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COMPARISON BETWEEN PRANDTL'S MIXING LENGTH MODEL AND DNS DATA FOR A CHANNEL FLOW UP TO $\mathrm{Re}_\tau \approx 5200$

BY

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Abstract. This paper presents a theoretical study of fully developed turbulent flow in a channel at friction Reynolds numbers of 1000, 2000 and 5200. For the momentum equation, Reynolds Averaged Navier-Stokes (RANS) approach was used, where the Reynolds stress was modeled using Prandtl's mixing length model with Van Driest's modification to capture the damping effect near wall. The velocity profile was computed by solving the non-linear ODE for momentum using Euler's method. The results were compared with the latest available DNS data.

Keywords: turbulent flow; channel flow; mixing length.

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1. Introduction

Turbulence represents the chaotic motion of the fluid in a given flow, characterized by an intense mixing between layers. The modeling of this phenomenon is of great importance, especially in engineering where parameters like friction coefficient and heat convection coefficient are needed for designing machineries and optimizing various processes.

Using the original Navier-Stokes equations to solve turbulence is very expensive in terms of computational power and unjustified for most engineering problems. In the last century, turbulence models were developed to approached turbulence in a statistical manner, as a good compromise between solutions accuracy and computational speed.

The main approach of modelling turbulence is by RANS (Reynolds Averaged Navier-Stokes) equations, which are similar with Navier-Stokes equations, but the dependent variables are time averaged quantities and in the conservation of momentum equation an additional term is introduced, named Reynolds-stress tensor. This tensor represents the additional apparent stress caused by velocity fluctuations. RANS equations are an undetermined system of equations, expressing the Reynolds-stress tensor in a way that the system is determined is called a closure problem. The most used method for closing the RANS equations is by Boussinesq hypothesis, in which the Reynolds-stress tensor is expressed as a linear relation between the strain-rate tensor and a coefficient of proportionality, called eddy viscosity or turbulent viscosity. This correlation is analogous with the one for viscous stress tensor, but here, the turbulent viscosity is a characteristic of the flow and not of the fluid. Boussinesq hypothesis is not valid in all cases (Schmitt, 2007), but it's a good approximation for the real phenomenon and is heavily used by engineers when modeling turbulence. Computing the turbulent viscosity is where the entire weight falls. The methods used for computing turbulent viscosity trigger an entire class of turbulence models, called turbulent viscosity models or eddy viscosity models. Depending on these methods, the main turbulent viscosity models are: algebraic, one-equation, two-equations models or models with more than two equations. Algebraic models, also called zero-equation models, are models where the turbulent viscosity is computed by an algebraic relation. In one- and two-equations models, the turbulent viscosity is computed with one respectively two additional PDE's.

In this paper, a comparison between the algebraic Prandtl mixing-length model and DNS data from (https://turbulence.oden.utexas.edu) is done. The justification is that, although the models with the additional PDE's are superior in terms of accuracy and applicability, the algebraic models have the advantage of an easier implementation in a numerical solver scheme.

A theoretical study of Prandtl's mixing length, published by Antonialli and Silveira-Neto (2018), validates the model by comparison with DNS data for

friction Reynolds up to $\mathrm{Re}_\tau \approx 5200$. Using the same approach, mean velocity is computed using Prandtl mixing-length model and Van Driest damping parameter. The theoretical data is compared with DNS data at friction Reynolds number of 1000, 2000 and 5200.

2. Prandtl Mixing Length Model

One of the first turbulence models was the mixing length model proposed by L. Prandtl (1925). He introduced the concept of mixing length, which is the distance a fluid parcel will travel between parallel fluid layers as a cause of velocity fluctuations before exchanging momentum. The Reynolds stress, σ_{xy}, as a function of mixing length is, (Van Driest, 1956):

$$\sigma_{xy} = \rho l^2 \left(\frac{d\bar{u}}{dy} \right)^2, \tag{1}$$

where ρ is the fluid density, l is the mixing length and $\bar{u}$ is the mean velocity in x direction. For a flow over a surface, the mixing length is:

$$l = \kappa y, \tag{2}$$

where κ is the Kármán constant with the typical value of $\kappa = 0.4$ (Van Driest, 1956) and y is the normal distance to the nearest wall. Later, Van Driest (1956) modified the correlation to capture the logarithmic variation of the Reynolds stress in the boundary layer. Eq. (1) becomes:

$$\sigma_{xy} = \rho \kappa^2 y^2 \left[1 - e^{(-y/A)} \right]^2 \left(\frac{d\bar{u}}{dy} \right)^2, \tag{3}$$

where A is the Van Driest damping parameter and has the expression:

$$A = A^+ \nu \left(\tau_w / \rho \right)^{-1/2}, \tag{4}$$

where A^+ is a dimensionless constant (with value of 26 for this analysis), ν is kinematic viscosity and τ_w is wall shear stress. Using turbulent viscosity μ_t,

$$\mu_t = \rho \kappa^2 y^2 \left[1 - e^{(-y/A)} \right]^2 \left(\frac{d\bar{u}}{dy} \right) \tag{5}$$

the Eq. (3) can be expressed as:

$$\sigma_{xy} = \mu_t \left(\frac{d\overline{u}}{dy} \right). \tag{6}$$

3. Comparison Method

Direct numerical simulation data for a channel flow at different friction Reynolds numbers are used to compare it with theoretical data. The data from DNS is normalized with friction velocity u_τ and kinematic viscosity ν. The superscript "+" denotes the normalized variables:

$$y^+ = y\frac{u_\tau}{\nu},\ u^+ = \frac{u}{u_\tau},\ \frac{du^+}{dy^+} = \frac{du}{dy}\frac{\nu}{u_\tau^2}. \tag{7}$$

The wall shear stress is computed using the friction velocity:

$$\tau_w = \rho u_\tau^2. \tag{8}$$

Using Prandtl mixing length model, the mean velocity profile is computed by solving the momentum equation for the particular case of channel flow presented in Fig. 1. The channel width is 2δ and U is the mean cross-section velocity. With the following assumptions:

$$\overline{v} = \overline{w} = 0,\ \frac{\partial \overline{u}}{\partial x} = \frac{\partial \overline{u}}{\partial z} = 0,\ \frac{\partial \overline{p}}{\partial y} = \frac{\partial \overline{p}}{\partial z} = 0,\ \nu = const. \tag{9}$$

The momentum equation becomes:

$$\frac{1}{\rho}\frac{d\overline{p}}{dx} = \left(\nu - \nu_t\right)\frac{d^2\overline{u}}{dy^2} \tag{10}$$

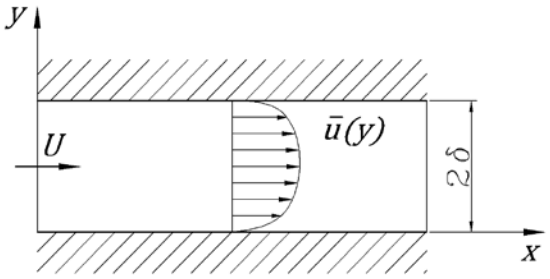


Fig. 1 – Channel geometry.

The boundary conditions are:

$$\bar{u}(0)=0,\ \frac{d\bar{u}}{dy}(\delta)=0. \tag{11}$$

Substituting turbulent viscosity ν_t with μ_t/ρ from Eq. (5), Eq. (10) becomes:

$$\frac{1}{\rho}\frac{d\bar{p}}{dx}=\frac{d}{dy}\left\{\nu\frac{d\bar{u}}{dy}-\kappa^2 y^2\left[1-\exp\left(\frac{-y}{A}\right)\right]^2\right\}. \tag{12}$$

Balancing the forces acting on the control volume, the mean pressure gradient $d\bar{p}/dx$ becomes, (Antonialli and Silveira-Neto, 2018):

$$\frac{d\bar{p}}{dx}=-\frac{u_\tau^2\rho}{\delta}. \tag{13}$$

Using the substitution (13), Eq. (12) is transformed into

$$\frac{u_\tau^2}{\delta}=\frac{d}{dy}\left\{\kappa^2 y^2\left[1-\exp\left(\frac{-y}{A}\right)\right]^2\left(\frac{d\bar{u}}{dy}\right)^2-\nu\frac{d\bar{u}}{dy}\right\}. \tag{14}$$

For easier comparison with the DNS data, Eq. (14) is normalized with parameters from (7). After simplifications, the normalized momentum equation is:

$$\frac{1}{\mathrm{Re}_\tau}=\frac{d}{dy^+}\left\{\kappa^2\left(y^+\right)^2\left[1-\exp\left(\frac{-y^+}{A^+}\right)\right]^2\left(\frac{du^+}{dy^+}\right)^2-\frac{du^+}{dy^+}\right\}, \tag{15}$$

and the normalized boundary conditions are:

$$u^+(0)=0,\ \frac{du^+}{dy^+}(\mathrm{Re}_\tau)=0. \tag{16}$$

Eq. (15) is integrated and the new form is

$$C+\frac{y^+}{\mathrm{Re}_\tau}=\kappa^2\left(y^+\right)^2\left[1-\exp\left(\frac{-y}{A}\right)\right]^2\left(\frac{du^+}{dy^+}\right)-\frac{du^+}{dy^+}. \tag{17}$$

Applying the boundary condition from (16), the constant of integration equals to -1.

$$\kappa^2\left(y^+\right)^2\left[1-\exp\left(\frac{-y}{A}\right)\right]^2\left(\frac{du^+}{dy^+}\right)-\frac{du^+}{dy^+}-\left(\frac{y^+}{\mathrm{Re}_\tau}-1\right)=0. \tag{18}$$

In terms of du^+/dy^+, Eq. (18) has two solution. Using the quadratic formula, the solutions are expressed as:

$$\left(\frac{du^+}{dy^+}\right)_{1,2}=\frac{1\pm\sqrt{1-4\kappa^2\left(y^+\right)^2\left[1-\exp\left(\frac{-y^+}{A^+}\right)\right]^2\left(\frac{y^+}{\mathrm{Re}_\tau}-1\right)}}{2\kappa\left(y^+\right)^2\left[1-\exp\left(\frac{-y^+}{A^+}\right)\right]^2}. \tag{19}$$

Comparing the $\left(du^+/dy^+\right)_{1,2}$ solutions with the DNS data, it is found that the correct solution is:

$$\frac{du^+}{dy^+}=\frac{2\left(\mathrm{Re}_\tau-y^+\right)}{\mathrm{Re}_\tau+\mathrm{Re}_\tau\sqrt{1-4\kappa^2\left(y^+\right)^2\left[1-\exp\left(\frac{-y}{A^+}\right)\right]^2\left(\frac{y^+}{\mathrm{Re}_\tau}-1\right)}}. \tag{20}$$

Eq. (20) is a first order, nonlinear, ordinary differential equation with no analytic solution. The solution was approximated using Euler method implemented in MATLAB, and a high accuracy was obtained by choosing a very small value for the Δy^+ step.

4. Results

The calculated normalized values of the mean velocity were compared with values from DNS database, (https://turbulence.oden.utexas.edu). The mean

velocity profile $\left(y^+ - u^+\right)$ is graphical represented from the wall to mid-channel, as well as for the inner region $\left(0 < y^+ < 200\right)$.

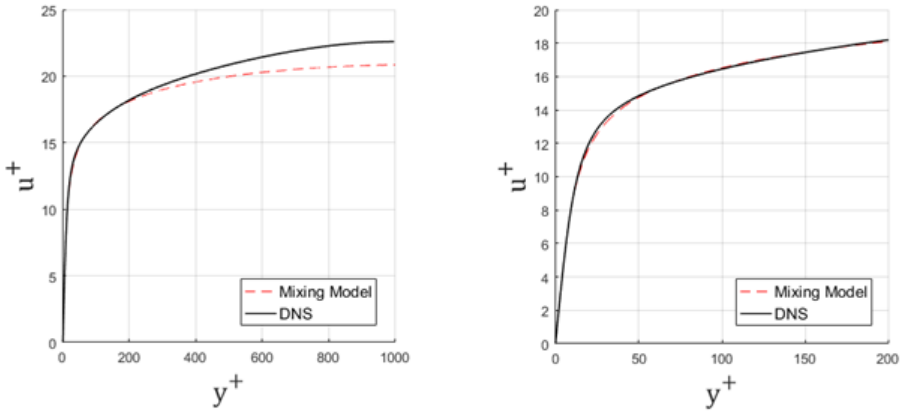


Fig. 2 – Mean velocity profile comparison, for $Re_\tau = 1000$ in (a) the 0-δ range and (b) the inner region.

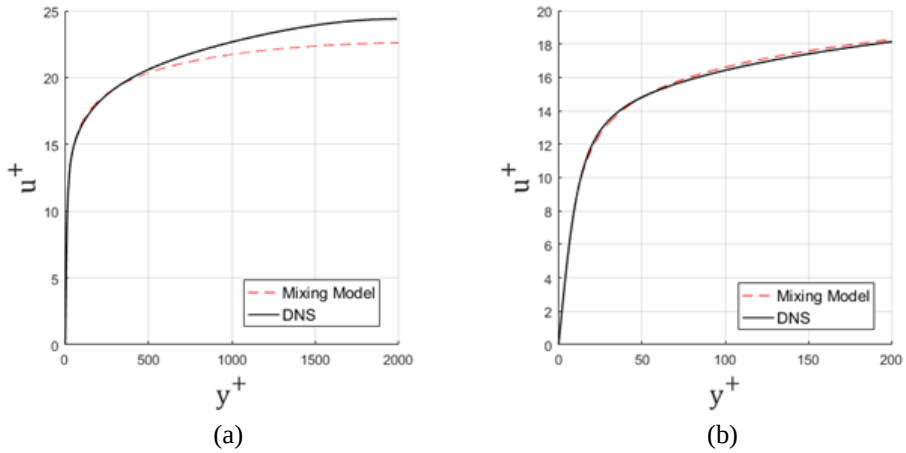
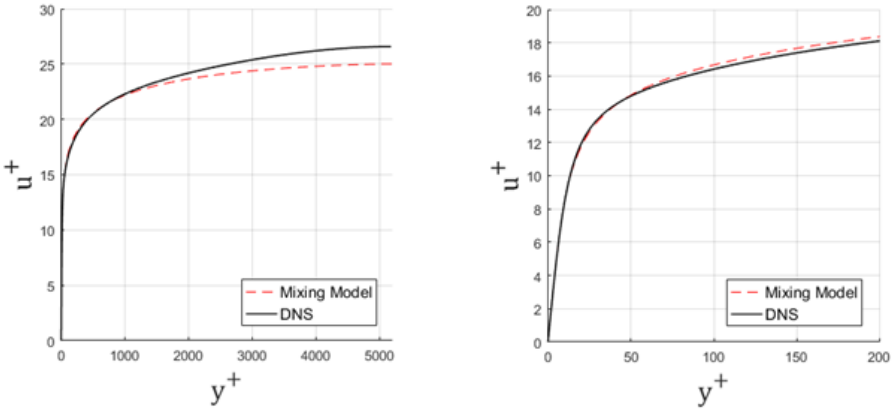


Fig. 3 – Mean velocity profile comparison in the 0-δ range (a) and in the inner region (b), for $Re_\tau = 2000$.



(a) (b)

Fig. 4 – Mean velocity profile comparison in the 0-δ range (a) and in the inner region (b), for $Re_\tau = 5200$.

As expected, the mixing length model provides good results, compared with DNS data in the inner region, close to the wall. It is observed that, even in the inner region, by increasing the friction Reynolds number, the theoretical mean velocity profile starts do drift away from the DNS data. Table 1 presents data obtained for an imposed maximum 5% acceptable error between the theoretical data and the DNS data.

Table 1

Maximum values for y/δ and y^+ at a maximum 5% error

Re_τ	y/δ	y^+
1000	*0.56*	*564*
2000	*0.58*	*1160*
5200	*0.7*	*3662*

5. Conclusion

The mixing-length model was developed during times when computer aided calculations were inexistent. Its simplicity permitted to model the turbulence effects in simple flows using the mathematical methods in early 20th century. Currently, the processing power of computers allowed to solve the RANS equations that use additional transport equation for quantities such as turbulent kinetic energy and turbulent dissipation, coupled with algorithms for wall treatment. It seems that the algebraic turbulence models have no practical use for today's engineering problems.

However, it's simplicity is an advantage because it may be easily implemented, as presented in this paper. The mixing length concept can be extended to other types of flow, as mixing length models is not a closed chapter, yet. Further research may transform the mixing-length models a useful engineering tool.

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COMPARAȚIE ÎNTRE MODELUL LUNGIMII DE AMESTEC PRANDTL ȘI DATE OBȚINUTE PRIN SIMULAREA NUMERICĂ DIRECTĂ PENTRU CURGEREA ÎNTR-UN CANAL CU RE_τ PÂNĂ LA 5200

(Rezumat)

Această lucrare prezintă un studiu teoretic al unei curgeri turbulente complet dezvoltate într-un canal, pentru Re_τ având valori de 1000, 2000 și 5200. Pentru ecuația de conservare a impulsului s-a folosit modelul RANS (Reynolds Averaged Navier-Stokes), iar pentru tensiunea turbulentă s-a folosit modelul lungimii de amestec Prandtl, cu modificarea lui Van Driest pentru includerea efectelor de amortizare a tensiunii turbulente în apropierea pereților. Profilul vitezei s-a determinat prin rezolvarea ecuației diferențiale cu derivate simple, neliniară, pentru conservarea impulsului. Rezolvarea s-a făcut folosind metoda Euler implementată în programul de calcul MATLAB. Rezultatele obținute au fost comparate cu ultimele simulări directe numerice existente în literatură.