

Existence, uniqueness, boundedness and stability of periodic solutions of a certain second-order nonlinear differential equation with damping and resonance effects

Everestus Obinwanne Eze¹, Uchenna Emmanuel Obasi¹,
Godwin Ezugorie², Enyiduru Ekwomchi Hannah¹

¹Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Umuahia, Nigeria, e-mails: obinwanneeze@gmail.com, sirurchobasi@gmail.com

²Department of Mathematics, Enugu State University Science and Technology Enugu, Nigeria, e-mail: hannahekwomchi173@gmail.com

SUMMARY

In this paper, some qualitative behaviors of solutions for certain second-order nonlinear differential equation with damping and resonance effects are considered. By employing Lyapunov's direct method, a complete Lyapunov function was used to investigate the stability of the system. Krasnoselskii's fixed point theorem was used to establish sufficient conditions that guaranteed the existence and boundedness of a unique solution. The results show that the equilibrium point was asymptotically stable. Furthermore, a test for periodicity was conducted using the Bendixson criterion, and the results showed that the solution of the second-order nonlinear differential equation is aperiodic, which extends some results from the literature.

Key words: Lyapunov function, Stability, Krasnoselskii's fixed point theorem, Periodic solutions, Duffing's equation

1. Introduction

Second-order nonlinear differential equations with damping are an essential tool for modeling physical problems that arise in different fields of endeavor, such as biology, chemistry, physics, mechanics, electronics, engineering, economics, atomic energy, and communication theory. Nonlinear differential equations are used to model damped and driven oscillators (Bender and Orszag, 1999), prediction of earthquake occurrence (Oyesanya, 2008), modeling of the brain (Zeeman, 1976), and crash analysis (Li and Li, 2014). Many authors have

proposed various methods in the literature discussing the qualitative behavior of solutions of nonlinear differential equations. In this area, mention can be made of the implicit function theorem, which confirms the existence of a unique solution, and the Lyapunov direct method, which involves the construction of a suitable positive definite function where the derivative of the positive definite quadratic form is negative definite.

The qualitative behaviors of solutions of differential equations of second order have been discussed extensively and are still receiving attention from researchers because of their practical applications to natural phenomena. For instance, see the works (Cartwright, 1956; Chen and Li, 2000; Eze et al., 2019a, 2019b, 2021; Habets and Metzen, 1989; Njoku and Omari, 2003; Pedro, 2004; Torres, 2000) and references cited therein.

The existence of solutions to ordinary differential equations is very much connected with the fixed point theorem. A number of cases arise in the literature where fixed point theorems have been used to generalize cases of nonlinear differential equations; see for instance the works (Ezeilo, 1974, 1975; Ressig, 1972, 1975; Ressig et al., 1974). On the use of the Leray–Schauder fixed point and integrated equation as a means to establish a priori bounds for nonlinear differential equations see (Ezeilo, 1960). One of the goals of the search for bounds of solutions to any differential equation is to achieve a global optimum of the solution in a closed and bounded domain. Many results on the boundedness of solutions of Duffing’s equation have been obtained; for instance see (Daxiong and Wenling, 2008; Ding, 1986; Lim and Kazda, 1964; Liu and Zanolin, 1998; Markus, 1969; Voitovich, 2011). Morris (1976) proved that each solution of the particular Duffing’s equation of the form:

$$\ddot{x} + 2x^3 = p(t) \tag{1}$$

is bounded for $t \in \mathbb{R}$. Osiogun *et al.* (2016) employed the implicit function theorem and Lyapunov functions to establish the existence and stability of periodic solutions for a second-order nonlinear differential equation. Furthermore, Torres (2003) discussed the existence of one-signed periodic solutions of second-order differential equations. Unfortunately, the problems of

existence, uniqueness, boundedness and asymptotic stability of periodic solutions of second-order nonlinear differential equations with damping and resonance effects are yet to be investigated. The aim of this paper is to extend the work found in (Osislogu et al., 2016). Consider

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = p(t) \quad (2)$$

where a, b, c are real constants and $p : [0, 2\pi] \rightarrow \mathbb{R}^n$ is continuous with

$$x(0) = x(2\pi)$$

$$\dot{x}(0) = \dot{x}(2\pi)$$

as periodic initial conditions. Here c is the damping coefficient and a is the resonance coefficient, and the dots indicate differentiation with respect to the independent variable t . Equation (2) is equivalent to the system of first-order differential equations

$$\dot{x} = y$$

$$\dot{y} = p(t) - cy - ax - bx^2 - 2x^3$$

where $p(t)$ is periodic, i.e. $[0, 2\pi] \rightarrow \mathbb{R}^+$ with $x(t + T) = x(t)$, and $a > 0, c > 0$ are constants on which appropriate conditions will be placed in order to solve equation (2). This work is motivated by study of the work in (Osislogu et al., 2016; Ezeilo, 1974). For the next section, we outline the mathematical tools that will be used in the paper. Our main results are stated and proved in section 3, and conclusions appear in section 4.

2. Preliminary results

Definition 2.1 (Stability): An equilibrium point x_e of a nonlinear system is said to be stable if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\bar{x} \in B(x_e, \delta) \Rightarrow \varphi(t, 0, \bar{x}) \in B(x_e, \varepsilon)$ for all $t \geq 0$.

Theorem 2.2 (Asymptotic Stability): Let $x = 0$ be an equilibrium point of the dynamical system $\dot{x} = f(x)$ and $x \in \Omega \subset \mathbb{R}^n$. Let $V: \Omega \rightarrow \mathbb{R}$ be continuously differentiable and

(i) $V(0) = 0$ and $V(x) > 0$ for all $x \in \Omega, x \neq 0$

(ii) $\dot{V}(x) \leq 0$, for all $x \in \Omega$

Then $x = 0$ is stable. Moreover if $\dot{V}(x) < 0$, for all $x \in \Omega, x \neq 0$ then $x = 0$ is asymptotically stable.

Definition 2.3. Consider the general nonlinear differential equation of the form

$$\dot{x} = f(t, x(t)) \quad \text{where } f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is continuous}$$

The function f in definition 2.3 is said to be T -periodic if for every $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and some $T > 0$, $f(t, x) = f(t + T, x)$ and $f(t, x) \neq f(t + T^*, x)$ for all $T^* < T$.

Theorem 2.4 (Bendixson criterion). Suppose that the domain $D \subset \mathbb{R}^2$ is simply connected. Then the equations $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ have a periodic solution if $\nabla \cdot (f, g) = 0$ in D .

Theorem 2.5 (Krasnoselskii's Fixed Point Theorem). Assume that F is a closed bounded convex subset of a Banach space X . Furthermore, assume that u_1 and u_2 are mappings from F into X such that the following conditions hold:

(i) $u_1 + u_2 \in F$,

(ii) u_1 is a contraction,

(iii) u_2 is continuous and compact.

Then $u_1 + u_2$ has a fixed point in F .

Theorem 2.6 (Eberlein-Šmulian). A subset of a Banach space X is relatively weakly compact if and only if it is relatively weakly sequentially compact. In particular, a subset of a Banach space X is weakly compact if and only if it is weakly sequentially compact.

Theorem 2.7 If $\{S(t); t \geq 0\}$ is a C_0 -semigroup then $\exists M \geq 1$ and $\alpha \in \mathbb{R}$ such that $\|S(t)\|_{L(X)} \leq M \exp(\alpha t)$ for each $t \geq 0$.

Proof: Since $\{S(t); t \geq 0\}$ is continuous. Suppose by contradiction, Let there exist a sequence $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ such that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, $\|S(t)\|_{L(X)} \rightarrow \infty$ as $n \rightarrow \infty$ then by the uniform boundedness principle, there exist $x \in X$ such that $(\|S(\alpha_n)x\|)_{n \in \mathbb{N}}$ is unbounded contradicting the fact that $S(t)$ is continuous at $t = 0$.

Theorem 2.8 (Gronwalls-Bellman's Inequality) Let f and g be continuous real-valued functions on some interval $[a, b]$, then

$$f(t) \leq K + \int_a^b f(s)g(s)ds$$

For some $K > 0$, It implies that $f(t) \leq K \exp[\int_a^b g(s)ds]$.

3. Results

We consider the Duffing equation of the form

$$\ddot{x} + c\dot{x} + ax + bx^2 + \beta x^3 = p(t)$$

Equation (2) can be rewritten in the following form:

$$\ddot{x} + c\dot{x} + ax = p(t) - bx^2 - \beta x^3 \quad (3)$$

The vectorization of equation (3) proceeds as follows.

Let $x = x_1$, $\dot{x}_1 = x_2$, $\ddot{x}_1 = \dot{x}_2$; then equation (3) gives

$$\ddot{x}_1 + c\dot{x}_1 + ax_1 = p(t) - bx_1^2 - \beta x_1^3 \quad (4)$$

An equivalent system to equation (4) is given by

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1 - cx_2 + p(t) - bx_1^2 - \beta x_1^3 \end{aligned} \right\} \quad (5)$$

In matrix form, equation (5) is written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & -c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ p(t) - bx_1^2 - \beta x_1^3 \end{pmatrix}$$

The above is of the form of a non-autonomous equation given by

$$\dot{\underline{x}}(t) = A\underline{x}(t) + g\underline{x}(t) \quad (6)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -a & -c \end{pmatrix}, \quad g\underline{x}(t) = \begin{pmatrix} 0 \\ p(t) - bx_1^2 - \beta x_1^3 \end{pmatrix} \text{ and } \underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The general solution of equation (6) is

$$\underline{x}(t) = \underline{C}_0 \exp(At) + \int_{t_0}^t \exp((t-s)A) g(s)ds \quad \forall t \in \mathbb{R} \quad (7)$$

or

$$x(t) = \underline{C_0} S(t) + \int_{t_0}^t s(t-u)g(u)du \text{ for } u \in \mathbb{R} \quad (8)$$

where $S(t)$ is a semigroup operator.

To generate a sequence of solutions in equation (7) we have that, for $t = t_1, t_2, \dots, t_k, \forall t_k \in \mathbb{R}$,

$$\begin{aligned} x(t_1) &= \underline{C_0} \exp(At_1) + \int_{t_0}^{t_1} \exp((t_1-s)A) g(s)ds \\ x(t_2) &= \underline{C_0} \exp(At_2) + \int_{t_0}^{t_2} \exp((t_2-s)A) g(s)ds \\ &\vdots \\ x(t_k) &= \underline{C_0} \exp(At_k) + \int_{t_0}^{t_k} \exp((t_k-s)A) g(s)ds \end{aligned}$$

Now assuming that $u_1(t)$ and $u_2(t)$ are solutions of equation (7), we have that

$$\begin{aligned} u_1(t) &= \underline{C_0} \exp(At_1) + \int_{t_0}^{t_1} \exp((t_1-s)A) g(s)ds \\ u_2(t) &= \underline{C_0} \exp(At_2) + \int_{t_0}^{t_2} \exp((t_2-s)A) g(s)ds \text{ for } t_1, t_2 \in \mathbb{R} \end{aligned}$$

Furthermore, we will show that $u_1(t) = u_2(t)$, giving uniqueness.

Suppose that $F \subset X$ where X is a Banach space and F is a subspace, and consider the convexity of its solutions. For $\alpha \in F$ say, then

Proof: Define a ball, $B = \{u_1(t) \in X: \|u_1(t)\| \leq 1\}$, which is convex. Then

$$\|\alpha u_1(t) + (1-\alpha)u_2(t)\| \leq \alpha + (1-\alpha) = 1 \in B$$

Hence the solution of the equation is convex.

Next, we verify the boundedness property of our solution. Let $u_1, u_2 \in X$; we show that $u_1 + u_2 \in F$

$$u_1(t_1) + u_2(t_2) = \underline{C_0} S(t_1) + \int_{t_0}^{t_1} S(t_1-u)g(u)du + \underline{C_0} S(t_2) + \int_{t_0}^{t_2} S(t_2-u)g(u)du$$

This is true by the uniform boundedness principle. Applying theorem 2.7 we have

$$\begin{aligned} u_1(t_1) + u_2(t_2) &= \underline{C_0} (S(t_1) + S(t_2)) + \int_{t_0}^{t_1} S(t_1-u)g(u)du + \int_{t_0}^{t_2} S(t_2-u)g(u)du \\ \|u_1(t_1) + u_2(t_2)\| &= \left\| \underline{C_0} (S(t_1) + S(t_2)) + \int_{t_0}^{t_1} S(t_1-u)g(u)du + \int_{t_0}^{t_2} S(t_2-u)g(u)du \right\| \\ &\leq \underline{C_0} \|S(t_1) + S(t_2)\| + \left\| \int_{t_0}^{t_1} S(t_1-u)g(u)du + \int_{t_0}^{t_2} S(t_2-u)g(u)du \right\| \end{aligned}$$

By the semigroup property,

$$S(t_1 - u) = S(t_1 - t_2) \cdot S(t_2 - u) \text{ and } \|S(t_1 - t_2)\| \leq 1. \quad (9)$$

We have

$$\begin{aligned} \|u_1(t_1) + u_2(t_2)\| &\leq \underline{C}_0 \|S(t_1) + S(t_2)\| + \left\| \int_{t_0}^{t_1} S(t_1 - t_2) \cdot S(t_2 - u) g(u) du + \right. \\ &\quad \left. \int_{t_0}^{t_2} S(t_2 - u) g(u) du \right\| \\ &\leq \underline{C}_0 \|S(t_1) + S(t_2)\| + \|S(t_1 - t_2)\| \int_{t_2}^{t_1} \|S(t_2 - u)\| \|g(u)\| du + \int_{t_0}^{t_2} \|S(t_2 - u)\| \|g(u)\| du \\ &\leq \underline{C}_0 \|S(t_1) + S(t_2)\| + M \int_{t_0}^{t_1} \|S(t_2 - u)\| du + M \int_{t_0}^{t_2} \|S(t_2 - u)\| du \\ &\leq \underline{C}_0 \|t_1 + t_2\| + M \left(\int_{t_0}^{t_1} \|S(t_2 - u)\| + \int_{t_0}^{t_2} \|S(t_2 - u)\| \right) du \\ &\leq \underline{C}_0 \|t_1 + t_2\| + M \left(\int_{t_0}^{t_1} \exp(\alpha(t_1 - u)) + \int_{t_0}^{t_2} \exp(\alpha(t_2 - u)) \right) du \end{aligned}$$

by theorem 2.7

$$= \underline{C}_0 \|t_1 + t_2\| + MK = \beta$$

where $K = \sup \left(\int_{t_0}^{t_1} \exp(\alpha(t_1 - u)) + \int_{t_0}^{t_2} \exp(\alpha(t_2 - u)) \right) du$, $\|u_1(t_1) + u_2(t_2)\| \leq \beta$. So we have boundedness for each $t_0, t_1, t_2 \in [0, \alpha]$ where $\alpha = \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$.

Next, we show that $u_1 = u_2$, that is, uniqueness of the solution.

But $u_1(t) = \underline{C}_0 S(t) + \int_{t_0}^t S(t - u) g(u) du$ where $S(t)$ is a semigroup

$$\begin{aligned} u_1(t) - u_2(t) &= \underline{C}_0 S(t) + \int_{t_0}^t S(t - u) g(u) du - (\underline{C}_0 S(t) + \int_{t_0}^t S(t - u) y(u) du) \\ &= \int_{t_0}^t S(t - u) g(u) du - \int_{t_0}^t S(t - u) y(u) du \\ &= \int_{t_0}^t S(t - u) [g(u) - y(u)] du \end{aligned}$$

Hence,

$$\begin{aligned} \|u_1(t) - u_2(t)\| &= \left\| \int_{t_0}^t S(t - u) [g(u) - y(u)] du \right\| \\ &\leq \int_{t_0}^t \|S(t - u) [g(u) - y(u)]\| du \\ &\leq \int_{t_0}^t \|S(t - u)\| \|g(u) - y(u)\| du \end{aligned}$$

and by theorem 2.7 and theorem 2.8 we have

$$\|u_1(t) - u_2(t)\| \leq 0$$

Therefore $u_1(t) = u_2(t)$, and since the uniqueness of the solution is satisfied, the closure is trivial. Hence $u_1 + u_2 \in F$.

Furthermore, we will show that $u_1(t)$ is a contraction, that is

$$\|u_1(t_1) - u_2(t_2)\| \leq K\|t_1 - t_2\| \text{ for } K \in [0,1), t_1, t_2 \in [0, \alpha]$$

From equation (8) we have that

$$u(t) = \underline{C}_0 S(t) + \int_{t_0}^t S(t-u)g(u)du \quad (10)$$

where $S(t)$ is a semigroup operator, $\underline{C}_0 > 0$, $S(t) = \exp(At)$ and $\|S(t_1 - t_2)\| \leq 1$.

Claim II: Equation (10) is a contraction

Proof:

$$\begin{aligned} u_1(t_1) - u_1(t_2) &= \underline{C}_0 S(t_1) + \int_{t_0}^{t_1} S(t_1-u)g(u)du - \underline{C}_0 S(t_2) + \int_{t_0}^{t_2} S(t_2-u)g(u)du \\ \|u_1(t_1) - u_1(t_2)\| &\leq \|\underline{C}_0 S(t_1) - \underline{C}_0 S(t_2)\| + \left\| \int_{t_0}^{t_1} S(t_1-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du \right\| \\ &= \underline{C}_0 \|S(t_1) - S(t_2)\| + \left\| \int_{t_0}^{t_1} S(t_1-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du \right\| \\ \|u_1(t_1) - u_1(t_2)\| &\leq \underline{C}_0 \|t_1 - t_2\| + \left\| \int_{t_0}^{t_1} S(t_1-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du \right\| \end{aligned}$$

Using equation (9), which has the same idea as Banach–Mazur distance in a super multiplicative metric space, we have

$$\begin{aligned} \|u_1(t_1) - u_1(t_2)\| &\leq \underline{C}_0 \|t_1 - t_2\| + \left\| \int_{t_0}^{t_1} S(t_1-t_2)S(t_2-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du \right\| \\ &= \underline{C}_0 \|t_1 - t_2\| + \left\| \int_{t_0}^{t_1} S(t_1-t_2)S(t_2-u)g(u)du + \int_{t_2}^{t_0} S(t_2-u)g(u)du \right\| \\ &= \underline{C}_0 \|t_1 - t_2\| + \left\| \int_{t_2}^{t_1} [S(t_1-t_2)S(t_2-u) + S(t_2-u)]g(u)du \right\| \\ &\leq \underline{C}_0 \|t_1 - t_2\| + \int_{t_2}^{t_1} \|S(t_1-t_2)S(t_2-u) + S(t_2-u)\| \|g(u)\| du \\ &\leq \underline{C}_0 \|t_1 - t_2\| + M \int_{t_2}^{t_1} \|S(t_1-t_2)S(t_2-u) + S(t_2-u)\| du \quad (11) \end{aligned}$$

where $\|g(u)\| \leq M$.

By theorem 2.7

$$\|u_1(t_1) - u_1(t_2)\| \leq \underline{C}_0 \|t_1 - t_2\| + 2M \int_{t_2}^{t_1} \|S(t_2-u)\| du$$

By theorem 2.8 equation (11) becomes

$$\|u_1(t_1) - u_1(t_2)\| \leq \underline{C}_0 \|t_1 - t_2\| \exp 2M\|t_1 - t_2\| \text{ where } t_1, t_2 \in [0, \alpha]$$

$$\begin{aligned}
 &= \underline{C_0} \|t_1 - t_2\| \\
 \|u_1(t_1) - u_1(t_2)\| &\leq \underline{C_0} \|t_1 - t_2\|
 \end{aligned} \tag{12}$$

Now take $\underline{C_0} = K \in [0,1)$; equation (12) becomes

$$\|u_1(t_1) - u_1(t_2)\| \leq K \|t_1 - t_2\| \tag{13}$$

Hence equation (13) is a contraction.

To show that u_2 is continuous, we proceed as follows. Recall that if $u_1 = u_2$ (uniqueness of the solution), then u_2 is also a contraction.

Given any $\varepsilon > 0 \exists \delta(\varepsilon)$ and $\|t_2 - t_1\| < \delta, \forall t_1, t_2 \in X$ such that $\|u_2(t_2) - u_2(t_1)\| < K\delta$. If $\delta = \frac{\varepsilon}{K+1}$ then

$$\|u_2(t_2) - u_2(t_1)\| \leq \frac{K\varepsilon}{K+1} < \varepsilon \tag{14}$$

By equation (14), $u_2(t_2)$ is continuous.

Finally we use theorem 2.6 to establish the compactness of u_2 . Assume that we are in a finite-dimensional space. Hein–Borel guarantees our result, since u_2 is closed and bounded hence compact. But since we are in an infinite-dimensional space (Banach space), by theorem 2.6, u_2 is sequentially compact, hence compact. Therefore $u_1 + u_2$ has a fixed point in F .

3.1. Stability analysis

Consider the general Duffing equation

$$\ddot{x} + c\dot{x} + g(t, x) = \lambda p(t) \tag{15}$$

When $g(t, x) = \lambda x + \beta x^3$, equation (15) becomes

$$\ddot{x} + c\dot{x} + \lambda x + \beta x^3 = \lambda p(t) \tag{16}$$

Equation (16) can be written as the equivalent system

$$\begin{aligned}
 \dot{x} &= y \\
 \dot{y} &= -cy - \lambda x - \beta x^3 + \lambda p(t)
 \end{aligned} \tag{17}$$

Define

$$2V = \frac{\delta}{\alpha c} \{[c^2 + c\lambda(\alpha + 1) + \alpha a\beta]x^2 + 2cxy + (\alpha + 1)y^2\} \quad (18)$$

where a, c, α, β and δ are all positive with $|x^2| \leq a$ and $\alpha = \lambda + a\beta$. Clearly $V(0,0) = 0$. Rearranging equation (18) we have

$$2V = \frac{\delta}{\alpha c} \{[c^2 + \alpha(\lambda c + a\beta)]x^2 + c\lambda x^2 + 2cxy + y^2 + \alpha y^2\}$$

On further simplification we have

$$\begin{aligned} 2V &= \frac{\delta}{\alpha c} \{[c^2 + \alpha(\lambda c + a\beta)]x^2 + c\lambda(x + y)^2 + (\alpha + 1 - \frac{c}{\lambda})y^2\} \\ 2V &= \frac{\delta}{\alpha c} \{[c^2 + \alpha(\lambda c + a\beta)]x^2 + c\lambda(x + y)^2 + (\frac{\lambda(\alpha+1)-c}{\lambda})y^2\} \end{aligned}$$

Clearly,

$$2V \geq \frac{\delta}{\alpha c} \{[c^2 + \alpha(\lambda c + a\beta)]x^2 + (\frac{\lambda(\alpha+1)-c}{\lambda})y^2\} \quad (19)$$

and using the Schwartz inequality $|xy| \leq \frac{1}{2}(x^2 + y^2)$, we have that

$$2V \leq \frac{\delta}{\alpha c} \{[c^2 + c\lambda(\alpha + 1) + \alpha a\beta + c]x^2 + (\alpha + 1 + c)y^2\} \quad (20)$$

From equation (19) and (20) we have for K_1 and K_2

$$K_1(x^2 + y^2) \leq V(x, y) \leq K_2(x^2 + y^2) \quad (21)$$

where $K_1 = \frac{\delta}{\alpha c} \times \min\{[c^2 + \alpha(c\lambda + a\beta)], (\frac{\lambda(\alpha+1)-c}{\lambda})\}$ and

$$K_2 = \frac{\delta}{\alpha c} \times \max\{[c^2 + c\lambda(\alpha + 1) + \alpha a\beta + c], (\alpha + 1 + c)\}$$

Inequality (21) shows the positivity of the function V . Differentiating equation (18) along the solution trajectories of system (17), we have

$$\begin{aligned} \dot{V} &= \frac{\delta}{\alpha c} \{[c^2 + c\lambda(\alpha + 1) + \alpha a\beta]x\dot{x} + c(x\dot{y} + \dot{x}y) + (\alpha + 1)y\dot{y}\} \\ \dot{V} &= \frac{\delta}{\alpha c} \{[c^2 + c\lambda(\alpha + 1) + \alpha a\beta]xy + cy^2 + [cx + (\alpha + 1)y](-cy - \lambda x - \beta x^3 + \lambda p(t))\} \end{aligned}$$

Simplifying using the conditions of the theorem, we have that when $p(t) \equiv 0$

$$\dot{V} = -\delta\{x^2 + y^2\} \leq 0$$

Hence the equilibrium point is asymptotically stable.

3.2. Response to damping

Setting $h(t) = 0$ in equation (2), the matrix form is given by $\dot{x} = Ax$ where $\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -a - bx - 2x^2 & -c \end{pmatrix}$ and $x = \begin{pmatrix} x \\ y \end{pmatrix}$. The characteristics equation is given by

$$\lambda^2 + \lambda c + z = 0$$

where $z = z(x) = a + bx + 2x^2$. The eigenvalues of equation (11) are given by $\lambda_1 = \frac{-c - \sqrt{c^2 - 4z}}{2}$ and $\lambda_2 = \frac{-c + \sqrt{c^2 - 4z}}{2}$. The eigenvalues depend on the damping term and resonance term, with no periodic coefficient. The general solution is given by $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. For different forms of damping, we consider the following cases.

Case I: When $c^2 - 4z < 0$, the damping term is small relative to z . As for the system, it will oscillate but will come to rest at equilibrium. The term under the square root is negative and the characteristic root becomes a complex root with a general solution $x(t) = c_1 e^{-\frac{ct}{2}} \cos(\mu t) + c_2 e^{-\frac{ct}{2}} \sin(\mu t)$, where $\mu = \sqrt{c^2 - 4z}/2$ is called the damped angular frequency of the system. At this point, the resonant frequency is approximately equal to the natural frequency of the system. This is the frequency of unforced vibration, which is given by $f_0 = \frac{\sqrt{a}}{2\pi}$. The system is able to store and easily transfer energy between two or more different storage modes, such as kinetic energy and potential energy. This happens when the resonance coefficient is greater than or equal to zero. Since the roots have nonzero imaginary parts, the system is underdamped.

Case II: When $c^2 - 4z > 0$, c is large relative to α . $c^2 - 4z$ is positive and the characteristic roots are real and distinct. In this case, the damping coefficient is high and the system will not oscillate. Since the characteristic roots are real and different, the system is overdamped.

Case III: When $c^2 - 4z = 0$, the damping coefficient is neither overdamped nor underdamped. The characteristic equation has a repeated root with general solution $x(t) = e^{-\frac{ct}{2}}(c_1 + c_2 t)$. This type of damping is called critical damping,

which produces the fastest return of the system to its equilibrium position. When $c = 0$, the remaining term is equal to zero and the response is not a sinusoid.

Let $f(x, y) = \dot{x}$ and $g(x, y) = \dot{y}$; then $\nabla \cdot (f, g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 - c = -c$ (negative definite). Hence by theorem 2.4, the Duffing equation has no periodic solution. This test confirmed that the nonlinear oscillators may be periodic in nature but do not obey periodic solutions.

3.3. Analysis of resonance term without damping

We examine the dynamics of the unforced system when $h(t) = 0$ in equation (1) and when $c = 0$ in equation (2). In equation (1) the coefficient of the cubic term is represented as β . The system described in equation (2) is conservative and all the orbits are described by an energy integral:

$$E(t) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}ax^2 + \frac{1}{2}x^4 = \text{constant}$$

where $E(t)$ is a constant on each orbit. Each such orbit is a closed oval, symmetric about the axis and the origin. In this case, the Duffing equation is a Hamiltonian system. The shape of $E(t)$ is shown in Fig. 1 for $a > 0$ and $a < 0$. The trajectory of $x = -(x, \dot{x})$ moves on the surface of $E(t)$ keeping $E(t)$ constant.

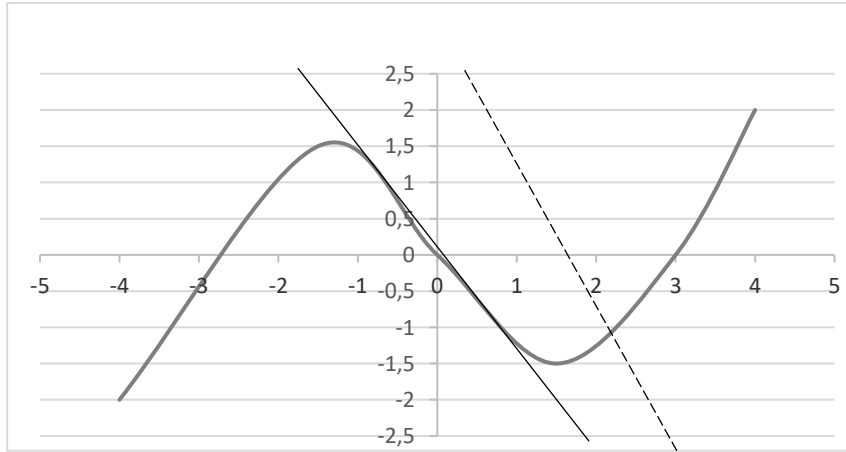


Figure 1. For $a > 0$, the Duffing oscillator can be interpreted as a forced oscillator with a nonlinear spring whose restoring force is written as $F = -ax - \beta x^3$

The amplitudes of the maximum velocity when $h(t)$ in equation (1) is decomposed into $c_1 \cos \mu t$ can easily be related to the energy

$$E(t) = \frac{1}{2}(\dot{x})^2 + \frac{1}{2}a(x)^2 + \frac{1}{2}(x)^4$$

When $E(t)$ is large, $\dot{x} = \sqrt{E(t)}$ and $x = \sqrt[4]{E(t)}$. Hence the orbit becomes increasingly elongated along the $\dot{x}$ -axis as $E(t)$ increases.

Case I: For $\beta > 0, a > 0, c > 0$ in equation (1) the only equilibrium is

$$\bar{x} \equiv (0,0)$$

and $E(t)$ satisfies

$$(i) E(t) = 0 \text{ if and only if } x = \bar{x}$$

$$(ii) E(t) > 0 \text{ if } x \neq \bar{x}$$

$$(iii) E(t) < 0 \text{ if } x \neq \bar{x}$$

Therefore, $E(t)$ is a Lyapunov function and $\bar{x}$ is globally asymptotically stable.

Case II: On the other hand, for $\beta > 0, a < 0, c > 0$ in equation (1), there are three equilibria as indicated below in Fig. 2,

(i) two of which are at the bottom of $E(t)$ and

(ii) one of which is at its peak. In this case, almost all the initial conditions converge to one of the equilibria at the bottom, except for the initial conditions on the stable manifold of the equilibrium at the peak.

The equilibria of the Duffing oscillator for $h(t) = A \cos kt$ or $A \sin kt$ in equation (1) when $b = 0, h(t) = 0$ or specifically where $k = 0$ can be obtained by substituting $\dot{x} = 0$ in equation (1), which gives

$$x(a + \beta x^2) = 0$$

Therefore the point $x = 0$ is always an equilibrium point.

Moreover, when $a + \beta x^2 = 0$, two equilibria are given by

$$x = \pm \sqrt{-a/\beta}$$

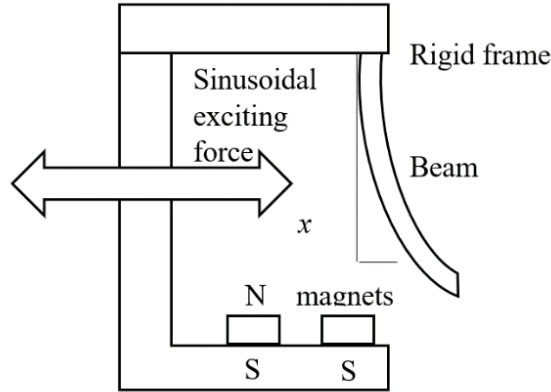


Figure 2. For $a < 0$, the Duffing oscillator can be regarded as a model of a periodically forced steel beam which is deflected towards two magnets

The stability of these equilibria can be understood by analyzing equation (1). For $b = 0$, $h(t) = 0$ or $k = 0$ equation (1) can be rewritten as

$$\frac{d}{dt} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -cy - ax - \beta x^3 \end{pmatrix}$$

and the Jacobian matrix $DF(x)$ of the right-hand side is calculated as

$$DF(x) = \begin{pmatrix} 0 & 1 \\ -ax - \beta x^2 & -c \end{pmatrix}.$$

Therefore, the eigenvalues of $DF(x)$ for the equilibrium $x = 0$ are

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4a}}{2}$$

and it is found that this equilibrium point is stable for $a \geq 0$ and unstable for $a < 0$.

Case III: On the other hand, the eigenvalues of the equilibria $x = \pm \sqrt{-a/\beta}$ are

$$\lambda = \frac{-c \pm \sqrt{c^2 - 8a}}{2}$$

and it is indicated that these equilibria were stable for $\beta > 0$ and $a < 0$ and unstable for $\beta < 0$ and $a < 0$.

4. Conclusion

The existence, uniqueness, boundedness, and stability of periodic solutions of second-order nonlinear differential equations with damping and resonance effects have been examined. An asymptotic stable equilibrium point was obtained using an appropriate Lyapunov function. Krasnoselskii's fixed point theorem was used to establish some qualitative properties of the differential equation using the general solution of the non-autonomous equation. The solution of the nonlinear differential equation was found to be aperiodic due to the presence of the damping term. The analysis of resonance showed that an increase in the resonance coefficient has no effect on the equilibrium point of the system, while a decrease in the resonance coefficient makes the system stable.

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