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An index for measuring departure from an anti-sum-symmetry model for square contingency tables with ordered categories

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Summary

In the analysis of square contingency tables, which are two-way contingency tables in which the row and column variables consist of the same classification, statistical models regarding the symmetry of row and column variables are often used rather than the independence. This study proposes an index for measuring the degree of departure from the anti-sum-symmetry model. The proposed index is constructed using the Kullback–Leibler divergence. The anti-sum-symmetry model is useful to evaluate whether symmetric and asymmetric structures exist with respect to the anti-diagonal of the table. We derive the plug-in estimator and large-sample confidence interval for the proposed index. The usefulness of the proposed index is demonstrated by applying it to real data.

Key words: anti-diagonal cell, asymmetry, Kullback–Leibler divergence, plug-in estimator

1. Introduction

A square contingency table is a two-way contingency table in which the row and column variables consist of the same classification. In the analysis of two-way contingency tables, we are usually interested in the independence between the row and column variables, but in square contingency table analysis, we are often interested in the symmetry or asymmetry between them. This is because, in square contingency tables, the observed frequencies tend to be concentrated near or in the main diagonal cells of the table, and the row and column variables are often not independent. Models based on the assumption of symmetry of row and column variables include the

symmetry model (Bowker, 1948), the marginal homogeneity model (Stuart, 1955), and the sum-symmetry model (Yamamoto, Tanaka and Tomizawa, 2013; Ando, 2022). Models assuming the asymmetry of row and column variables include the conditional symmetry model (McCullagh, 1978), the extended marginal homogeneity model (Tomizawa, 1984; Tomizawa, 1993), and the conditional sum-symmetry model (Yamamoto et al., 2013; Ando, 2021a; Ando, 2021b).

Consider the data in Table 1. Tables 1a and 1b are the results of grip strength tests for men and women aged 15 to 69, respectively; source: National Health and Nutrition Examination Survey 2011–2012 (https://wwwn.cdc.gov/Nchs/Nhanes/2011-2012/MGX_G.htm). The row and column variables are the grip strength of the right and left hand, respectively, and they are categorized from the highest level (1) to the lowest level (5). These levels are determined based on the Muscle Strength Procedure Manual of the National Health and Nutrition Examination Survey.

Table 1. Data from grip strength test on men and women aged 15–69; source: National Health and Nutrition Examination Survey 2011–2012 (https://wwwn.cdc.gov/Nchs/Nhanes/2011-2012/MGX_G.htm)

	Left hand					
Right hand	(1)	(2)	(3)	(4)	(5)	Total
(a) For men						
Excellent (1)	215	124	46	14	2	401
Very good (2)	37	143	165	74	16	435
Good (3)	7	45	156	166	51	425
Fair (4)	2	20	62	226	210	520
Poor (5)	1	2	16	61	495	575
Total	262	334	445	541	774	2356
(b) For women						
Excellent (1)	380	229	49	17	2	677
Very good (2)	34	204	198	80	23	539
Good (3)	5	55	159	161	74	454
Fair (4)	1	11	37	106	168	323
Poor (5)	0	2	12	45	256	315
Total	420	501	455	409	523	2308

Ando (2021c, 2023) pointed out that for these data it may be appropriate to evaluate whether symmetric and asymmetric structures exist with respect to the anti-diagonal cells of the table rather than the main diagonal

cells of the table. This is because, for grip strength data such as Table 1, many people are right-handed (Intage group self-designed survey, 2023) and the grip strength of the dominant hand is usually higher than that of the non-dominant hand. Ando (2021c, 2023) proposed the anti-sum-symmetry (ASS) and anti-conditional sum-symmetry (ACSS) models, which have respectively symmetric and asymmetric structures for the anti-diagonal of the table.

When the ASS model does not fit the presented data well, analysts are interested in applying the ACSS model or in measuring the degree of departure from the ASS model. If the ACSS model fits the presented data well, then analysts can evaluate the degree of departure from the ASS model using the asymmetric parameter of the ACSS model. However, when the ACSS model does not fit the presented data well, analysts cannot make an evaluation. As the ACSS model does not fit the data set in Table 1a well (Ando, 2023), we are interested in proposing an index that can evaluate the degree of departure from the ASS model even when the ACSS model does not fit the presented data well.

When the ASS model does not fit well for several data sets, it is not appropriate to compare degrees of departure from the model using the values of the likelihood ratio chi-square statistic for several data sets with different sample sizes. This is because the test statistics depend on the sample size. The proposed index can appropriately compare degrees of departure from the ASS model even in these situations.

For the analysis of square contingency tables, many works have been done on indices for measuring the degree of departure from the model of interest. For example, studies on indices for measuring the degree of departure from the symmetry model include Tomizawa, Seo and Yamamoto (1998), Tomizawa, Miyamoto and Hatanaka (2001), Ando, Tahata and Tomizawa (2017), and Ando, Momozaki, Masusaki and Tomizawa (2024); those for the marginal homogeneity model include Tomizawa and Makii (2001), Tomizawa, Miyamoto and Ashihara (2003), Tahata, Iwashita and Tomizawa (2006, 2008), Tahata, Tajima and Tomizawa (2006), Iki, Tahata and Tomizawa (2012), Ando, Noguchi, Ishii and Tomizawa (2021), and Ando, Fujimoto and Tomizawa (2024); and a study for the sum-symmetry model is Yamamoto, Aizawa and Tomizawa (2016).

The remainder of this paper is organized as follows. Section 2 introduces the ASS model and the ACSS model, and proposes an index for measuring the degree of departure from the ASS model. In section 3 we derive the plug-

in estimator and large-sample confidence interval for the proposed index. In section 4 we demonstrate the usefulness of the proposed index by applying it to real data. Section 5 contains concluding remarks.

2. Proposed index for measuring departure from anti-sum-symmetry model

Consider $R \times R$ square contingency tables with ordered categories. Let X and Y be row and column variables, respectively, and $\Pr(X = i, Y = j)$ $(= \pi_{ij})$ be the cell probability that an observation will fall in the (i, j)th cell of the $R \times R$ contingency table $(i = 1, 2, \ldots, R; j = 1, 2, \ldots, R)$. Also let

$$\begin{aligned} A_t &= \Pr(X + Y = t, X + Y < R + 1) \quad \text{and} \\ B_t &= \Pr(X + Y = 2(R + 1) - t, X + Y > R + 1) \quad \text{for } t = 2, 3, \ldots, R, \end{aligned}$$

where the cells satisfying $X + Y = R + 1$ lie on the anti-diagonal of the table.

Ando (2021c) proposed the ASS model and the ACSS model defined by

$$A_t = B_t \quad \text{for } t = 2, 3, \ldots, R \quad \text{and} \quad A_t = \tau B_t \quad \text{for } t = 2, 3, \ldots, R,$$

respectively, where the parameter τ is unspecified. The asymmetric parameter τ shows the degree of departure from the ASS model. This is because, when τ is equal to one, the ACSS model is identical to the ASS model.

When the ASS model does not fit well for several data sets, we are interested in comparing the degrees of departure from the ASS model. It is not appropriate to compare degrees of departure from the model using values of the likelihood ratio chi-square statistic G^2 for data sets with different sample sizes. This is because the test statistics depend on the sample size. Furthermore, the ASS model imposes no restriction on the anti-diagonal cell probabilities. So it seems natural that indices for measuring degree of departure from the ASS model and their ranges do not depend on the anti-diagonal cell probabilities.

When the ACSS model fits well for several data sets, we can compare degrees of departure from the ASS model using the estimates of the asymmetry parameter τ. However, when the ACSS model does not fit well for one data set, this procedure may not be appropriate. Therefore, it is necessary to propose an index that can measure the degree of departure from the ASS model even when the ACSS model does not fit the presented data well.

Assume that $A_t + B_t > 0$ for $t = 2, \ldots, R$. We propose an index for measuring the degree of departure from the ASS model as follows:

$$\Phi = \frac{1}{\log 2} \min_{\{C_t^*, D_t^*\}} I\left(\{A_t^*, B_t^*\}; \{C_t^*, D_t^*\}\right),$$

where $C_t^* = D_t^*$ for $t = 2, 3, \ldots, R$, and

$$I\left(\{A_t^*, B_t^*\}; \{C_t^*, D_t^*\}\right) = \sum_{t=2}^{R} \left[A_t^* \log\left(\frac{A_t^*}{C_t^*}\right) + B_t^* \log\left(\frac{B_t^*}{D_t^*}\right) \right],$$

with

$$0 \log 0 = 0, \quad A_t^* = \tfrac{A_t}{\delta}, \quad B_t^* = \tfrac{B_t}{\delta}, \quad \delta = \textstyle\sum_{s=2}^{R} (A_s + B_s),$$

$$C_t^* = \tfrac{C_t}{\Delta}, \quad D_t^* = \tfrac{D_t}{\Delta}, \quad \Delta = \textstyle\sum_{s=2}^{R} (C_s + D_s),$$

$$C_t = \textstyle\sum_{\substack{i+j<R+1 \\ i+j=t}} q_{ij}, \quad D_t = \sum_{\substack{i+j>R+1 \\ i+j=2(R+1)-t}} q_{ij},$$

$$\textstyle\sum_{i=1}^{R} \sum_{j=1}^{R} q_{ij} = 1 \quad \text{and} \quad q_{ij} > 0 \quad \text{for } i, j = 1, 2, \ldots, R.$$

Namely, the index Φ indicates the minimum Kullback–Leibler divergence between $\{A_t^*, B_t^*\}$ and $\{C_t^*, D_t^*\}$ with the structure of the ASS model. When $C_t^* = D_t^* = 2^{-1}(A_t^* + B_t^*)$ for $t = 2, 3, \ldots, R$, the Kullback–Leibler divergence $I\left(\{A_t^*, B_t^*\}; \{C_t^*, D_t^*\}\right)$ is minimum. Therefore, the index Φ is also expressed as

$$\Phi = \frac{1}{\log 2} \sum_{t=2}^{R} \left[A_t^* \log\left(\frac{A_t^*}{2^{-1}(A_t^* + B_t^*)}\right) + B_t^* \log\left(\frac{B_t^*}{2^{-1}(A_t^* + B_t^*)}\right) \right].$$

The index Φ has the following properties: (i) the range of Φ is zero to one, (ii) the index Φ is equal to zero if and only if $A_t = B_t$ for $t = 2, 3, \ldots, R$ (i.e., the ASS model holds), and (iii) the index Φ is equal to one if and only if $A_t = 0$ and $B_t > 0$ or $A_t > 0$ and $B_t = 0$ for $t = 2, 3, \ldots, R$.

Under the ACSS model (i.e., $A_t = \tau B_t$ for $t = 2, 3, \ldots, R$), the index Φ is further expressed as

$$\begin{aligned}
\Phi &= \frac{1}{\log 2} \sum_{t=2}^{R} \left[\tau B_t^* \log\left(\frac{\tau B_t^*}{2^{-1}(\tau B_t^* + B_t^*)}\right) + B_t^* \log\left(\frac{B_t^*}{2^{-1}(\tau B_t^* + B_t^*)}\right) \right] \\
&= \frac{1}{\log 2} \sum_{t=2}^{R} \left[\tau B_t^* \log\left(\frac{\tau}{2^{-1}(\tau + 1)}\right) + B_t^* \log\left(\frac{1}{2^{-1}(\tau + 1)}\right) \right] \\
&= \frac{1}{(\tau + 1) \log 2} \left[\tau \log\left(\frac{2\tau}{\tau + 1}\right) + \log\left(\frac{2}{\tau + 1}\right) \right].
\end{aligned}$$

Figure 1 sketches the value of Φ corresponding to the value of τ in the range $0.1 \leqslant \tau \leqslant 10$ under the ACSS model. Under the ACSS model, the index Φ increases monotonically as τ approaches zero or infinity.

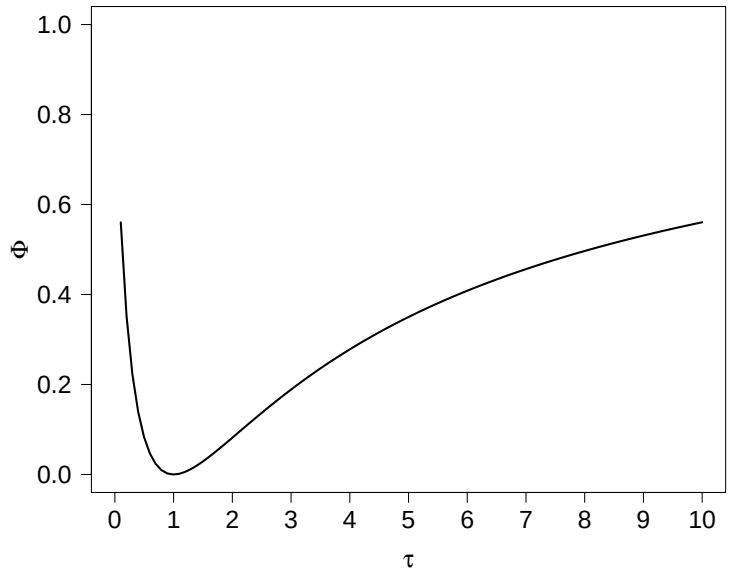

Figure 1. Shape of the index Φ corresponding to the value of τ in the range $0.1 \leqslant \tau \leqslant 10$ under the anti-conditional sum-symmetry model

Taking the limit as $\tau \to 0$ or $\tau \to \infty$, the index Φ is as follows:

$$\lim_{\tau \to 0} \Phi = 1 \quad \text{and} \quad \lim_{\tau \to \infty} \Phi = 1.$$

From Figure 1, even if the ACSS model does not fit the presented data well, the value of the index Φ can be interpreted by converting it to the degree of asymmetry (i.e., τ) under the ACSS model.

3. Estimator and confidence interval for the proposed index

Let x_{ij} and p_{ij} $(= N^{-1}x_{ij})$ be the observed frequency and sample proportion in the (i, j)th cell of the table, respectively, where N $(= \sum \sum x_{ij})$ is sample

size. Also let $\boldsymbol{\pi}, \boldsymbol{x}$ and $\boldsymbol{p}$ be $R^2 \times 1$ vectors as follows:

$$\begin{aligned}\boldsymbol{\pi} &= (\pi_{11}, \ldots, \pi_{1R}, \pi_{21}, \ldots, \pi_{2R}, \ldots, \pi_{R1}, \ldots, \pi_{RR})^\top, \\ \boldsymbol{x} &= (x_{11}, \ldots, x_{1R}, x_{21}, \ldots, x_{2R}, \ldots, x_{R1}, \ldots, x_{RR})^\top \quad \text{and} \\ \boldsymbol{p} &= N^{-1}\boldsymbol{x} = (p_{11}, \ldots, p_{1R}, p_{21}, \ldots, p_{2R}, \ldots, p_{R1}, \ldots, p_{RR})^\top,\end{aligned}$$

where the symbol "$\top$" denotes the transpose of a vector or matrix.

Assume that $\boldsymbol{x}$ is distributed in a multinomial distribution with parameters $(N, \boldsymbol{\pi})$. From Bishop, Fienberg and Holland (2007, section 14.6.3), $\sqrt{N}(\boldsymbol{p} - \boldsymbol{\pi})$ approaches asymptotically (as $N \to \infty$) the multivariate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{Diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^\top$, where $\mathbf{0} = (0, \ldots, 0)^\top$ is the $R^2 \times 1$ zero vector and $\mathbf{Diag}(\boldsymbol{\pi})$ is the diagonal matrix with the elements of $\boldsymbol{\pi}$.

Let $(\partial\Phi/\partial\boldsymbol{\pi}^\top)$ denote the $1 \times R^2$ vector whose sth entry is the partial derivative of Φ with respect to the sth component of $\boldsymbol{p}$ evaluated at $\boldsymbol{p} = \boldsymbol{\pi}$, that is,

$$\left(\frac{\partial\Phi}{\partial\boldsymbol{\pi}^\top}\right)_s = \left.\frac{\partial\Phi}{\partial p_s}\right|_{\boldsymbol{p}=\boldsymbol{\pi}},$$

where p_s is the sth component of $\boldsymbol{p}$. Then we obtain

$$\hat{\Phi} = \Phi + \left(\frac{\partial\Phi}{\partial\boldsymbol{\pi}^\top}\right)(\boldsymbol{p} - \boldsymbol{\pi}) + o(\|\boldsymbol{p} - \boldsymbol{\pi}\|),$$

where the plug-in estimator $\hat{\Phi}$ is given by Φ with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$, and is a consistent estimator.

Let G^2_{ASS} denote the likelihood ratio chi-square statistic for testing the goodness of fit of the ASS model. To many readers, it may seem that $N^{-1}G^2_{ASS}$ and $\hat{\Phi}$ (i.e., $(N^*)^{-1}G^2_{ASS}$ where $N^* = (2\log 2)\sum_{k+l\neq R+1} x_{kl}$) are reasonable indices for measuring the degree of departure from the ASS model. However, $\hat{\Phi}$ rather than $N^{-1}G^2_{ASS}$ would be useful for comparing the degrees of departure from the ASS model in several tables. This is because the range of $N^{-1}G^2_{ASS}$ depends on the anti-diagonal proportions, i.e., $0 \leqslant N^{-1}G^2_{ASS} \leqslant N^{-1}N^*$ $(= (2\log 2)(1 - N^{-1}\sum_{k+l=R+1} x_{kl}))$; however, $\hat{\Phi}$ always ranges between 0 and 1 without depending on the anti-diagonal proportions.

From the delta method (Bishop et al., 2007, section 14.6.3), $\sqrt{N}(\hat{\Phi} - \Phi)$ approaches asymptotically (as $N \to \infty$) a normal distribution with mean

zero and variance

$$\begin{aligned}\sigma^2[\hat{\Phi}] &= \left(\frac{\partial\Phi}{\partial\boldsymbol{\pi}^\top}\right)(\mathbf{Diag}(\boldsymbol{\pi})-\boldsymbol{\pi}\boldsymbol{\pi}^\top)\left(\frac{\partial\Phi}{\partial\boldsymbol{\pi}^\top}\right)^\top \\ &= \sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)^2+\left\{\sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)\right\}^2,\end{aligned} \tag{1}$$

where

$$\frac{\partial\Phi}{\partial\pi_{kl}}=\begin{cases}\dfrac{1}{\delta\log 2}\log\left(\dfrac{2A_{k+l}}{A_{k+l}+B_{k+l}}\right)-\dfrac{1}{\delta}\Phi, & \text{for } k+l<R+1,\\ 0, & \text{for } k+l=R+1,\\ \dfrac{1}{\delta\log 2}\log\left(\dfrac{2B_{2(R+1)-(k+l)}}{A_{2(R+1)-(k+l)}+B_{2(R+1)-(k+l)}}\right)-\dfrac{1}{\delta}\Phi, & \text{for } k+l>R+1.\end{cases}$$

The square root of the second term on the right-hand side of equation (1) can be expressed as follows:

$$\begin{aligned}\sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right) &= \sum_{k+l<R+1}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)+\sum_{k+l>R+1}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)\\ &= \sum_{t=2}^{R}\left[\sum_{\substack{k+l<R+1\\ k+l=t}}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)+\sum_{\substack{k+l>R+1\\ k+l=2(R+1)-t}}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)\right]\\ &= \Phi-\Phi=0.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\sigma^2[\hat{\Phi}] &= \sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)^2=\sum_{k+l<R+1}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)^2+\sum_{k+l>R+1}\pi_{kl}\left(\frac{\partial\Phi}{\partial\pi_{kl}}\right)^2\\ &= \sum_{k+l<R+1}\pi_{kl}\left[\frac{1}{\delta\log 2}\log\left(\frac{2A_{k+l}}{A_{k+l}+B_{k+l}}\right)\right]^2\\ &\quad+\sum_{k+l>R+1}\pi_{kl}\left[\frac{1}{\delta\log 2}\log\left(\frac{2B_{2(R+1)-(k+l)}}{A_{2(R+1)-(k+l)}+B_{2(R+1)-(k+l)}}\right)\right]^2-\frac{1}{\delta}\Phi^2.\end{aligned}$$

The asymptotic normality of $\sqrt{N}(\hat{\Phi}-\Phi)$ is not applicable when $\Phi = 0$ and $\Phi = 1$. This is because $\sigma^2[\hat{\Phi}]$ is equal to zero. Therefore, when the index Φ takes a value close to 0 or 1, the accuracy with which $\sqrt{N}(\hat{\Phi}-\Phi)$ approaches a normal distribution asymptotically (as $N \to \infty$) may become worse.

We will derive the large-sample confidence interval for Φ. The plug-in estimator $\widehat{\sigma^2[\hat{\Phi}]}$ is given by $\sigma^2[\hat{\Phi}]$ with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$. As the plug-in estimator $\widehat{\sigma^2[\hat{\Phi}]}$ converges in probability to $\sigma^2[\hat{\Phi}]$, $\sqrt{N}(\hat{\Phi}-\Phi)$ approaches asymptotically (as $N \to \infty$) a normal distribution with mean zero and variance $\widehat{\sigma^2[\hat{\Phi}]}$. Therefore, $\hat{\Phi} \pm z_{\alpha/2}\widehat{\sigma[\hat{\Phi}]}/\sqrt{N}$ is an approximate $100(1-\alpha)\%$ confidence interval for Φ, where $z_{\alpha/2}$ is the percentage point from the standard normal distribution that corresponds to a two-tail probability equal to α.

4. Application to real data

Consider the data set in Table 1, which presents a cross-classification of grip strength levels for right and left hands. These data are from a grip strength test on women aged 15–69.

Table 2 gives the values of the likelihood ratio chi-square statistic G^2 for each of the ASS model and the ACSS model. As the ASS model does not fit well for Table 1a and b, we are interested in comparing the degrees of departure from the ASS model.

Table 2. Values of the likelihood ratio chi-square statistic G^2 for each model applied to the data of Table 1

Applied models	Degrees of freedom	G^2
For the data set of Table 1a		
Anti-sum-symmetry	4	162.02*
Anti-conditional sum-symmetry	3	38.57*
For the data set of Table 1b		
Anti-sum-symmetry	4	43.98*
Anti-conditional sum-symmetry	3	3.50

* indicates significance at the 0.05 level.

Next, we consider applying the ACSS model, comparing degrees of departure from the ASS model using the asymmetry parameter τ of the ACSS

model. However, as the ACSS model does not fit well for Table 1a, this procedure cannot be applied. The proposed index Φ can compare degrees of departure from the ASS model even in these situations.

From Table 3, as the confidence intervals for Φ overlap, it is not possible to conclude that there is a significant difference in the degree of departure from the ASS model between the data sets of Table 1a and Table 1b.

Table 3. The estimate of the index Φ, the approximate standard error for $\hat{\Phi}$ (i.e., $\widehat{\sigma[\hat{\Phi}]}/\sqrt{N}$), and the approximate 95% confidence interval for Φ (i.e., $\hat{\Phi} \pm z_{\alpha/2}\widehat{\sigma[\hat{\Phi}]}/\sqrt{N}$) for the data in Table 1

Estimate	Standard error	Confidence interval (Lower limit, Upper limit)
For the data set of Table 1a		
0.0344	0.0106	(0.0136, 0.0552)
For the data set of Table 1b		
0.0095	0.0043	(0.0011, 0.0180)

5. Concluding remarks

This study has proposed the index Φ for measuring the degree of departure from the ASS model. The proposed index Φ would be useful when the analyst wants to measure using Kullback–Leibler divergence the degree of departure from the ASS model (i.e., $A_t = B_t \quad \text{for } t = 2, 3, \ldots, R$) toward the structure of maximum asymmetry of the ASS model (i.e., $A_t = 0$ and $B_t > 0$ or $A_t > 0$ and $B_t = 0$ for $t = 2, 3, \ldots, R$).

When the ASS model does not fit well for several data sets, we are interested in comparing the degrees of departure from the ASS model. It is not appropriate to compare degrees of departure from the model using the values of likelihood ratio chi-square statistic G^2 for data sets with different sample sizes. This is because the test statistics depend on the sample size. To many readers it may seem that $N^{-1}G^2_{ASS}$ and $\hat{\Phi}$ (i.e., $(N^*)^{-1}G^2_{ASS}$, where $N^* = (2\log 2)\sum_{k+l\neq R+1} x_{kl}$) are reasonable indices for measuring the degree of departure from the ASS model. However, $\hat{\Phi}$ rather than $N^{-1}G^2_{ASS}$ would be useful for comparing the degrees of departure from the ASS model in several tables. This is because the range of $N^{-1}G^2_{ASS}$ depends on the anti-diagonal proportions, i.e., $0 \leqslant N^{-1}G^2_{ASS} \leqslant N^{-1}N^*$ $(= (2\log 2)(1 -$

$N^{-1} \sum_{k+l=R+1} x_{kl}$)); however, $\hat{\Phi}$ always ranges between 0 and 1 without depending on the anti-diagonal proportions.

When the ACSS model fits well for several data sets, we can compare degrees of departure from the ASS model using the asymmetry parameter τ of the ACSS model. However, when the ACSS model does not fit well for one data set, this procedure may not be appropriate. From Figure 1, even if the ACSS model does not fit the presented data well, the value of the index Φ can be interpreted by converting it to the degree of asymmetry (i.e., τ) under the ACSS model.

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