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ON THE STUDY AND POSSIBLE APPLICATIONS OF MINIMALLY COMPLEX CHAOS

BY

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Abstract. The nature of chaos is elusive and disputed, however it can be connected to sensitivity to initial conditions caused by nonlinearity of the equations describing chaotic phenomena. A nowhere-near comprehensive list of such equations can still be shown: the Boltzmann equation, Ginzburg-Landau equation, Ishimori equation, Korteweg-de Vries equation, Landau-Lifshitz-Gilbert equation, Navier-Stokes equation, and many more. This disproportionality between input and output creates an analytically-difficult situation, one that is complicated both algebraically and numerically – however, the study of equations that are both simple and chaotic may yield useful connections between algebraic complexity and chaos. Such connections can be used to determine the simplest possible chaotic function, which can be used as a "chaotic operator" for various non-chaotic or chaotic functions, thus reducing the problem of chaos to one based strictly on algebra.

Keywords: turbulence; chaos; nonlinearity; attractor.

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1. Introduction

The study of the behaviour of the atmosphere, and of other complex systems, has been a long and difficult endeavour; numerous previous studies have attempted to quantify and to explain complex and seemingly-random phenomena. In attempting to do so, they have consistently found that the typical tools of mathematics and physics are insufficient in solving the problem: notions such as "non-linearity", "chaos", "randomness" and yet also "coherence", blur the lines and create confusion. However, the coupling and expanding of such notions means that many of these questions could be, at least partially, settled – and practical solutions can be found so that one day, finally, we may understand and even use to our full advantage many of these complicated phenomena.

2. Common and Uncommon Chaotic Systems

The nature of chaos regarding algebraic complexity is still unclear, especially since multiple cases of relatively simple differential equation systems have been found to exhibit highly chaotic behaviour. Following the Ruelle-Takens theory of turbulence, it is found that such systems regularly produce what have been named "strange attractors", which are attractors with a fractal structure. Referring once again to simplicity producing chaos, Lorenz's studies must be brought up next. The Lorenz system was initially derived from an Oberbeck-Boussinesq approach to atmospheric dynamics – which is to say that the atmosphere is a mechanically incompressible but thermally compressible system. Obtaining the equations is a question of developing a two-dimensional flow of fluid of uniform depth H, with an imposed temperature difference ΔT, under gravity g, with buoyancy α, thermal diffusivity κ, and kinematic viscosity υ:

$$\frac{\partial}{\partial t}(\nabla^2\phi) = \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}(\nabla^2\psi) + \upsilon\nabla^2(\nabla^2\phi) + g\alpha\frac{dT}{dx} \tag{1}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial z}\frac{\partial\psi}{\partial x} - \frac{\partial\theta}{\partial x}\frac{\partial\psi}{\partial z} + \kappa\nabla^2 T + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} \tag{2}$$

where ψ is a stream function, defined such that the velocity components $u = (u, w)$ of the fluid motion are (Tabor, 1989):

$$u = \frac{\partial\psi}{\partial z}, w = -\frac{\partial\psi}{\partial x} \tag{3}$$

Then, it can be noticed that periodic solutions of the form:

$$\psi = \psi_0 \sin\left(\frac{\pi a x}{H}\right)\sin\left(\frac{\pi z}{H}\right) \tag{4}$$

grow for Rayleigh numbers larger than a critical value, $Ra > Ra_c$ (Tabor, 1989). By including the terms X, Y, Z where X is proportional to convective intensity, Y to the temperature difference between descending and ascending currents, and Z to the difference in vertical temperature profile from linearity, the following equations are obtained:

$$\dot{X} = \sigma(Y - X), \dot{Y} = -XZ + rX - Y, \dot{Z} = XY - bZ \tag{5}$$

And where:

$$\sigma \equiv \frac{\upsilon}{\kappa} = Prandl\ number = Pr \tag{6}$$

$$r \equiv \frac{Ra}{Ra_c} = \frac{Rayleigh\ number}{critical\ Rayleigh\ number} \tag{7}$$

And b is a geometric factor. The Rayleigh number can be defined in the following manner:

$$Ra = \frac{time\ scale\ for\ thermal\ transport\ via\ diffusion}{time\ scale\ for\ thermal\ transport\ via\ convection} = Gr \cdot Pr \tag{8}$$

Where:

$$Pr = \frac{momentum\ diffusivity}{thermal\ diffusivity} \tag{9}$$

and Gr, also named the Grashof number is:

$$Gr = \frac{buoyancy}{viscosity} \tag{10}$$

The initial parameters used are $b = \frac{8}{3}$, $\sigma = 10$, $r = 28$, and the onset of chaos is at $r \gtrsim 24.74$ (Grassberger and Procaccia, 2004). If $r < 0$ there results only one equilibrium point at the origin; this corresponds to no convection at all. The chosen parameters give a fractal dimension of $D_F \cong 2.06$ (Grassberger and Procaccia, 2004). The critical points at $(0, 0, 0)$ correspond to no convection while the critical points found at $\left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$ mean steady convection.

The system is represented in its entirety in Fig. 1, with the initial points $(1,1,1)$ and the original initial parameters. 5000 iterations of the equations used

to construct the model are plotted to represent the attractor, each iteration being considered as a time step equal to 0.01.

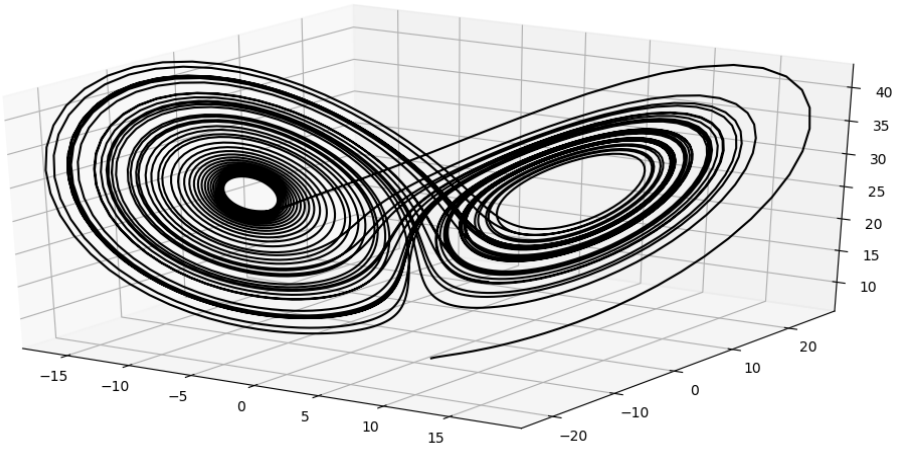


Fig. 1 − Lorenz system after 5000 iterations; initial parameters: $b = \frac{8}{3}$, $\sigma = 10$, $r = 28$; initial points: $(1, 1, 1)$.

The Lorenz attractors have been a source of confusion and fascination in the fields of chaos and nonlinearity, mostly because it is a relatively simple system. Being composed of three first-order differential equations, and having only three parameters, one cannot call it "complicated" from an algebraic perspective. The equations describing the Malkus wheel, which were the first attempt to transpose the Lorenz attractor equations into reality, are slightly different, yet even simpler than the Lorenz system:

$$\dot{x} = y - fx,\ \dot{y} = -\lambda y + xz,\ \dot{z} = -\lambda z - xy + \lambda \tag{11}$$

where x represents the angular velocity, y and z are center of mass coordinates (Lorenz and Haman, 1996). This time, the system presents two parameters: λ which quantifies the cup leakage and f which is the inverse of the time constant for the slowdown of the wheel. With certain values, the system proves itself to be chaotic as well as easy to construct as an experimental setting.

3. The Minimal Algebraic Complexity of Chaotic Systems

An even simpler Lorenz-type system can be constructed through rescaling to create a "diffusionless Lorenz system", however any other

simplification attempts has been proven to lead to a loss of "chaoticity" for this system type (Sprott, 2010; van der Schrier and Maas, 2000). Even so, a different system can be considered with more basic qualities *i.e.* a single quadratic nonlinearity:

$$\dot{x} = -y - z,\ \dot{y} = x + ay,\ \dot{z} = b + z(x - c) \tag{12}$$

This is the Rössler system, where the original parameters are $a = b = 0.2,\ c = 5.7$ (Rössler, 1976). Its dimension is $D_F = 2.0132$ for the given parameters, but the maximum possible dimension is $D_F = 2.1587$ with $a = 0.6276$, $b = 0.798$, $c = 2.0104$ (Rössler, 1979). Now, for many years, this and the Lorenz systems were regarded as the simplest examples of chaos in autonomous dissipative systems of ordinary differential equations. However, a "less famous" simpler system exists, also credited to Rössler:

$$\dot{x} = -y - z,\ \dot{y} = x,\ \dot{z} = a(y - y^2) - bz \tag{13}$$

This system, called the "Rössler prototype-4", presents only six terms, a single quadratic nonlinearity, and only two parameters, exhibiting chaos for $a = b = 0.5$ (Rössler, 1979). It, in turn, is one of 18 other similar systems found through an extensive numerical search – some of these have very interesting properties (Sprott, 2010). The third of them:

$$\dot{x} = -y,\ \dot{y} = x + z,\ \dot{z} = xz + 3y^2 \tag{14}$$

Wherein $x_0 = 0.6$, has the distinct property of being dissipative but time-reversible with a symmetric pair of strange attractors that exchange role when time is reversed (Sprott, 2010). This interesting attribute is not to be neglected: the fact that a system can be time-reversible yet dissipative at the same time hints towards soliton-like qualities. The fourth:

$$\dot{x} = yz,\ \dot{y} = x - y,\ \dot{z} = 1 - xy \tag{15}$$

Wherein $x_0 = -1$ and $y_0 = 1$, has been shown to exhibit bistability, hysteresis, and highly incoherent phase dynamics.

Given the richness of content that one can access when investigating such systems, a natural question arises: is it possible to redefine the systems as equations containing just one differentiable parameter, and is it possible to construct a hierarchy of such equations? To answer this question, one must consider the fact that, in all of these systems, we are presented with three first-

order ordinary differential equations, the equivalent of which would be a single third order ordinary differential equation (Chlouverakis and Sprott, 2005). This is called a "jerk" equation, which is generally of the form:

$$\dddot{x} = f(\ddot{x}, \dot{x}, x) \tag{16}$$

Where the "jerk", in general, is the third derivative of position with respect to time. Given that the previously illustrated systems have been shown to be some of the simplest possible differential equation systems that produce chaos, we would expect that the jerk would be the lowest possible position derivative for which an ordinary differential equation with smooth continuous functions can exhibit chaos (Sprott, 2010). Any explicit ordinary differential equation can be cast in the form of a system of coupled first-order ordinary differential equations, but the reverse is not always true. However, both the Lorenz and Rössler systems can be written in jerk form (Sprott, 2010):

$$\dddot{x} + \left(1 + \sigma + b - \frac{\dot{x}}{x}\right) + \left[b(1 + \sigma + x^2) - (1 + \sigma)\frac{\dot{x}}{x}\right]\dot{x} - b\sigma(r - 1 - x^2)x = 0 \tag{17}$$

And:

$$\dddot{y} + (c - a)\ddot{y} + (1 - ac)\dot{y} + cy - b - (\dot{y} - ay)(\ddot{y} - a\dot{y} + y) = 0 \tag{18}$$

Both of these equations, although functional, are quite unwieldy – a study has shown, however, that all of the 18 previously mentioned chaotic systems can be represented in a similar manner, with jerk equations of increasing complexity (Eichhorn *et al*., 1998). The simplest of these is:

$$\dddot{x} + a\ddot{x} - \dot{x} + x = 0 \tag{19}$$

Which is chaotic for $a = 2.02$, and for $x(0) = 5$, $\dot{x}(0) = 2$, and $\ddot{x}(0) = 0$, presenting a highest Lyapunov exponent equal to 0.0486 (Eichhorn *et al*., 1998; Sprott, 2010). Now, if one accepts that quadratic forms are the simplest possible nonlinear forms, and the jerk is the lowest derivative for which chaos occurs in such systems, then the previous equation must be the algebraically-simplest possible continuous chaotic system. Any polynomial with fewer terms would have no adjustable parameters, disabling the range of its possible dynamics – in fact, it has been proven quite rigorously that there can be no simpler chaotic system (Fu and Heidel, 1997).

Interestingly, the above-mentioned system can actually be solved; assuming $a = 2.02$:

$$x(t) = c_1 e^{-2.56253t} + e^{0.271263t}[c_2 \sin(0.562722t) + c_3 \cos(0.562722t)] \tag{20}$$

We may now attempt to actually determine the values of the three constants, given our chaotic initial values:

$$x(0) = c_1 + c_3 = 5 \tag{21}$$

Thus:

$$c_1 = 5 - c_3 \tag{22}$$

Then:

$$\dot{x}(0) = -2.5623c_1 + 0.271263c_3 + 0.562722c_2 = 2 \tag{23}$$

Which yields:

$$c_3 = 5.22716 - 0.19859c_2 \tag{24}$$

Finally:

$$\ddot{x}(0) = 6.56538c_1 - 0.2430724c_3 + 0.30531c_2 = 0 \tag{25}$$

Which, by replacement, will give us the values of the constants and the final form of the function:

$$x(t) = 0.10378 \cdot e^{-2.56253t} + e^{0.271263t}[4.89622 \cdot \sin(0.562722t) + 1.66644 \cdot \cos(0.562722t)] \tag{26}$$

The equation above thus has the "honour" of being the solution to the simplest possible continuous chaotic system. It might seem tiresome and trivial to write out these values with so many decimals, however, one must remember that this is a chaotic function that we are working with: almost every single decimal counts. Here, "chaoticity" becomes apparent upon consulting the semilogarithmic graph of the equation: ignoring negative values, the system seems to be oscillating in ever-increasing orbits; however, neither the period, nor the shapes, nor the scaling of these orbits seem to have a periodic character (Fig. 2).

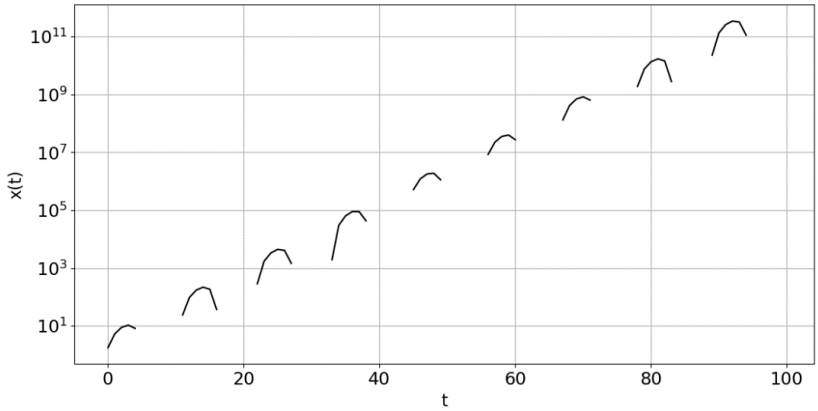


Fig. 2 − Semilogarithmic plot of the solution of the simplest possible chaotic system.

4. Conclusion

The fact that the obtained equation is not unimodal complicates our analysis; the Schwartzian of a unimodal map, if negative, automatically points to chaoticity. In any case, resulting from the discussion above, there can be no chaotic system involving only two first-order, ordinary differential equations (Coddington and Levinson, 1955). The fact that such simple "mathematical nuclei of chaos" even exist is fascinating, and it might be used to produce a future theory of "chaoticity" as resulting from the coupling of minimally-complex "units" of chaotic equations; first, however, one would need to have a conclusive answer to the following question: does more complexity automatically equal more chaos?

In conclusion, the existence of minimally-complex chaotic equations reveals multiple possibilities in chaos theory, perhaps through the development of a "chaotic operator" developed through Green's functions that, when applied to a non-chaotic function produces a chaotic one, and vice versa when applied inversely. The postulated existence of such an operator would algebraically explain chaotic behaviours of all kind as operator manifestations, and would reveal much about methods to control and even harness chaoticity in various processes.

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STUDIUL ŞI IDENTIFICAREA APLICAŢIILOR FUNCŢIILOR HAOTICE MINIMAL COMPLEXE

(Rezumat)

Natura haosului este disputată şi dificil de descris, dar acesta poate fi conectat cu senzitivitatea la condiţiile iniţiale cauzate de nonlinearitatea ecuaţiilor care descriu fenomene haotice. O listă incompletă a acestor ecuaţii este: ecuaţia Boltzmann, ecuaţia Ishimori, ecuaţia Korteweg-de Vries, ecuaţia Landau-Lifshitz-Gilbert, ecuaţia Navier-Stokes, şi multe altele. Această disproporţionalitate între date iniţiale şi rezultate creează o situaţie dificilă analitic, una care este complicată din punct de vedere algebric şi numeric – totuşi, studiul ecuaţiilor care sunt atât simple cât şi haotice poate oferi conexiuni folositoare între noţiunea de complexitate algebrică şi haos. Asemenea conexiuni pot fi folosite pentru a determina cea mai simplă funcţie haotică posibilă, care poate fi folosită drept un "operator haotic" pentru multe tipuri de funcţii haotice sau non-haotice, astfel reducând problema haosului strict la o dimensiune algebrică.