial c}, A_3 = 2b\frac{\partial}{\partial a} + c\frac{\partial}{\partial b}. \tag{44}$$

The commutation eqs. (21) are satisfied by these operators, which is obvious that the two algebras are realizations of the same algebra, that is, of the SL(2ℝ) group, having the structural constants given by eq. (22). These two algebras dictate the properties of the Lie algebra, its emergence in one, two or three dimensions being purely accidental.

Taking into account (43), the group is based on two variables, i.e. *a*, *b*, *c* are dependent on

two parameters. Their functional form is given in [5, 6] regarding a certain interpretation of the situation, from a physical point of view. Now, the link previously made will be highlighted: the group generated by eq. (44) being an associated group, we can say that Gaussians can be considered families of invariant manifolds with three-parameters of the SL(2ℝ) group [7].

5. Simultaneously invariant functions to the action of the groups. The linear harmonic oscillator case

For now, the point of interest is the fact that the group (36) was made through induction as a group of the parameters a, b, c induced by group (14), with the invariance condition of the functions $H(x, y)$ given by (35). Reciprocally, the groups (14) and (36) being given for the variables (x, y) and the parameters (a, b, c), the functions $H(x, y)$ are the families of invariant manifolds with three parameters. Thus, by generalization, important physical applications result. These applications give some physical meaning to the parameters a, b, c, as well as some interesting connections between certain classical cases and their quantum correspondences.

The general theory of the parametric families of invariant manifolds allows the obtainment of these families as solutions of Stoka equations [7]:

$$X_i f(x, a) + A_i f(x, a) = 0. \tag{45}$$

Here, X_i are infinitesimal generators of the variable groups and A_i are generators of the parameter group. It is necessary for the two groups to be isomorph, i.e. their Lie algebrae belonging to one and the same algebra, for the present case being (20) and (44). In this case, taking into account (20) and (44) for X_i and A_i, and detailing the Stoka equations (45), it is indeed possible to obtain families of invariant manifolds with three parameters – the previous $H(x, y)$ function, but only with the transitivity condition (43).

It is important to take into account that the Stoka equations offer the possibility to construct invariant quantities *a priori* through certain groups and depending on certain parameters, with the condition that the variable group and parameter group have to be isomorph. More generally, given two isomorph groups, it is possible to identify the families of common invariant manifolds, without labeling *a priori*, any one of the two groups as a parameter group.

Starting from the equation of motion, it can be presented as an example for a given parameterization – an ensemble of harmonic oscillators. In the unidimensional case, we know the equation of the harmonic oscillator:

$$\ddot{q} + w^2 q = 0, \tag{46}$$

where q is the coordinate. The solution of eq. (46) can be written in the form:

$$q(t) = ze^{i(wt+\phi)} + \bar{z}e^{-i(wt+\phi)}, \tag{47}$$

where z is a complex amplitude, $\bar{z}$ is the complex conjugate , ϕ is a specific phase and t is a time parameter. Thus, z and ϕ are specific to each oscillator in an interval, possibly in an ensemble, which has the differential equation of motion (46) and, as a consequence, the same frequency w.

In the case of an assembly of "Planck oscillators" interacting with electromagnetic radiation, the analysis led to the law of energy density distribution as a function of frequencies, given by Planck in 1900. This esemble of harmonic oscillators can be described by a continuous group in three variables with three parameters. Thus, it is observed that the ratio of the two fundamental solutions of eq. (46), ratio denoted by *k*,

$$k = 2e^{i(wt+\phi)}, \tag{48}$$

is a solution of Schwartz equation [8]

$$\{k, t\} = 2w^2, \tag{49}$$

where it was denoted by

$$\{k, t\} = (\frac{k''}{k'})' - \frac{1}{2}(\frac{k''}{k'})^2, \tag{50}$$

the Schwartzian of function k with respect to variable t, the accent signifying the derivation with respect to the variable.

Equation (49) has the property of being invariant with respect to the homographic transformation of the function, since the function:

$$\sigma(t) = \frac{\alpha k + \beta}{\gamma k + \delta} \tag{51}$$

satisfies the same equations

$$\{\sigma, t\} = 2w^2. \tag{52}$$

The transformations (51) form a continuous group that contains three real parameters on the complex straight line and the parameter k has a projective character. Thus, for each oscillator, a specific projective parameter defined by the equation is introduced:

$$\sigma(t) = \frac{z+\bar{z}k}{1+k},$$

which satisfies the eq. (52). There must be a real homographic relationship between the parameters of the different oscillators in the ensemble:

$$\sigma_1(t) = \frac{\alpha\sigma(t) + \beta}{\gamma\sigma(t) + \delta}. \tag{53}$$

It can be said that the group (53) can be considered a "synchronization" group between various oscillators, the amplitudes of each oscillator participating in the "synchronization" process, in the sense that the amplitudes are correlated (not only their phases).

The synchronization manifested through the delay of the oscillator phases on the ensemble, must be only a totally particular case. Indeed, group (53) implies for $z, \bar{z}, k$ the following parameter group:

$$z_1 = \frac{\alpha z + \beta}{\gamma z + \delta}, k_1 = \frac{\gamma\bar{z} + \delta}{\gamma z + \delta}k \tag{54}$$

Equation (54) shows that the phase of k is only shifted with a quantity depending on the amplitude of the oscillator, at the passing between different members of the ensemble. Furthermore, the amplitude of the oscillator is homographically affected.

Now, considering for the group (54) the parametrization (16), the following infinitesimal generators of the group (54) are obtained:

$$B_1 = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, B_2 = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}, B_1 = z^2\frac{\partial}{\partial z} + \bar{z}^2\frac{\partial}{\partial \bar{z}} + (z - \bar{z})k\frac{\partial}{\partial k}, \tag{55}$$

with the commutation relations

$$[B_1, B_2] = B_1, [B_2, B_3] = B_3, [B_3, B_1] = -2B_2, \tag{56}$$

which reveal the same structure as the one of Lie algebra of group SL(2$\mathbb{R}$). Thus, the Lie algebra of the group (54) is actually a Lie algebra of the SL(2$\mathbb{R}$) group. In fact, the group (54) only represents another action of the group SL(2$\mathbb{R}$), realized in three variables $z, \bar{z}, k$.

Therefore, the conditions of Stoka Theorem are met [7] and it is possible to identify the simultaneously invariant functions to the action of the groups (20) and (55), as solutions to Stoka equations:

$$X_i f(x, y, z, \bar{z}, k) + B_i f(x, y, z, \bar{z}, k) = 0, i = 1,2,3. \tag{57}$$

By explaining (57) with the help of (20) and (55) it is possible to easily obtain, through successive reductions, the simultaneously invariant functions with the form:

$$f(\mu,\nu)=const., \tag{58}$$

where μ and ν have the expressions:

$$\mu=-\frac{i(z-\bar{z})}{(x-zy)(x-\bar{z}y)},\nu=k\frac{x-\bar{z}y}{x-zy}, \tag{59}$$

where ν is a complex unimodular and μ is real parameter.

A certain type of such invariant functions can be written as linear combinations:

$$p\mu=m\left(\nu+\frac{1}{\nu}\right)+2n, \tag{60}$$

where m,n,p are three real and arbitrary constants.

Taking into account (59), eq. (60) is transformed in:

$$mk^{-1}Z^2+2nZ\bar{Z}+mk\bar{Z}^2=p, \tag{61}$$

where

$$Z=\frac{x-\bar{z}y}{\sqrt{-i(z-\bar{z})}}, \tag{62}$$

admitting that

$$-i(z-\bar{z})>0. \tag{63}$$

Equation (61) represents a family of conics. If the condition below is met, they become ellipses:

$$m^2-n^2<0, \tag{64}$$

Condition (64) always is satisfied if

$$m=Qsh2r,n=Qch2r, \tag{65}$$

where Q is a real constant.

A very important particular case should be taken into account, i.e. when z is purely imaginary, retaining, without restricting the generality, the value $z=i$.

Comparing the quadratic form (61) with H(x,y) from (29), we obtain the following values for *a*, *b*, *c*:

$$a=Q(ch2r+sh2r\cos\phi),b=-Qsh2r\sin\phi,c=Q(ch2r-sh2r\cos\phi), \tag{66}$$

where ϕ is a fixed value of k.

Starting from the standard quadratic form of the coefficients *a*, *b*, *c*, these relations show us that these coefficients are located on the trajectories of the group (36) and are obtained by finite

transformations generated by the infinitesimal transformations of the group (44). So, the constant Δ in (42) characterizing the transitivity varieties of the group (31) is equal to the square of Q. An important observation is that the Gaussian thus obtained is only a particular case of distribution that can be obtained, with the additional condition that it must fulfill the principle of maximum multifractal informational entropy with quadratic constraints. Using group theory, we can select, based on some criteria, the Stoka solutions that can be much more general.

6. The multifractal informational energy in Onicescu sense. A natural variant of an uncertainty relationship

The multifractal informational energy in Onicescu formulation [9] for the normal multifractal distribution (9) is defined through the relation:

$$E(\mu) = \int \rho^2 (x, y, \mu) dxdy. \tag{67}$$

Specifically, through (9) and (11), at any scale resolution, it results:

$$E(a, b, c) = \frac{\sqrt{ac - b^2}}{4\pi} \tag{68}$$

Two important consequences result from (68). First, this is an indicator of the distribution spread, because at any scale resolution, the quantity:

$$A = \frac{2\pi}{\sqrt{ac - b^2}} \tag{69}$$

Equation (69) represents the areas of ellipses of equal probability, H(x,y)=const. Therefore, at any scale resolution, as the gaussian (9) is more grouped, then the higher is its multifractal informational energy. Secondly, through (69) it results that, at any resolution scale, on the manifold of statistical hypotheses referring to the parameters a, b, c given through the group SL(2$\mathbb{R}$), the statistical inference takes place at stationary multifractal informational energy, its constant value characterizing the transitivity variations of the respective group. Indeed, in order for the quadratic form (11) to be positively defined, it is necessary and sufficient for D from (13) to satisfy the condition:

$$D(\mu) \equiv [\delta q(\mu)]^2 [\delta p(\mu)]^2 - cov^2(q, p, \mu) \geq 0, \tag{70}$$

which implies

$$[\delta q(\mu)]^2[\delta p(\mu)]^2 \geq cov^2(q,p,\mu). \tag{71}$$

This relation offers a natural variant of an uncertainty relationship

$$\Delta q(\mu)\Delta p(\mu) \geq |cov(q,p,\mu)|. \tag{72}$$

In such a framework, taking into account (12) and (13) it results, for any resolution scale, the relation:

$$ac - b^2 = \frac{1}{D}, \tag{73}$$

from which, with (68), there exists:

$$E(\rho) = \frac{1}{4\pi\sqrt{D}}. \tag{74}$$

If the multifractal informational energy of the repartition $\rho(x,y,\mu)$ is stationary, ρ itself must be a constant and can be conveniently chose in the form:

$$[\delta q(\mu)]^2[\delta p(\mu)]^2 - cov^2(q,p,\mu) = 4m_0^2\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}, \tag{75}$$

from where the multifractal uncertainty relation results:

$$\Delta q(\mu)\Delta p(\mu) \geq 2m_0\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}, \tag{76}$$

where m_0 is the rest mass of the complex system entity which ”supports” the dynamics. In particular, for dynamics on monofractal manifolds described through Peano curves in $D_F = 2$ at Compton scale resolution $\lambda = h/2m_0$, where h is the Planck constant, (76) is reduced to the uncertainty relation of quantum mechanics

$$\Delta q\Delta p \geq h. \tag{77}$$

The relation (70) allows an interesting connection with a quantum case reffering to coherent states [10, 11]. From relations (12) with a, b, c given by (66), it is immediately found that:

$$\begin{aligned} a(\mu)D(\mu) &= [\delta q(\mu)]^2 = QD(\mu)[ch2r + sh2rcos\Omega], \\ -b(\mu)D(\mu) &= cov(q,p,\mu) = -QD(\mu)sh2rsin\Omega, \\ c(\mu)D(\mu) &= [\delta p(\mu)]^2 = QD(\mu)[ch2r - sh2rcos\Omega], \end{aligned} \tag{78}$$

from where, through comparison with (73) it results

$$a(\mu)c(\mu) - b^2(\mu) = Q^2 \tag{79}$$

and it is chosen

$$Q = \frac{\hbar}{2}. \tag{80}$$

These ensembles or equivalent quantum states were given by Stoler [10, 11] as states which satisfy coherence conditions in all orders, being very important in quantum optics calculations and for the description of measurement processes [12]. For these states, the uncertainty relation is not minimal, but rather it emerges from (78) - (80) in the form given by Stoler:

$$(\Delta q)^2(\Delta p)^2 = \frac{\hbar^2}{4}(1 + sh^2 2r sin^2\Omega). \tag{81}$$

The uncertainty is minimal only for $\Omega = 0$, which, when taking into account the way $a, b, c,$ were obtained from (66) reverts to the statement that all the oscillators found in the ensemble have the same phase initially, i.e. 0. In this hypothesis, at any time after the initial one, the uncertainty relation no longer fulfills the minimality condition at the same time with the covariance of the ensemble, which becomes different from zero. The relations (66) show the way in which the values of the constraints referring to maximum principle of the informational entropy will be fixed. It reverts to the concretization of a model to which these quantities can relate, as it was the case for harmonic oscillators.

7. Conclusions

The main conclusions of the present paper are:

A new formulation of the informational entropy on multifractal manifolds is proposed.

In the context of the new formulation of the informational entropy, some connections with known results from the dynamics of complex systems are discussed: correlations of SL(2ℝ) groups with Hamiltonian formalism, simultaneous invariances between two groups (spatial coordinate group and parametric group), linear harmonic oscillator/oscillator ensemble dynamics from the perspective of SL(2ℝ) groups, uncertainty relations on multifractal manifolds etc.

Moreover, the existence of a multifractal perspective in the complex dynamics analysis allows direct correspondences with deep learning algorithms (i.e. deep learning is part of a larger family of machine learning methods based on artificial neural networks with representation learning).

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