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Relative exponential sum-symmetry model and orthogonal decomposition of the sum-symmetry model for ordinal square contingency tables

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Summary

This study proposes a new exponential sum-symmetry model for square contingency tables with same row and column ordinal classifications. In the existing exponential sum-symmetry (ESS) model, the probability that the sum of row and column levels is t, where the row level is less than the column level, is Δ^{t-2} times higher than the probability that the sum of row and column levels is t, where the row level is greater than the column level. On the other hand, in the proposed ESS model, the ratio of these two probabilities is $\Delta^{t/3}$. In other words, in the existing ESS model, the ratio of the two probabilities varies exponentially depending on the absolute gap between t and 2, while in the proposed ESS model, the ratio of the two probabilities varies exponentially depending on the relative gap between t and 3, although in both ESS models, the ratio of the two probabilities is Δ when t is the minimum value (i.e., $t = 3$). Moreover, this study introduces a new decomposition theorem for the sum-symmetry model using the proposed ESS. The proposed decomposition theorem satisfies asymptotic equivalence for the test statistic.

Key words: absolute gap, asymmetry, asymptotic equivalence, relative gap, symmetry, test statistic

1. Introduction

Consider $R \times R$ square contingency tables with same row and column ordinal classifications. Let π_{ij} denote the probability that an observation will fall in the (i, j)th cell of the table, for $i = 1, \ldots, R; j = 1, \ldots, R$.

Bowker (1948) proposed the symmetry (S) model defined by

$$\pi_{ij} = \pi_{ji} \quad \text{for } i < j.$$

This model indicates the symmetric structure of the cell probabilities with respect to the main-diagonal cells of the table.

Agresti (1983) proposed the linear diagonals-parameter symmetry (LDPS) model defined by

$$\pi_{ij} = \delta^{j-i}\pi_{ji} \quad \text{for } i < j.$$

This model indicates the asymmetric structure of the cell probabilities with respect to the main-diagonal cells of the table. The LDPS model with $\delta^{j-i} = 1$, for all $i < j$, is identical to the S model. In the LDPS model, the odds π_{ij}/π_{ji}, for $i < j$, equal δ^{j-i}. In other words, in the LDPS model, the odds π_{ij}/π_{ji} vary exponentially depending on the absolute gap, that is the distance from the main diagonal (i.e., $j - i$), and vary exponentially depending on the relative gap between the distance from the main diagonal and its minimum (i.e., 1).

Let X and Y be the row and column variables, respectively. Further, let $T = X + Y$. We focus on the following probabilities:

$$A_t = \Pr(T = t, X < Y) = \sum_{(i,j)\in a_t}\sum \pi_{ij} \quad \text{and}$$
$$B_t = \Pr(T = t, X > Y) = \sum_{(i,j)\in b_t}\sum \pi_{ij} \quad \text{for } t = 3, 4, \ldots, 2R-1,$$

where

$$a_t = \{(i,j) \mid i + j = t, i < j\} \quad \text{and} \quad b_t = \{(i,j) \mid i + j = t, i > j\}.$$

Yamamoto et al. (2013) proposed the sum-symmetry (SS) model defined by

$$A_t = B_t \quad \text{for } t = 3, 4, \ldots, 2R-1.$$

Yamamoto et al. (2016) proposed the exponential sum-symmetry (ESS) model defined by

$$A_t = \Delta^{t-2}B_t \quad \text{for } t = 3, 4, \ldots, 2R-1.$$

The ESS model with $\Delta^{t-2} = 1$, for all $t = 3, 4, \ldots, 2R-1$, is identical to the SS model. In the ESS model, the odds A_t/B_t vary exponentially depending on the absolute gap between t and 2, and the odds A_t/B_t equal Δ when t is the minimum value (i.e., $t = 3$). However, as with the LDPS model,

the odds do not vary exponentially depending on the relative gap between t and 3.

This study proposes a model such that the odds A_t/B_t vary exponentially depending on the relative gap between t and 3. Moreover, this study introduces a new decomposition theorem for the SS model using the proposed model. The proposed decomposition theorem satisfies asymptotic equivalence for the test statistic.

The rest of this paper is organized as follows. Section 2 introduces a new model, and provides a decomposition theorem for the SS model. Section 3 demonstrates the utility of the proposed model and decomposition theorem by applying them to real data. Sections 4 and 5 contain discussion and concluding remarks, respectively.

2. Proposed model and decomposition of the SS model

This study proposes the relative ESS (RESS) model defined by

$$A_t = \Delta^{t/3} B_t \quad \text{for } t = 3, 4, \ldots, 2R-1.$$

The RESS model with $\Delta^{t/3} = 1$, for all $t = 3, 4, \ldots, 2R-1$, is identical to the SS model. In the RESS model, the odds A_t/B_t vary exponentially depending on the relative gap between t and 3, and the odds A_t/B_t equal Δ when t is the minimum value (i.e., $t = 3$).

Ando (2021a) introduced the global sum-symmetry (GSS) and weighted GSS (WGSS) models, defined by

$$\sum_{t=3}^{2R-1} A_t = \sum_{t=3}^{2R-1} B_t \quad \text{and} \quad \sum_{t=3}^{2R-1} (t-2) A_t = \sum_{t=3}^{2R-1} (t-2) B_t,$$

respectively. It must be noted that the GSS model is equivalent to the global symmetry proposed by Read (1977). Yamamoto et al. (2016) gave a decomposition theorem—the SS model holds if and only if both the ESS and GSS models hold; however, this decomposition theorem does not satisfy asymptotic equivalence for the test statistic. Thus, the value of the likelihood ratio chi-squared statistic of the SS model is not asymptotically equivalent to the sum of those of the ESS and GSS models. To address this issue, Ando (2021a) proposed a decomposition theorem—the SS model holds if and only if both the ESS and WGSS models hold. Additionally, Ando (2021a) showed

asymptotic equivalence for the test statistic—the value of the likelihood ratio chi-squared statistic of the SS model is asymptotically equivalent to the sum of those of the ESS and WGSS models.

We propose the sum mean equality (SME) model defined by

$$\sum_{t=3}^{2R-1} tA_t = \sum_{t=3}^{2R-1} tB_t.$$

This study introduces decomposition theorems for the SS model using the RESS model. We obtain the following lemma.

Lemma 2.1. *The SS model holds if and only if both the RESS and SME models hold.*

Proof. It is clear that the necessary condition is satisfied: If the SS model holds, then both the RESS and SME models hold. We need to establish that the sufficient condition also holds: If both the RESS and SME models hold, then the SS model holds.

Since both the RESS and SME models hold, we obtain the following equality:

$$\sum_{t=3}^{2R-1} t\Delta^{t/3}B_t = \sum_{t=3}^{2R-1} tB_t \Longleftrightarrow \sum_{t=3}^{2R-1} (\Delta^{t/3} - 1)tB_t = 0. \tag{1}$$

From the equality (1), it is necessary to satisfy $\Delta^{t/3} = 1$ for all $t = 3, 4, \ldots, 2R-1$. This is because the t and B_t are always positive. Therefore, we obtain $\Delta = 1$ (i.e., $A_t = B_t$ for $t = 3, \ldots, 2R-1$). The proof is complete. □

As for Lemma 2.1, we obtain the following lemmas.

Lemma 2.2. *The SS model holds if and only if both the RESS and GSS models hold.*

Lemma 2.3. *The SS model holds if and only if both the RESS and WGSS models hold.*

The numbers of degrees of freedom for testing the goodness-of-fit of the SS, ESS, RESS, GSS, WGSS and SME models are $2R-3$, $2R-4$, $2R-4$, 1, 1 and 1, respectively. It must be noted that the number of degrees of freedom for the SS model is equal to the sum of those for the RESS and SME models.

Let f_{ij} denote the observed frequency in the (i,j)th cell of the table $(i=1,\ldots,R; j=1,\ldots,R)$, with a sample size N $(=\sum\sum f_{ij})$. We assume multinomial sampling over the cells of the square contingency table. In other words, the observed frequencies $\{f_{ij}\}$ have a multinomial distribution with parameters that are cell probabilities $\{\pi_{ij}\}$.

Each model can be tested for goodness-of-fit by means of, for example, the likelihood ratio chi-squared statistic with the corresponding degrees of freedom. The likelihood ratio chi-squared statistic of the model M is given by

$$G^2(\mathrm{M}) = 2\sum_{i=1}^{R}\sum_{j=1}^{R} f_{ij}\log\left(\frac{f_{ij}}{\hat{e}_{ij}}\right),$$

where $\hat{e}_{ij}$ is the maximum likelihood estimate of the expected frequency e_{ij} under the model M.

Assume that the model M_1 holds if and only if both the models M_2 and M_3 hold and the following asymptotic equivalence holds:

$$G^2(\mathrm{M}_1) \simeq G^2(\mathrm{M}_2) + G^2(\mathrm{M}_3), \tag{2}$$

where the number of degrees of freedom for the model M_1 is equal to the sum of those for the models M_2 and M_3. Darroch and Silvey (1963) stated that (i) when the expression (2) holds, if both the models M_2 and M_3 are accepted (at the α significance level) with high probability, then the model M_1 will be accepted. However, (ii) when the expression (2) does not hold, it will most likely give rise to an incompatible situation wherein both the models M_2 and M_3 are accepted with high probability, but the model M_1 is rejected with high probability. Examples of this have been provided by Darroch and Silvey (1963) and Tahata et al. (2011). Therefore, it is preferable that the decomposition theorem satisfies the expression (2).

We obtain the following theorem.

Theorem 2.1. *The following asymptotic equivalence holds:*

$$G^2(\mathrm{SS}) \simeq G^2(\mathrm{RESS}) + G^2(\mathrm{SME}).$$

Proof. Let $\boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \ldots, \pi_{1R}, \ldots, \pi_{R1}, \pi_{R2}, \ldots, \pi_{RR})^T$, and let $\boldsymbol{A}^T$ denote the transpose of the matrix (or vector) $\boldsymbol{A}$. Under the RESS model,

$$\Delta = \frac{A_3}{B_3}.$$

Therefore, the RESS model is expressed as follows:

$$\boldsymbol{h}_1(\boldsymbol{\pi}) = \mathbf{0}_{2R-4},$$

where

$$\boldsymbol{h}_1(\boldsymbol{\pi}) = (h_{1,4}(\boldsymbol{\pi}), h_{1,5}(\boldsymbol{\pi}), \ldots, h_{1,2R-1}(\boldsymbol{\pi}))^T$$

with

$$h_{1,t}(\boldsymbol{\pi}) = (B_3)^{t/3} A_t - (A_3)^{t/3} B_t,$$

and $\mathbf{0}_d$ is a $d \times 1$ vector, with all components being zero.

The SME model is expressed as

$$\boldsymbol{h}_2(\boldsymbol{\pi}) = \mathbf{0}_1,$$

where

$$\boldsymbol{h}_2(\boldsymbol{\pi}) = \sum_{t=3}^{2R-1} tA_t - \sum_{t=3}^{2R-1} tB_t.$$

From Lemma 2.1, the SS model is expressed as

$$\boldsymbol{h}_3(\boldsymbol{\pi}) = \left(\boldsymbol{h}_1(\boldsymbol{\pi})^T, \boldsymbol{h}_2(\boldsymbol{\pi})^T\right)^T = \mathbf{0}_{2R-3}.$$

Let $\boldsymbol{H}_s(\boldsymbol{\pi})$ $(s = 1, 2, 3)$ denote the matrix of partial derivatives of $\boldsymbol{h}_s(\boldsymbol{\pi})$ with respect to $\boldsymbol{\pi}$ (i.e., $\boldsymbol{H}_s(\boldsymbol{\pi}) = \partial \boldsymbol{h}_s(\boldsymbol{\pi})/\partial \boldsymbol{\pi}^T$). Let $\boldsymbol{\Sigma}(\boldsymbol{\pi})$ be $\mathrm{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^T$, where $\mathrm{diag}(\boldsymbol{\pi})$ is a diagonal matrix, with the ith component of $\boldsymbol{\pi}$ as the ith diagonal element. We denote $\boldsymbol{p}$ as $\boldsymbol{\pi}$, with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$, where $p_{ij} = f_{ij}/N$. Using the delta method, $\sqrt{N}(\boldsymbol{h}_3(\boldsymbol{p}) - \boldsymbol{h}_3(\boldsymbol{\pi}))$ has an asymptotically normal distribution, with mean $\mathbf{0}_{2R-3}$ and covariance matrix

$$\boldsymbol{H}_3(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_3(\boldsymbol{\pi})^T = \begin{bmatrix} \boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_1(\boldsymbol{\pi})^T & \boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T \\ \boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_1(\boldsymbol{\pi})^T & \boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T \end{bmatrix}.$$

All the elements of $\boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T$ are equal to 0, under the SS model. This is because we have the following equalities:

$$\frac{\partial h_{1,t}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}^T} \mathrm{diag}(\boldsymbol{\pi}) \frac{\partial \boldsymbol{h}_2(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = 0 \quad \text{for } t = 4, \ldots, 2R-1,$$

$$\boldsymbol{\pi}^T \frac{\partial \boldsymbol{h}_2(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_{t=3}^{2R-1} tA_t - \sum_{t=3}^{2R-1} tB_t.$$

Therefore, we obtain

$$\begin{aligned}
&\boldsymbol{h}_3(\boldsymbol{\pi})^T[\boldsymbol{H}_3(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_3(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_3(\boldsymbol{\pi}) \\
&\quad = \boldsymbol{h}_1(\boldsymbol{\pi})^T[\boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_1(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_1(\boldsymbol{\pi}) \\
&\qquad + \boldsymbol{h}_2(\boldsymbol{\pi})^T[\boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_2(\boldsymbol{\pi}).
\end{aligned}$$

Since the Wald statistic is asymptotically equivalent to the likelihood ratio statistic—see, for example, Rao (1973, Sec. 6e. 3), Darroch and Silvey (1963), and Aitchison (1962)—we obtain Theorem 2.1. The proof is complete. □

From Theorem 2.1, we see that while Lemma 2.1 satisfies the expression (2), Lemmas 2.2 and 2.3 do not satisfy the expression (2). This is because the SME model is different from the GSS and WGSS models.

3. Application to data

We consider the dataset in Table 1, taken from Tan (2017, p. 78). Table 1 is a 4×4 square contingency table recording the right and left eye distance vision of 635 patients from an eye clinic. Yamamoto et al. (2013) stated that it is natural to evaluate an individual's vision level as the sum of the right and left eye grades for data such as those in Table 1.

Table 2 gives the values of G^2 for each of the SS, ESS, RESS, GSS, WGSS and SME models. This table shows that the RESS model fits well, but the other models fit poorly. We point out that the goodness of fit of the RESS model dramatically improves compared to the SS and ESS models.

Under the RESS model, the maximum likelihood estimate of the parameter Δ is 0.682. Thus, the maximum likelihood estimates of $\Delta^{t/3}$, for all of $t = 3, 4, 5, 6, 7$, are less than 1. Therefore, a patient's left eye grade is estimated to be better than his/her right eye grade.

We want to analyze the cause of the fact that the SS model does not fit the dataset in Table 1. From Lemma 2.1, we can infer that the SS model does not hold for the dataset in Table 1 because of the SME model rather than the RESS model.

From Table 2, we see that (i) the value of the likelihood ratio chi-squared statistic of the SS model (i.e., 52.817) is nearly equal to the sum of those of the RESS and SME models (i.e., 52.272), (ii) the value of the likelihood ratio chi-squared statistic of the SS model is less than the sum of those of the RESS and GS models (i.e., 58.225), and (iii) the value of the likelihood

Table 1. Dataset of right and left eye distance vision data of 635 patients from an eye clinic; source Tan (2017, p. 78).

	Left eye grades				
Right eye grades	(1)	(2)	(3)	(4)	Total
Highest grade (1)	50	21	24	35	130
	(50)	(25.537)	(29.997)	(37.567)	
(2)	42	22	12	42	118
	(37.463)	(22)	(12.880)	(39.338)	
(3)	56	32	52	16	156
	(50.003)	(30.886)	(52)	(10.449)	
Lowest grade (4)	67	82	20	62	231
	(64.667)	(84.662)	(25.552)	(62)	
Total	215	157	108	155	635

Note: Maximum likelihood estimates of expected frequencies under the RESS model are shown in parentheses in the second line.

Table 2. Values of the likelihood ratio chi-squared statistic (G^2) for the models applied to the data of Table 1.

Models	Degrees of freedom	G^2
SS	5	52.817*
ESS	4	13.288*
RESS	4	7.829
GSS	1	50.396*
WGSS	1	38.709*
SME	1	44.443*

Note: The symbol $*$ implies significance at the 5% level.

ratio chi-squared statistic of the SS model is greater than the sum of those of the RESS and WGSS models (i.e., 46.538).

4. Discussion

As a extension of the ESS model, Yamamoto et al. (2016) proposed the two-parameter sum-symmetry (TSS) model defined by

$$A_t = \Gamma \Delta^{t-2} B_t \quad \text{for } t = 3, 4, \ldots, 2R-1.$$

Ando (2021b) proposed a decomposition theorem—the SS model holds if and only if all of the TSS, GSS and WGSS models hold. Additionally, Ando (2021b) showed asymptotic equivalence for the test statistic—the value of

the likelihood ratio chi-squared statistic of the SS model is asymptotically equivalent to the sum of those of the TSS and DGSS models. A DGSS model is a model that simultaneously satisfies both the GSS and WGSS models.

In line with the relationship between the ESS and TSS models, we consider a extension of the RESS model, namely

$$A_t = \Gamma\Delta^{t/3}B_t \quad \text{for } t = 3, 4, \ldots, 2R-1.$$

This model is also expressed as

$$A_t = \Gamma_1\Delta_1^{t-2}B_t \quad \text{for } t = 3, 4, \ldots, 2R-1,$$

where $\Gamma_1 = \Gamma\Delta_1^2$ and $\Delta_1 = \Delta^{1/3}$. Therefore, this model is equivalent to the TSS model.

Additionally, the DGSS model is equivalent to a model that simultaneously satisfies both the GSS and SME models, although the SME model is different from the WGSS model. The WGSS model is also expressed as

$$\sum_{t=3}^{2R-1} tA_t + 2\sum_{t=3}^{2R-1} A_t = \sum_{t=3}^{2R-1} tB_t + 2\sum_{t=3}^{2R-1} B_t.$$

Therefore, under the GSS model (i.e., $\sum_{t=3}^{2R-1} A_t = \sum_{t=3}^{2R-1} B_t$), the WGSS model is equivalent to the SME model (i.e., $\sum_{t=3}^{2R-1} tA_t = \sum_{t=3}^{2R-1} tB_t$).

5. Concluding remarks

This study has proposed the RESS model, in which the odds A_t/B_t vary exponentially depending on the relative gap between t and 3, and the odds A_t/B_t equal Δ when t is the minimum value (i.e., $t = 3$). Additionally, this study has established a decomposition theorem, stating that the SS model holds if and only if both the RESS and SME models hold (Lemma 2.1), and has shown that the value of the likelihood ratio chi-squared statistic of the SS model is asymptotically equivalent to the sum of those of the RESS and SME models (Theorem 2.1).

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