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USE OF THE ALTERNATING DIRECTIONS METHOD IN CYLINDRICAL COORDINATES, APPLIED TO THE TWO-DIMENSION UNSTEADY CONDUCTION HEAT TRANSFER

BY

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Abstract. Finding analytical solutions for the unsteady conduction heat transfer equation is generally very difficult due to the intrinsic complexity of the problem. In most cases, the equation has no analytical solutions. A numerical approach that surmounts these difficulties consists of the finite difference technique. The alternate directions finite-difference technique especially suits to the two-dimension heat conduction. The paper considers a hollow cylinder consisting of a conduction heat transfer medium, and the two-direction unsteady heat propagation within this medium is studied by means of the alternate directions method. The approach simulates the heat conduction process during a charge-discharge cycle of a single borehole heat storage.

Keywords: finite difference technique, unsteady heat conduction, two-dimension heat propagation, cylindrical coordinates.

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1. Introduction

Even in the simplest cases of transient conduction heat transfer, such as the one-dimension heat diffusion in simple geometries (plane or cylindrical), analytical solutions of the heat transfer equation may become extremely complicated and difficult to handle. For instance, the unsteady heat conduction in cylinders results in analytical solutions involving the use of Bessel functions, which are particularly laborious to use (Incropera *et al*., 2007; Korenev, 2021; Wang *et al*., 2008).

Two- or three-dimension conduction transient heat transfer, especially when dealing with Neumann boundary conditions results in much more complicated partial differential equations, which have no analytical solutions.

A practical numerical approach in such cases is the use of finite differences. This way, the partial derivatives present in the conduction heat transfer equation are turned into finite differences, at the cost of obtaining an approximate solution, but the accuracy of the result can be controlled in a certain measure by decreasing the time step value at each iteration, combined with a finer mesh. There are two variants of the finite difference method, which depend on how one considers the time step (Tannehill *et al.*, 1997). The explicit scheme calculates the present time temperature in the current node of the mesh as a function of the past time step temperatures of the three nodes (the current one, and the neighboring two ones). The advantage of the explicit approach consists of the fact that the new nodal temperatures are sequentially calculated from simple linear algebraic equations. The procedure is however time-consuming, because of the limitation imposed by the stability and convergence criterion, which imposes upper values of the time and space steps (Causon and Mingham, 2010). The implicit method involves writing (for each node) of a linear algebraic equation that expresses the three new nodal temperatures as a function of the past time step temperature in the central node. A three-unknown equation results and thus a set of equations must be constructed, the solution of which will be the new (present time) nodal temperatures. The advantage of this technique is that there are no restrictions upon the time and space steps, but it comes with a price, *i.e*., the complications arisen by the rather complicated algorithms of solving sets of linear algebraic equations (Causon and Mingham, 2010).

When a two-direction transient conduction heat transfer equation is approximated by finite difference equations, if the implicit technique is used, the matrix of the equations set is a sparse one and turning it into a diagonal matrix necessary to obtain the solution is very laborious. However, two simplified approaches have been developed, that transform the original sparse matrix into two band ones, similar to the one that results in the case of one-dimension conduction heat transfer (Godunov and Reabenki, 1977; LeVeque,

2007). The two approaches are the *Alternating Direction Technique*, and the *Decomposition Method*.

The present paper deals with the *Alternating Directions Technique*, applied to the transient conduction heat diffusion in two dimensions (radial and longitudinal) in a cylindrical domain with an axial channel through which a hot and a cold fluid flow alternately.

2. Mathematical Model

Consider a long cylindrical channel of length L and inner radius R_0, surrounded by a cylindrical domain of radius R_d. The initial temperature of the domain is T_∞, considered uniform.

Fig. 1 schematizes the domain and the mesh attached in order to apply the finite difference scheme.

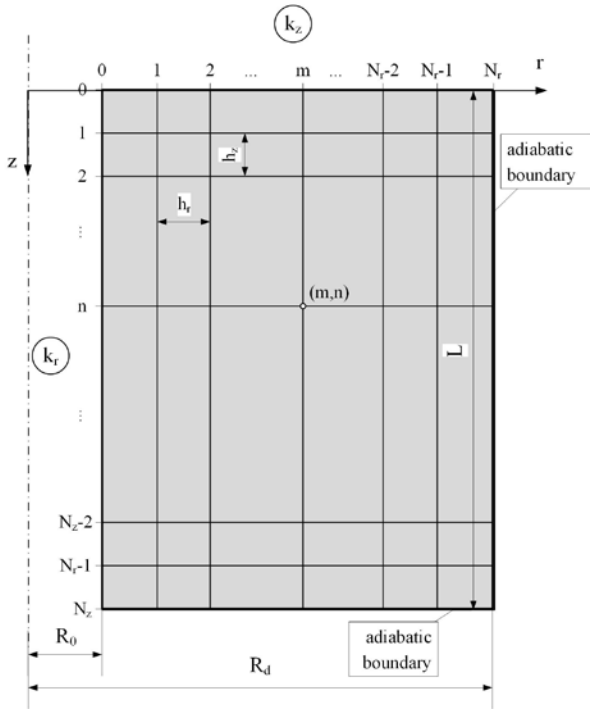


Fig. 1 – The domain and the mesh.

The transient conduction heat transfer equation in cylindrical coordinates with radial and axial propagation is (Incropera *et al.*, 2007; Wang *et al.*, 2008):

- in radial direction:

$$\frac{\partial \theta}{\partial \tau} = \frac{\lambda}{\rho c}\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2}\right) \tag{1}$$

- in axial direction:

$$\frac{\partial \theta}{\partial \tau} = \frac{\lambda}{\rho c}\frac{\partial^2 \theta}{\partial z^2} \tag{2}$$

In Eqs. (1) and (2) and onwards, the dimensionless temperature θ is defined as:

$$\theta = \frac{T - T_\infty}{T_H - T_\infty} \tag{3}$$

where T is the generic temperature, and T_H is the hot fluid temperature (see also section 4).

The boundary conditions are:

- at $r = R_0$:

$$k_r \theta_0 = -\lambda\left(\frac{\partial \theta}{\partial r}\right)\Big|_{r=R_0} \tag{4}$$

where θ_0 is the dimensionless temperature at the surface of the central channel.

- at $z = 0$:

$$k_z\left(\theta_a - \theta_0\right) = -\lambda\left(\frac{\partial \theta}{\partial z}\right)\Big|_{z=0} \tag{5}$$

where θ_a is the dimensionless temperature of the fluid in contact with the front surface of the domain.

- at $r = R_d$ (adiabatic boundary):

$$\left(\frac{\partial \theta}{\partial r}\right)\Big|_{r=R_d} = 0 \tag{6}$$

- at $z = L$ (adiabatic boundary):

$$\left(\frac{\partial \theta}{\partial z}\right)\Big|_{z=L} = 0 \tag{7}$$

In the above equations, ρ stands for the density, *c* for the specific heat, and λ for the thermal conductivity of the material of the cylinder.

3. Finite Difference Approach

Turning the partial derivatives into finite differences requires the domain discretization by means of a mesh. As schematized in Fig. 1, the mesh consists of N_r nodes in the radial direction with radial step h_r, respectively N_z nodes in the axial direction with axial step h_z.

The *Alternating Directions Technique* (ADT) derives from the Crank-Nicolson scheme in one direction (Recktenwald, 2004) and was developed by J. Douglas and H. H. Rachford (Douglas and Rachford, 1956).

With the finite difference operator D^2 corresponding to the second order partial derivatives, the conduction heat transfer Eq. (1) can be formally rewritten for the current time step Δτ (numer *p*), (LeVeque, 2007; Godunov and Reabenki, 1977) as follows:

$$\frac{\theta_{m,n}^{p}-\theta_{m,n}^{p-1}}{\Delta\tau}=a\left(D_r^2\theta_{m,n}^{p}+D_z^2\theta_{m,n}^{p-1}\right) \tag{8}$$

In Eq. (8), *a* stands for the thermal diffusivity of the domain material.

ADT splits the time step Δτ into two intervals of equal magnitude $\Delta\tau/2$, and thus two corresponding stages will result:

– The intermediary stage occurring from the preceding time $(p-1)\Delta\tau$ to the intermediary time $(p-1/2)\Delta\tau$. For this step of the procedure, an implicit approach in the radial direction is considered, concurrently with an explicit one in the axial direction. Consequently, in Eq. (8), D_r^2 refers to the nodal temperatures at the end of the phase $(p-1/2)\Delta\tau$ in the radial direction (presently unknown), while D_z^2 applies to the nodal temperatures in the axial direction, already determined in the previous step of the procedure, $(p-1)\Delta\tau$.

– The final stage: from $(p-1/2)\Delta\tau$ to $p\Delta\tau$. The treatment is explicit for the radial direction, respectively implicit for the axial one. The explicit approach uses the intermediate temperatures that have been determined in the previous (intermediary) stage. The outcome of this stage will represent the final temperatures for the current time step *p*.

Consequently, Eq. (8) written for the two stages becomes respectively:

– For the intermediary stage (unknown temperatures: superscript *i*):

$$\frac{\theta_{m,n}^{i}-\theta_{m,n}^{p-1}}{\frac{\Delta\tau}{2}}=a\left(\underbrace{D_r^2\theta_{m,n}^{i}}_{\text{implicit}}+\underbrace{D_z^2\theta_{m,n}^{p-1}}_{\text{explicit}}\right) \tag{9}$$

– For the final stage (unknown temperatures: subscript *p*):

$$\frac{\theta_{m,n}^{p}-\theta_{m,n}^{i}}{\frac{\Delta\tau}{2}}=a\left(\underbrace{D_r^2\theta_{m,n}^{i}}_{\text{explicit}}+\underbrace{D_z^2\theta_{m,n}^{p}}_{\text{implicit}}\right) \tag{10}$$

The operator D^2 applied to the nodal temperatures results as follows:
– In the radial direction:

$$D_r^2\theta_{m,n}=\frac{1}{h_r^2}\left(\frac{c_m-1}{c_m}\theta_{m-1,n}-2\theta_{m,n}+\frac{c_m+1}{c_m}\theta_{m+1,n}\right) \tag{11}$$

where:

$$c_m=2\left(N_r\frac{R_0}{R_d-R_0}+m\right) \tag{12}$$

− In the axial direction:

$$D_z^2\theta_{m,n}=\frac{1}{h_z^2}\left(\theta_{m-1,n}-2\theta_{m,n}+\theta_{m+1,n}\right) \tag{13}$$

In Eqs. (11) and (13), the superscripts of the nodal temperatures must be selected in accordance with the type of operator D^2 (implicit, respectively explicit, according to the stage of the procedure). In the implicit operator, there are three unknown nodal temperatures (which need to be determined), whereas the explicit operator contains known nodal temperatures that had been determined in the previous stage. As a result, in each stage, one must determine the temperatures that appear in the implicit operator and thus, a set of linear algebraic equations results, that has to be solved.

After a series of manipulations, Eqs. (11) and (13) yield respectively:

$$-\theta_{m-1,n}^{i}+\sigma_{r,m}\theta_{m,n}^{i}-\gamma_{r,m}\theta_{m+1,n}^{i}=\beta_{r,m} \tag{14}$$

$$-\theta^{p}_{m,n-1} + \sigma_z \theta^{p}_{m,n} - \theta^{p}_{m,n+1} = \beta_{z,\mathrm{n}} \tag{15}$$

where:

$$\sigma_{r,m} = 2\frac{c_m}{c_m - 1}\left(1 + \frac{h_r^2}{a\Delta\tau}\right) \tag{16}$$

$$\gamma_{r,m} = \frac{c_m + 1}{c_m - 1} \tag{17}$$

$$\beta_{r,m} = \frac{c_m}{c_m - 1}\left(\frac{h_r}{h_z}\right)^2 \left[\theta^{p-1}_{m,n-1} - 2\left(1 - \frac{h_z^2}{a\Delta\tau}\right)\theta^{p-1}_{m,n} + \theta^{p-1}_{m,n+1}\right] \tag{18}$$

$$\sigma_z = 2\left(1 + \frac{h_z^2}{a\Delta\tau}\right) \tag{19}$$

$$\beta_{z,n} = \left(\frac{h_z}{h_r}\right)^2 \left[\frac{c_m - 1}{c_m}\theta^{i}_{m-1,n} - 2\left(1 - \frac{h_r^2}{a\Delta\tau}\right)\theta^{i}_{m,n} + \frac{c_m + 1}{c_m}\theta^{i}_{m+1,n}\right] \tag{20}$$

Eqs. (14) and (15) written for all current nodes in both radial and axial directions respectively make up two sets of linear equations the solutions of which represent the nodal temperatures at the end of the two stages of the ADT procedure.

4. Results and Discussion

The *Alternating Directions Technique* has been used to perform a numerical simulation of the unsteady conduction heat transfer in cylindrical coordinates in the case of a cylindrical domain consisting of a 20 meters long central channel, 25 centimetres across. The channel is surrounded by a cylindrical domain of radius R_d = 5.125 m. The considered temperatures are T_H = 100°C, T_∞ = 10°C, T_L = 5°C. The thermal properties of the domain material are: $\lambda = 0.25\ \mathrm{Wm^{-1}K^{-1}}$, $a = 0.18\cdot10^{-6}\ \mathrm{m^2s^{-1}}$.

The convection heat transfer coefficient between the flowing fluid and the channel inner surface is $k_\mathrm{r} = 1000\ \mathrm{Wm^{-2}K^{-1}}$. The convection heat transfer coefficient between the domain frontal surface and the other fluid is $k_\mathrm{z} = 10\ \mathrm{Wm^{-2}K^{-1}}$.

The considered mesh for the cylindrical domain has $N_r = 50$ and $N_z = 50$ nodes respectively.

In order to apply the ADT, a computer code has been developed by the authors and it has been used to generate the nodal temperatures at specified time intervals: 30, 60, 120, and 180 days, for the stages of the process: heat addition (charging), and heat removal (discharging).

During charging, a fluid of temperature $T_H > T_\infty$ flows along the central channel and transfers heat by convection (heat transfer coefficient: k_r). Concurrently, heat exchange by convection occurs between another fluid of temperature T_a (heat transfer coefficient: k_z) and the frontal surface $z = 0$. The boundaries at $r = R_d$ and at the lower end ($z = L$) are adiabatic.

During discharging, a cold fluid of temperature $T_L < T_\infty$ flowing along the central channel extracts heat by convection from the cylindrical domain, the heat transfer coefficient being k_r. The convection heat transfer at the frontal surface remains identical to that during the charging stage.

In the computer code, the final nodal temperatures for the charging (heat addition) stage are stored in a data file, used to initiate the run for the discharge (heat removal) stage. Output data supplied by the computer codes have been used to plot the temperature field at the selected time intervals.

The charging stage temperature plots are displayed in Figs. 2 through 5.

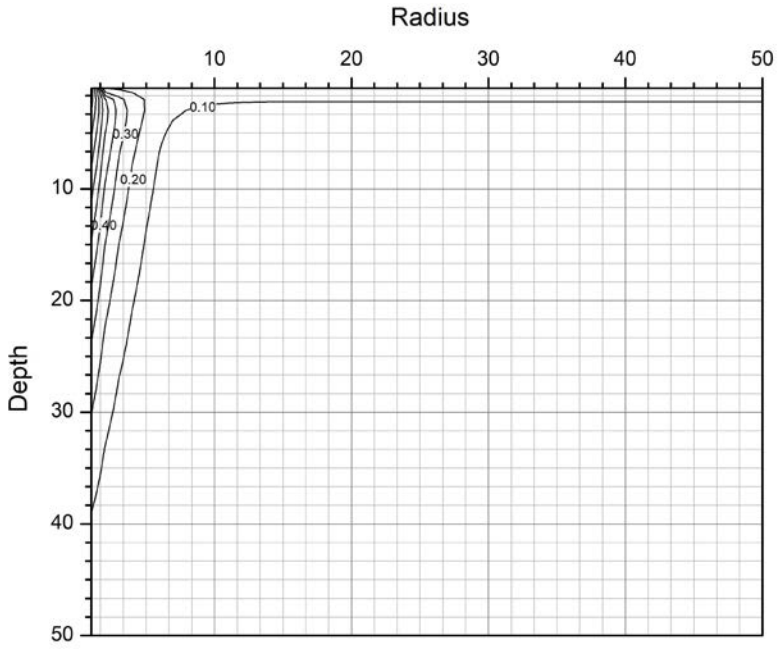


Fig. 2 – Temperature plot – charging, after 30 days.

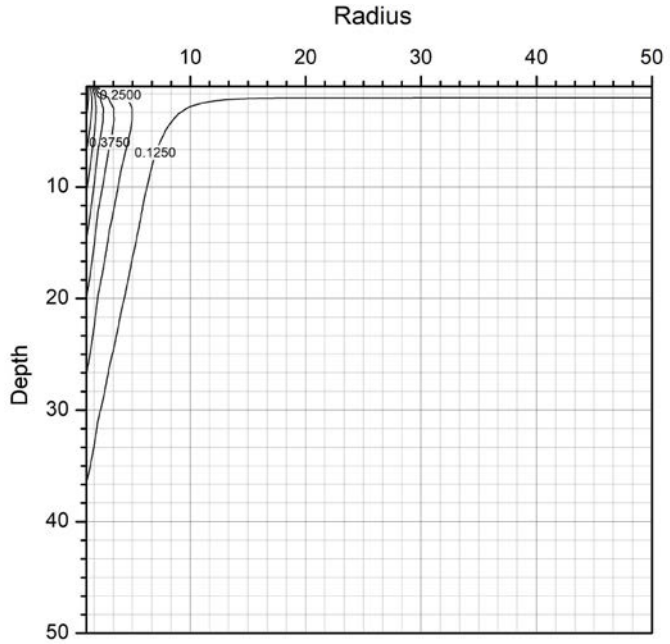


Fig. 3 – Temperature plot – charging, after 60 days.

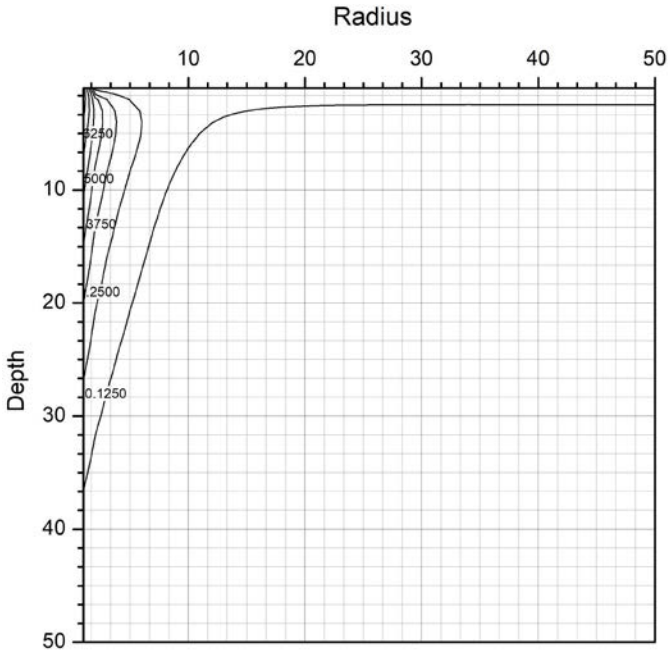


Fig. 4 – Temperature plot – charging, after 120 days.

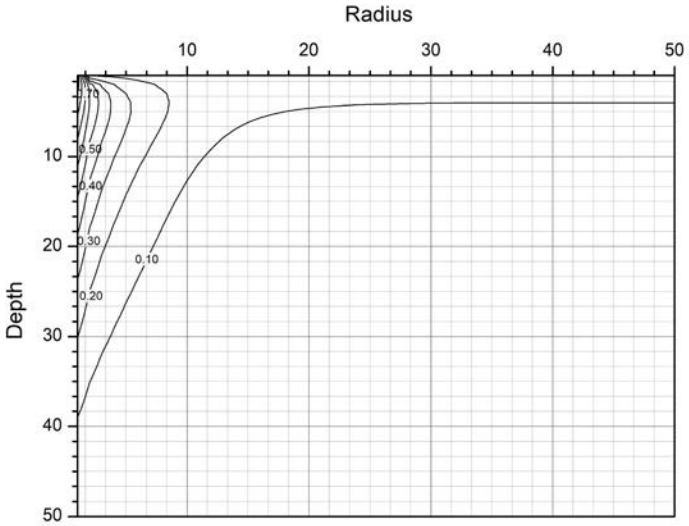


Fig. 5 – Temperature plot – charging, after 180 days.

The low thermal conductivity of the cylinder material explains why the penetration radius is so small: about 10 nodes after 60 days and about 15 nodes after 180 days.

Figs. 6 through 9 display the temperature during discharging for elapsed times of 30, 60, 120, and 180 days.

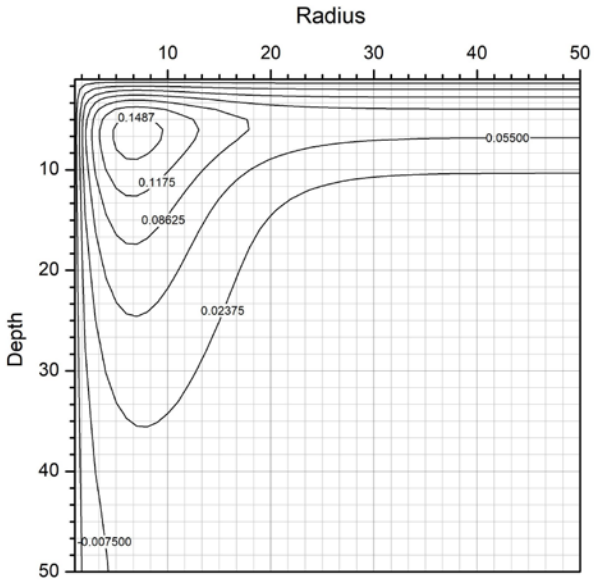


Fig. 6 – Temperature plot – discharging, after 30 days.

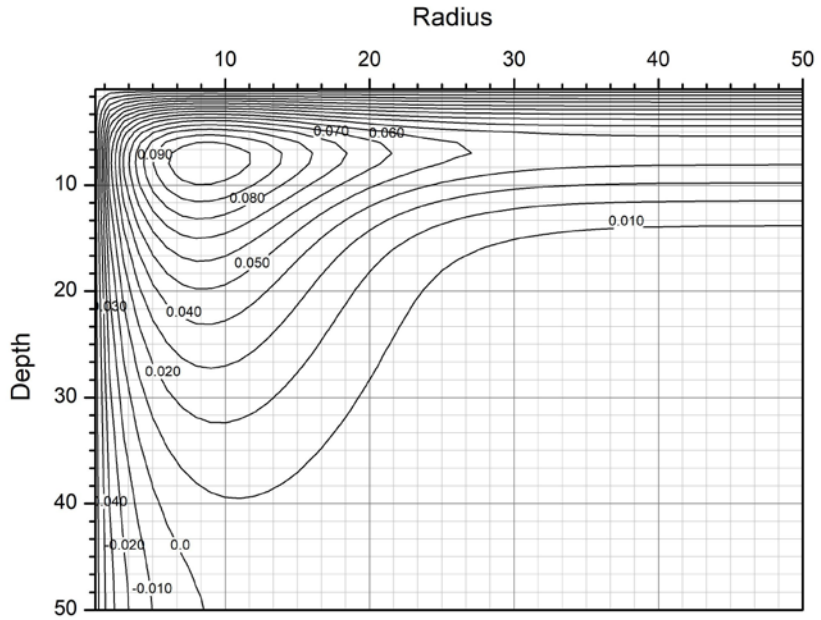


Fig. 7 – Temperature plot – discharging, after 60 days.

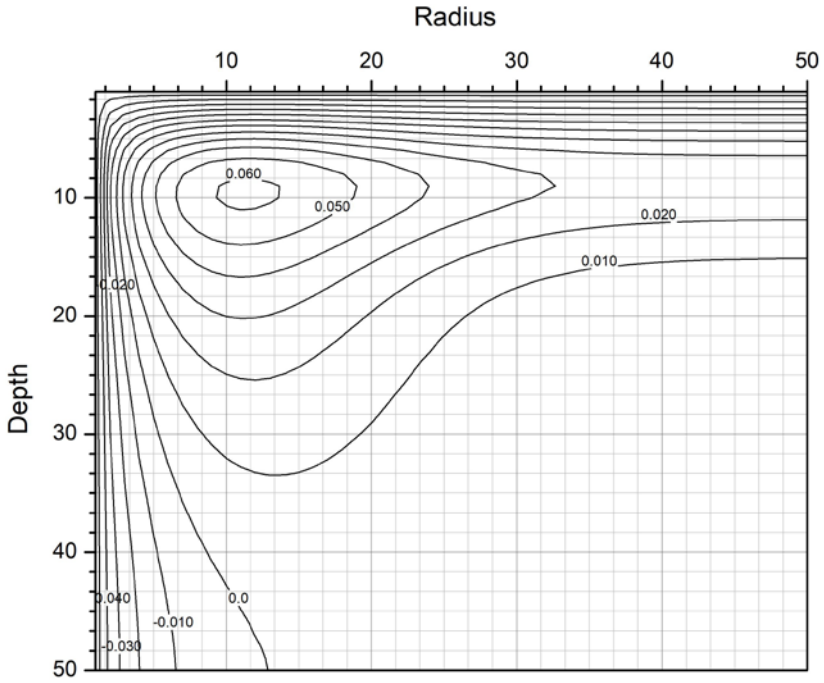


Fig. 8 – Temperature plot – discharging, after 120 days.

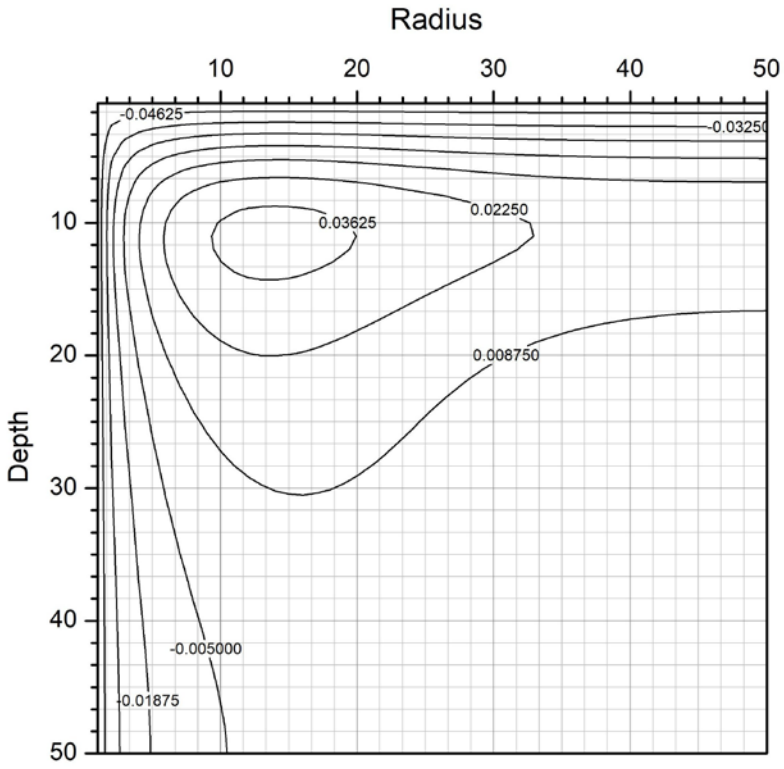


Fig. 9 – Temperature plot – discharging, after 180 days.

The discharge plots show a warmer core (the closed curves) that can be explained by the simultaneous heat removal occurring along the radial and axial directions.

The numerical results are in accordance with the adiabatic "internal" boundaries of the domain. This hypothesis was imposed by the fact that a semi-infinite domain in the radial direction is impossible to take into account when using the finite-difference approach, the only option being to consider a radius long enough as to make it impossible for the thermal perturbation to reach the boundary in a reasonable time.

5. Conclusions

The *Alternating Directions Technique* applied to two-directions unsteady heat conduction proves to be a useful tool to be used in the context of finite-differences methods. The numerical example considered in the paper deals with the operation of a single borehole heat storage system during a year time span consisting of a charge-discharge cycle and it confirms the

effectiveness of the method and its versatility in determining the temperature distribution across thermal storage. Further research will be oriented towards considering variable temperature heat transfer fluids in contact with the cylindrical domain and the use of the ADT to study and design seasonal ground thermal storage systems.

REFERENCES

Causon D.M., Mingham C.G., 2010 *Introductory Finite Difference Methods for PDEs*, Ventus Publishing Aps, Frederiksberg, 2010.

Douglas J., Rachford H.H., *On the numerical solution of heat conduction problems in two and three space variables*. Trans. AMS, **82**, 421-439, 1956.

Godunov S.K., Reabenki V.S., *Finite Difference Computing Schemes* (Romanian translation) Editura Tehnică, Bucharest, 1977.

Incropera F.P., DeWitt D.P., Bergman T.L., Lavine A.S., *Fundamentals of Heat and Mass Transfer*, 6th Edition, John Wiley and Sons, N.Y., 2007.

Korenev B.G., *Bessel Functions and Their Applications* (transl. from German), CRC Press, Taylor and Francis Group, 2021.

LeVeque R., *Finite Difference Methods for Ordinary and Partial Differential Equations*, Society for Industrial and Applied Mathematics, Philadelphia, 2007.

Recktenwald G.W., *Finite difference approximations to the heat equation*, Mechanical Engineering, **10,** 1-27 (2004).

Tannehill J.C., Anderson D.A., Pletcher R.H., *Computational Fluid Mechanics and Heat Transfer* (second Edition), Taylor & Francis, New York, 1997.

Wang L., Zhou X., Wei X., *Heat Conduction. Mathematical Models and Analytical Solutions*, Springer-Verlag, Berlin Heidelberg, 2008.

UTILIZAREA METODEI DIRECŢIILOR ALTERNANTE ÎN COORDONATE CILINDRICE, CU APLICAŢII ÎN TRANSFERUL DE CĂLDURĂ CONDUCTIV ÎN REGIM NESTAŢIONAR

(Rezumat)

Găsirea soluţiilor analitice ale ecuaţiei transferului de căldură conductiv nestaţionar este în general extrem de dificilă, datorită complexităţii intrinsece a problemei. În mare majoritate a cazurilor, aceste soluţii nu există. Depăşirea acestor dificultăţi este posibilă prin utilizarea metodelor numerice, cum ar fi metoda diferenţelor finite. Metoda cu diferenţe finite a direcţiilor alternante este în special aplicabilă în cazul conducţiei bidimensionale. În lucrare se consideră un cilindru gol în care propagarea căldurii se face bidimensional în regim nestaţionar, iar studierea procesului se face prin metoda direcţiilor alternante. În exempul numeric prezentat se consideră un ciclu încărcare-descărcare al unui sistem de stocare termică în sol cu foraj vertical.