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TOPOLOGICAL METHODS IN SIGNAL PROCESSING

TOPOLOŠKE METODE U PROCESIRANJU SIGNALA

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Abstract: This article gives an overview of the applications of algebraic topology methods in signal processing. We explain how the notions and invariants such as (co)chain complexes and (co)homology of simplicial complexes can be used to gain insight into higher-order interactions of signals. The discussion begins with some basic ideas in classical circuits, continues with signals over graphs and simplicial complexes, and culminates with an overview of sheaf theory and the connections between sheaf cohomology and signal processing.

Keywords: signal processing, graph, simplicial complex, chain complex, homology, cohomology, sheaf, Laplacian, spectral analysis

Sažetak: Ovaj članak daje pregled primjena metoda algebarske topologije u procesiranju signala. Objašnjavamo kako se pojmovi i invarijante kao sto su (ko)lančani kompleksi i (ko)homologija simplicijalnih kompleksa mogu koristiti za razumijevanje interakcija signala višeg reda. Diskusija počinje sa nekim osnovnim idejama klasičnih strujnih kola, nastavlja se sa signalima defisanim na grafovima i simplicijalnih kompleksa, a kulminira pregledom teorije snopova i veze između kohomologije snopa i obrade signala.

Ključne riječi: procesiranje signala, graf, simplicijalni kompleks, lančani kompleks, homologija, kohomologija, snop, Laplaceova matrica, spektralna analiza

INTRODUCTION

Signal processing is concerned with incorporating and interpreting a set of measurements into coherent and useful information about a system. These measurements are typically taken over time or space, but, in recent years, it has become clear that it is desirable to understand signals over more complicated structures. Neural, social, and sensor networks are just some of the examples where signals interact and depend on each other in more complicated ways than the standard theory could accommodate.

To deal with this, one direction in which the theory developed was *graph signal processing* where the signal is now measured over the vertices of a graph and the edges are meant to encode the interactions between those signals. Despite its success in fields such as sensor networks, biological networks, image processing, and machine learning (a comprehensive review of developments and challenges in graph signal processing can be found in [18]), this theory had a shortcoming, and that was that the edges of a graph could only encode pairwise relations between the signals. Possible triple or higher oder interactions could not be accounted for in a system with such a simple underlying structure.

The natural next step, and a subject or much recent investigation, is to study signals over *simplicial complexes* [3], [4], [12]. These are generalizations of graphs where, rather than using just vertices and edges, one also puts together triangles, tetrahedra, and their high-dimensional generalizations (simplices) to create an object called a simplicial complex. Such an object is both geometric and combinatorial, and this dual nature endows it with a rich structure that can be exploited from many points of view. Simplicial complexes are an excellent framework for studying signals with multiple interactions. For example, if signals over, say, three vertices interact in some way, that can be represented with a triangle as the underlying structure. For four vertices, the representation is a tetrahedron, etc.

Bringing simplicial complexes into the picture is also where it becomes useful to bring in emloy topology. Topology has a lot to say about simplicial complexes (and more general spaces) since it can extracts those features that are unchanged under and independent of deformations. The notion of *homology* is particularly useful in this context as it captures the existence of essential "holes" in the space, and its dual, *cohomology*, carries even more structure. As we will demonstrate, these concepts turn out to have remarkable connections to some standard tools from signal processing such as the Laplacian, which is crucial in spectral signal analysis.

It might seem counterintuitive that topology, which is known for its lack of geometric rigidity, would be useful in signal processing which is often constrained by geometry.

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But in many real-life situations, relationships between nearby signals are what matters most, and topology does pay attention to local information. Furthermore, topology is also able to put together local information into a global picture via its invariants, and such approach is precisely the novelty that topology is introducing into signal processing.

This expository paper aims to provide just the highlights of the role of topology in signal processing. We barely scratch the surface but provide ample references for further reading. We progress both historically and mathematically, skipping the details and rigorous proofs for brevity and readability.

In Section 2, we begin by introducing the basic notions from topology - spaces, maps, homeomorphisms, simplicial complexes, chain complexes, and (co)homology. Section 3 then treats some classical results about circuits in the language of (co)homology. This is meant to indicate that using algebraic topology in electrical engineering, and signal processing in particular, is not a new notion and that it has existed, albeit in somewhat of a background role, for some time.

The following three sections are natural gradual generalizations: We move from particular kinds of signals on circuits to arbitrary signals defined over a graph in Section 4. This is where we review some of the main features of the spectral study of the graph Laplacian. We then move to signals over simplicial complexes in Section 5, where topology really starts to come into play. In particular, the relationship between spectral analysis and cohomology of the underlying simplicial complex is elucidated.

The final generalization is that to the language of sheaves. Sheaf theory allows for the possibility that signals are not single, but multiple measurements at each vertex. They are also designed to facilitate the passage from local to global information. Sheaf-theoretic language allows for a vast unication of many of the concepts found in signal processing. The main reference for this part of the paper is [23].

1. A BRIEF INTRODUCTION TO TOPOLOGY

In this section, we give an extremely brief introduction to the basic notions of (algebraic) topology. Some standard texts on the subject are [1], [11], [17].

1.1. Topological Spaces and Maps

A topological space X is a set endowed with a structure of *open subsets*, namely a collection of subsets of X that contains the empty set and X itself, and is closed under arbitrary unions and finite intersections.

If a set X has a metric d on it, then the metric can be used to induce a topology on X by first defining an *open ball of radius* ϵ of a point $x \in X$ as the set consisting of all points $y \in X$ such that $d(x; y) < \epsilon$ and then using all open balls as basis for a topology on X (so open sets of X are all unions of all possible balls).

Example 1.1. The n-dimensional Euclidean space $\mathbb{R}^n$ has the standard topology induced by the usual Euclidean distance function.

A subset A of a topological space X is a *subspace* if it is endowed with the topology where open sets are of the form $A \cap U$ where U is open in X. This is the *induced topology* on A.

Example 1.2. The n-dimensional sphere S^n and the n-dimensional torus $S^1 \times \cdots \times S^1$ (where "$\times$" stands for the usual cartesian product of sets) are thought of as subsets of $\mathbb{R}^{n+1}$ and are topologized as such.

A function $f : X \to Y$ between topological spaces X and Y is *continuous* if the preimage of an open set in Y is an open set in X (i.e. f pulls back open sets to open sets). For a metric space and the induced topology, this coincides with the usual definition of continuity from analysis.

A continuous function between topological spaces is called a *map*.

Topology is the study of those properties of spaces that are invariant under continuous deformations. Depending on the context, there are several ways to define a deformation, but the most basic one is a *homeomorphism*; a map $f : X \to Y$ that is a bijection and has a continuous inverse. If there is a homeomorphism between X and Y, then those spaces are said to be homeomorphic and we write $X \cong Y$.

A weaker notion than a homeomorphism is *homotopy equivalence*; instead of requiring that f has an inverse, we only ask that there be a map $g : X \to Y$ such that $f \circ g$ and $g \circ f$ are *homotopic* to the identity map (rather than equal to the identity map). This means that there exist a one-parameter family of maps starting with $f \circ g$ (resp. $g \circ f$) and ending with the identity map on Y (resp. X). We write $X \simeq Y$ if X and Y are homotopy equivalent.

Spaces that are homeomorphic or homotopy equivalent are thought of as the same from the topological point of view. Here are some basic examples.

Example 1.3.

- A circle and a square are homeomorphic.
- A sphere S^n with a point removed is homeomorphic to $\mathbb{R}^n$.
- If $m \neq n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.
- An open interval in $(a, b) \subset \mathbb{R}$ and $\mathbb{R}$ are homeomorphic.
- Euclidean spaces $\mathbb{R}^n$ are homotopy equivalent for all $n \geq 0$ ($\mathbb{R}^0$ is a one-point space).
- The standard torus $S^1 \times S^1$ with a point removed is homotopy equivalent to two circles touching at a point (figure-8).

1.2. Simplicial Complexes

One issue with the general definitions above is that making concrete computations with them is dificult. One way to get around this is to restrict attention to spaces that look like (i.e. are homeomorphic to) spaces that are put together from triangles, tetrahedra, and their generalizations. This turns out to cover most spaces that are ordinarily encountered in topology (and certainly in any practical applications of topology). At the same, this point of view affords a combinatorial approach to their study.

To make this more precise, we make the following definition.

Definition 1.4. For $n \geq 0$, a (*geometric*) *n-simplex* Δ^n is the convex set spanned by $n + 1$ affinely independent points $\{v_0, v_1, ..., v_n\}$ in $\mathbb{R}$ for some $N \geq n$. Independence means that the points are not contained in any hyperplane, i.e. the vectors $v_i - v_n$, $1 \leq i \leq n$ are linearly independent.

A 0-simplex Δ^0 is thus a point, 1-simplex Δ^1 is a line segment, 2-simplex Δ^2 is a (filled-in) triangle, 3-simplex is a (solid) tetrahedron, etc. An n-simplex is said to have *dimension n*.

The definition does not specify how the $n + 1$ points are embedded in $\mathbb{R}^N$ but this does not matter since all n-simplices are homeomorphic.

A simplex is topologized as a subspace of $\mathbb{R}^N$.

A *face* of an n-simplex is the convex set spanned by some subset of its vertices. A tetrahedron thus has 4 triangle faces, 6 line segment faces, and 4 vertex faces.

Definition 1.5. A (geometric) simplicial complex X consists of a collection of simplices such that

- If a simplex is in X, so is its every face;
- Simplices in X intersect only in common faces.

Figure 1 gives examples of simplicial complexes. Note that the simplices in a simplicial complex need not be of the same dimension. Even though the top figure depicts a disconnected space, a simplicial complex will for our purposes always be connected. Thus a simplicial complex can be thought of as some number of tetrahedra of various dimensions glued together along common faces.

The *dimension* of a simplicial complex X is the highest dimension of a simplex appearing in X. A connected simplicial complex of dimension 1 is thus a collection of edges that meet along some vertices, i.e. it is precisely a graph, as illustrated in the bottom picture of Figure 1.

For a fixed n, the collection of n-simplices of X together with their faces is called the n-skeleton of X. This is a subcomplex of X.

Many spaces are homeomorphic to simplicial complexes. For example, Figure 2 exhibits the circle S^1 as the simplicial complex consisting of three 1-simplices (any number of 1-simplices could have been used, as long as they form a polygon) and S^2 as the simplicial complex consisting of four 2-simplices (triangles) put together into a (hollow) tetrahedron.

All the information, up to homeomorphism, about a simplicial complex, is contained in the following combinatorial information: If the vertices of X are labeled $V = \{v_0, v_1, ..., v_k\}$, then each n-simplex in X can be labeled by some subset of those vertices, denoted by $[v_{i0}, ..., v_{in}]$. Conversely, specifying subsets of V tells us precisely how to build the simplicial complex X. For example, if we are given the subset $[v_2, v_4, v_7]$, then we can say that $[v_2, v_4, v_7]$ is a 2-simplex in X.

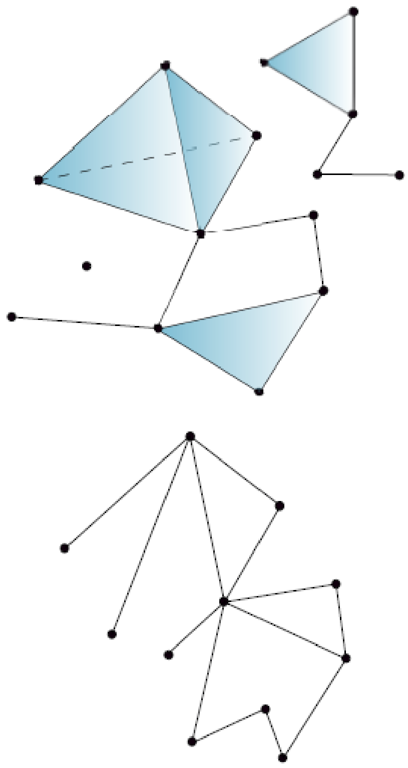


Figure 1: Examples of a simplicial complex of dimension 3 (top) and of dimension 1 (bottom). A connected 1- dimensional complex is a graph.

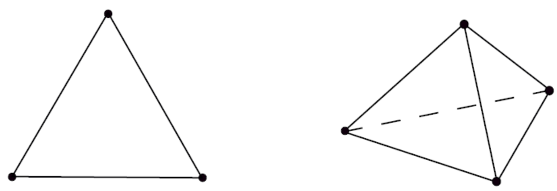


Figure 2: Simplicial complexes homeomorphic to the circle S^1 and the 2-sphere S^2.

If $[v_{i0}, ..., v_{in}]$ is a simplex, then all subsets of $\{v_{i0}, ..., v_{in}\}$ are also simplices because of the requirement that, if a simplex is in X, so is its every face. Intersections of subsets that represent simplices also have to represent simplices because of the requirement that simplices must meet along common faces. We can thus make the following definition, in parallel to Definition 1.5.

Definition 1.6. An (*abstract*) *simplicial complex* X is a set of points (vertices) $V = \{v_0, v_1,..., v_n\}$ and a collection of subsets of V called simplices such that

- The singletons $\{v_i\}$, $0 \leq i \leq k$, are in the collection;
- If a subset is in the collection, so are all its subsets;
- If two subsets are in the collection, so is their intersection.

It should be clear that an abstract complex determines a geometric complex up to homeomorphism; it suffices to select k affinely independent points in $\mathbb{R}^N$ and proceed to fill in the simplices according to the recipe of the subsets that are supplied by the abstract complex. One can of course go the other way as well, and simply just retain the sets describing the vertex structure of the simplices in a geometric complex while forgetting its geometry. The two notions of geometric and abstract complexes are thus interchangeable.

Built into the simplex notation $[v_{i0}, ..., v_{in}]$ is the order in which vertices are listed. A choice of such an order is called an *orientiation*. If we list the elements differently, then we will say that the orientation is the same if the new list differs from the original list by an even permutation of vertices. Otherwise the orientation is different. There are thus always two possible orientations of a simplex.

A simplicial complex is completely determined by how its faces t together. More formally, generalizing the notion of the adjacency matrix for graphs, we have the following construction: Given a simplicial complex X of dimension d, the *incidence or connection matrices* $\mathbf{A}_i$, $0 \leq i \leq d$, associated to X are the matrices whose entries are

$$\mathbf{A}_n(\alpha,\beta) = \begin{cases} 0, & \text{if } \Delta_\alpha^{n-1} \text{ is not a face of } \Delta_\beta^n \\ 1, & \text{if } \Delta_\alpha^{n-1} \text{ is a face of } \Delta_\beta^n \text{ and orientations match} \\ -1, & \text{if } \Delta_\alpha^{n-1} \text{ is a face of } \Delta_\beta^n \text{ and orientations do not match} \end{cases} \tag{1}$$

We can then consider the Laplacians of X:

$$\begin{aligned} &\mathbf{L}_0 = \mathbf{A}_1\mathbf{A}_1^T; \\ &\mathbf{L}_n = \mathbf{A}_n\mathbf{A}_n^T + \mathbf{A}_{n+1}\mathbf{A}_{n+1}^T, \quad 1 \leq n \leq d-1; \\ &\mathbf{L}_d = \mathbf{A}_d\mathbf{A}_d^T. \end{aligned} \tag{2}$$

The Laplacians completely capture the structure of X. Their eigenvalues are very informative and the spectral analysis of the Laplacians is hence very useful. We will say more about this in the following sections.

1.3. Chain Complexes and Homology

At the broadest level, algebraic topology attempts to develop *invariants*, namely algebraic objects that can be associated to spaces in such a way that, if two spaces are homeomorphic or homotopy equivalent, these invariants are the same. The most basic topological invariants are *homotopy* and *homology* groups. The former are easy to define: The *nth homotopy group of a space* X, $\pi_n(X)$, $n \geq 0$, is the set of homotopy equivalence classes of maps $S^n \to X$. However, these groups are notoriously difficult to compute.

On the other hand, homology groups $\mathrm{H}_n(X)$, $n \geq 0$, are harder to define but easier to compute. These are more useful for our purposes. The intuition behind them is that they measure the structure of "holes" in a space. For example, $\mathrm{H}_0(X)$ counts the number of components of X, $\mathrm{H}_1(X)$ counts the number of "circular holes" of X, and $\mathrm{H}_2(X)$ gives the number of "2-spherical holes" of X. In general, $\mathrm{H}_n(X)$ keeps track of "n-dimensional holes" of X.

We will give a brief description of the definition of homology for simplicial complexes and will make a comment about how to define it for general spaces, although we will not need the more general definition. For details, see, for example, [11, Chapter 2].

Definition 1.7. Suppose $n \geq 0$. Let $\mathrm{C}_n(X;\mathbb{R})$, called the group of *$n$-chains* of X, be the free group generated by the open n-simplices of X (which we will by abuse of notation also denote by Δ^n). An n-chain, i.e. an element of $\mathrm{C}_n(X;\mathbb{R})$, is thus a formal finite sum

$$\sum_\alpha a_\alpha \Delta_\alpha^n \tag{3}$$

where $a_\alpha \in \mathbb{R}^N$.

There is a map

$$\partial_n : \mathrm{C}_n(X;\mathbb{R}) \to \mathrm{C}_{n-1}(X;\mathbb{R}) \tag{4}$$

called the *boundary operator* defined as follows: For a simplex $[v_0, ..., v_n]$,

$$\partial_n(|v_0,\ldots,v_n|) = \sum_{i=0}^n (-1)^i [v_0,\ldots,v_{i-1},v_{i+1},\ldots v_n].$$

Thus, for example, for a 2-simplex $[v_0, v_1, v_2]$,

$$\partial_2([v_0,v_1,v_2]) = [v_1,v_2] - [v_0,v_2] + [v_1,v_2]$$

The boundary operator can be extended linearly to $\mathrm{C}_n(X{:}\mathbb{R})$, and this is how the map (4) is finally defined.

It is a simple combinatorial exercise to show that the boundary of a boundary is zero, i.e. we have

Proposition 1.8.

$$\partial_{n-1} \circ \partial_n = 0 \tag{5}$$

A shortened notation for this result is $\partial^2 = 0$. This makes sense intuitively since, for example, the boundary of a 2-simplex (triangle) is the three edges of the triangle, but that closed path of three edges does not itself have a boundary.

It is also not hard to see that, in terms of the incidence matrices (1), equation (5) can be restated as

$$\mathbf{A}_{n-1}\mathbf{A}_n = 0 \tag{6}$$

Because $\partial^2 = 0$, the collection of n-chains and boundary maps forms *a chain complex*

$$\begin{aligned} &\ldots \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X;\mathbb{R}) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X;\mathbb{R}) \to \ldots \\ &\ldots \xrightarrow{\partial_2} \mathrm{C}_1(X;\mathbb{R}) \xrightarrow{\partial_1} \mathrm{C}_0(X;\mathbb{R}). \end{aligned} \tag{7}$$

Note that the condition $\partial^2 = 0$ also means that the image of each boundary map is necessarily contained in the kernel of the next one, i.e.

$$\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n),$$

where ker stand for the kernel and im for the image. Elements of the image of ∂_{n+1} are called n-*boundaries* and those in the kernel of ∂_n are called n-*cycles*.
We are now finally ready to define homology.

Definition 1.9. For X a simplicial complex, the *nth homology group of X* is the quotient group

$$\mathrm{H}_n(X:\mathbb{R}) = \frac{n-\text{cycles}}{n-\text{boundaries}} = \frac{\ker(\partial_n)}{im(\partial_{n+1})}$$

The interpretation of this is that $\mathrm{H}_n(X;\mathbb{R})$ records n-dimensional holes in the sense that those n-chains that look like they bound something, but they do not, appear in the homology, i.e. they are not "killed" in the quotient.

For example, the closed path of three edges that bounds a 2-simplex would not get recorded in $\mathrm{H}_1(X;\mathbb{R})$ since that path has an "inside". But if that path appears just as a path of 1-simplices without the presence of a 2-simplex in its interior, that becomes a non-trivial element of $\mathrm{H}_1(X;\mathbb{R})$. This is because there is an essential "1-dimensional hole" that is bounded by those 1-simplices.

Each homology group $\mathrm{H}_n(X;\mathbb{R})$ is isomorphic to the vector space $\mathbb{R}^r$ for some $r \geq 0$. This number r is called the *rank* of the homology group, or the nth *Betti* number B_n. The rank records the number of holes of the appropriate dimension. The rank of the zeroth homology group is the number of connected components of X.

It should be clear from the definitions that, if X is a d-dimensional complex, $\mathrm{H}_d(X;\mathbb{R}) = 0$ since such a complex does not have nontrivial chains of dimension higher than d.

Example 1.10. Looking back at Figure 2, which up to homeomorphism depicts the circle S^1 and the sphere S^2, we have

$$\mathrm{H}_0(S^1;\mathbb{R}) = \mathbb{R}, \quad \mathrm{H}_1(S^1;\mathbb{R}) = \mathbb{R}$$
$$\mathrm{H}_0(S^2;\mathbb{R}) = \mathbb{R}, \quad \mathrm{H}_1(S^2;\mathbb{R}) = \mathbb{R} \quad \mathrm{H}_2(S^2;\mathbb{R}) = \mathbb{R}$$

Homology also has an interpretation in terms of the incidence matrices (1). Namely, it turns out that the Laplacians (2) contain all the homology information. Recall that the nth Betti number B_n is the rank of $\mathrm{H}_n(X;\mathbb{R})$.

Theorem 1.11. We have

$$B_n = \dim(\ker(\mathbf{L}_n)) \tag{8}$$

where ker stands for the usual kernel (or nullspace) of the Laplacian matrix $\mathbf{L}_n$ and dim is the dimension of this subspace.

To define the homology groups for an arbitrary space that is not a simplicial complex, we take all possible maps

$$\sigma^n : \Delta^n \to X$$

as the generating set for the n-chains (i.e. replace Δ^n_α by σ^n_α in the definition of $C_n(X:\mathbb{R})$).

Example 1.12. Figure 3 shows a (hollow) torus T^2 whose homology groups are

$$\mathrm{H}_0(T^2;\mathbb{R}) = \mathbb{R}, \quad \mathrm{H}_1(T^2;\mathbb{R}) = \mathbb{R}^2$$
$$\mathrm{H}_2(T^2;\mathbb{R}) = \mathbb{R}, \quad \mathrm{H}_{\geq 3}(T^2;\mathbb{R}) = 0$$

The two circles that can be taken as generators of $\mathrm{H}_1(T^2;\mathbb{R})$ are also pictured. Each of them bounds a 1-dimensional hole that is essential to the topology of T^2.

There is a dual notion of *cohomology* of a space X which associates to it a graded ring. It is often more computable than homology and carries more structure. First define the *group of n-cochains of X* as

$$\mathrm{C}^n(X:\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_n(X:\mathbb{R}), \mathbb{R}) \tag{9}$$

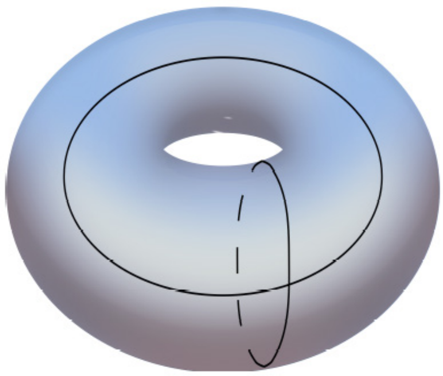


Figure 3: A torus T^2 showing the two generators of the first homology.

namely the group of linear functions from $\mathrm{C}_n(X:\mathbb{R})$ to $\mathbb{R}$. In short, this is the *dual group* of the *n*-chains. Since these are linear maps, it suffces to define an $f \in \mathrm{C}^n(X:\mathbb{R})$ on the simplices of a chain. The boundary ∂_n is replaced by the natural dual coboundary map

$$\partial^n : C^n(X;\mathbb{R}) \to C^{n+1}(X;\mathbb{R})$$

Definition 1.13. For X a simplicial complex, the *nth cohomology group of* X is the quotient group

$$\mathrm{H}^n(X:\mathbb{R}) = \frac{n-\text{cocycles}}{n-\text{coboundaries}} = \frac{\ker(\partial^n)}{im(\partial^{n-1})} \tag{10}$$

Since homology and cohomology of a simplicial complex are finitely generated, the ranks of these groups are the same for each n. In other words, the Betti numbers are the same for homology and cohomology, and we will interchangeably use this notation and terminology. In short,

$$B_n = \mathrm{rank}(\mathrm{H}_n(X;\mathbb{R})) = \mathrm{rank}(\mathrm{H}^n(X;\mathbb{R}))$$

As is always the case with duality, there is a natural bilinear pairing between homology and cohomology given by

$$\mathrm{H}^n(X;\mathbb{R}) \times \mathrm{H}_n(X;\mathbb{R}) \to \mathbb{R}$$

$$\left(f, c = \sum_\alpha a_\alpha \Delta_\alpha^n\right) \to \langle f, c\rangle = f(c) \tag{11}$$

$$= \sum_\alpha a_\alpha f(\Delta_\alpha^n)$$

This will be useful in the next section.

2. CLASSICAL CIRCUITS AND HOMOLOGY

The idea that the language of (co)homology is useful in electrical engineering is not new. In fact, some basic ideas about classical circuits can be expressed in this way. This notion appears as early as 1923 [28] and is by now well-established and covered in a number of standard sources, such as [2], [8], [14], [26].

For example, if one thinks of a circuit as a 1-dimensional complex, i.e. a graph, Kirchhoff's voltage and current laws can be restated as saying that voltage is a 1-coboundary and current is a 1-cycle.

To explain a little, regard an elementary circuit as a connected, oriented (or directed) graph X. In other words, X is an oriented connected 1-dimensional simplicial complex consisting of the set of vertices (or nodes) $V = \{v_1,..., v_n\}$ (the 0-simplices), and edges (or branches, representing conducting wires) $E = \{e_1,..., e_m\}$ (the 1-simplices).

A 1-chain $I \in C_1(X:\mathbb{R})$ is called the *current* and a 1-cochain $V \in \mathrm{C}^1(X:\mathbb{R})$ is called the voltage. Unravelling what this means, I is really the sum

$$I = \sum_{\alpha=1}^{n} a_\alpha e_\alpha$$

where a_α is the current on the edge e_α (with appropriate signs reflecting the orientation of the edges).

On the other hand, V is the sum

$$V = \sum_{\alpha=1}^{n} b_\alpha e'_\alpha$$

where b_α is the voltage on the edge e_α and e'_α is the cochain which, by duality, equals 1 when evaluated on e_α and it equals zero on all other edges (see (11)).

Recall that we have (co)boundary operators

$$\partial_1 : \mathrm{C}_1(X;\mathbb{R}) \to \mathrm{C}_0(X;\mathbb{R})$$
$$\partial^0 : \mathrm{C}^0(X;\mathbb{R}) \to \mathrm{C}^1(X;\mathbb{R}).$$

We then have

Theorem 2.1.

- **Kirchhoff's First Law:** *Voltage V is a 1- coboundary, namely there is a 0-cochain P such that*

$$V = \partial^0 P$$

- **Kirchhoff's Second Law:** *Current I is a 1-cycle, namely*

$$\partial_1 I = 0$$

The first law thus says that there exists a functional P on the vertices of X (linearly extended to formal sums of vertices) whose coboundary is the voltage. This functional is called the potential. The second law says that the currents along the edges at each vertex add up to zero.

One can also regard resistance as a function

$$r : E \to \mathbb{R}_+$$

that defines an inner product on $C_1(X:\mathbb{R})$, i.e. a function $C_1(X:\mathbb{R}) \times C_1(X:\mathbb{R}) \to \mathbb{R}$ by linearly extending the assignment

$$\langle e_i, e_j\rangle = \begin{cases} r(e_i), & i \neq j; \\ 0, & i = j. \end{cases}$$

But an inner product like that defines a map

$$R : C_1(X;\mathbb{R}) \to \mathrm{Hom}(C_1(X;\mathbb{R}), \mathbb{R}) = C^1(X;\mathbb{R})$$
$$c \to R_c$$

where R_c is the function

$$R_c : C_1(X;\mathbb{R}) \to \mathbb{R}$$
$$c' \to \langle c, c'\rangle$$

This is of course the usual bilinear pairing of a vector space with its dual.

In particular, we can see where the current $I \in C_1(X:\mathbb{R})$ is sent under this map. Recall that the voltage V is an element of $C^1(X:\mathbb{R})$. The situation where

$$V = R(I) \quad \text{or} \quad V = RI$$

precisely corresponds to *Ohm's Law*.

With this line of investigation, various other results about classical circuits can easily be reframed and investigated.

3. SIGNALS OVER GRAPHS

The story of classical circuits and homology can be naturally generalized in a number of directions. As mentioned in the Introduction, one can work in the more general framework of signals defined over a graph, i.e. a 1-dimensional complex, rather than restricting the attention to voltage and current. A rich field of graph signal processing (GSP) arose from this more general framework where a graph is the structure over which data is defined and signals measurements are taken at each vertex. Vehicle movement, internet traffic, cell phone measurements, IoT sensor measurement, and social media relations are just some of the examples. In fact, a special case of GSP is the standard discrete time signal processing (DPS) where a signal is adjacent to exactly two neighbors. This can be modeled by taking the graph that is a cycle (a closed loop of vertices and edges). Other familiar situations fall under the umbrella of GSP; for example, a graph that is a rectangular lattice grid models digital imaging, etc.

We now very briefly review some of the most salient features of GSP to motivate the generalization to simplicial complexes. More details can be found in [18], [20], [27].

Given a graph (V,E) where V is the set of vertices and E the set of edges as before, a *signal* is simply a function

$$s^0 : V \longrightarrow \mathbb{R} \quad (\text{or } \mathbb{C}) \tag{12}$$

(the reason for the superscript in s^0 will become clear in the next section). If the signal at vertex i is s_i, then one can represent s as a vector or as a formal sum, i.e.

$$(a_0,\ldots,a_n) \leftrightarrow \sum_{i=0}^{n} a_i v_i \tag{13}$$

But, according to (3) and (9), the second representation is simply a 0-cochain on X. Signals on X thus precisely correspond to 0-cochains on X, and GSP can thus in a sense be regarded as the study of $C^0(X;\mathbb{R})$.

Now recall the notion of the incidence matrices of a simplicial complex from (1). In case of a graph, we only have $\mathbf{A}_1$, which is of couse the usual adjacency matrix. Another matrix that is relevant is the *degree matrix* $\mathbf{D}_1$, a diagonal matrix whose ith diagonal entry is the degree, or valence, of v_i (taken with signs according to the orientation of edges at v_i). It is not hard to see that the Laplacian $\mathbf{L}_0 = \mathbf{A}_1\mathbf{A}_1^T$ from (2) is also equal to

$$\mathbf{L}_0 = \mathbf{A}_1 - \mathbf{D}_1$$

One common modification in signal processing is that edges might be assigned weights. In that case, the entries of the adjacency matrix $\mathbf{A}_1$ carry the positive and negative weight values instead of the usual ±1. The weight might reflect the actual distance between signal sources, or the expected similarity between sources, or some other information. For the sake of brevity, we will ignore the weighted case, although not much would change in the rest of the discussion if we did not.

The spectral study of $\mathbf{L}_0$ is one of the cornerstones of GSP. Namely, consider the eigenvalue decomposition

$$\mathbf{L}_0 = \mathbf{U}_0 \Lambda_0 \mathbf{U}_o^T \tag{14}$$

where $\mathbf{U}_0$ is the orthonormal matrix of eigenvectors and Λ_0 is the diagonal matrix of eigenvalues. The eigenvalues are important in the study of the clustering of the graph. Clustering is essentially a phenomenon that has to do with connected components. From the point of view of (co)homology, clustering is thus precisely detected by $\mathrm{H}^0(X;\mathbb{R})$, as the rank of the zeroth (co)homology gives the number of connected components of a space.

From this, one can define the *graph Fourier transform* (*GFT*) of the signal s as

$$\mathbf{X}_0 = \mathbf{U}_o^T s. \tag{15}$$

The vector $\mathbf{X}_0$ represents the projections of the signal s onto the eigenvectors of the Laplacian. I.e. GFT is a decomposition of the diagonal onto the eigenvectors of Λ_0. This is analogous to the classical Fourier transform that decomposes the signal onto an orthogonal basis of eigenfunctions (for more details, see, for example [25]).

Solving for s in the above (and using $U_0^{-1} = U_0^T$) gives

$$s = \mathbf{U}_0\mathbf{X}_0$$

This can be thought of as an expansion of the signal in terms of the eigenvalues of the Laplacian.

It is interesting to observe that, even though a Fourier transform is a tool from analysis, in this context it also depends on the topological structure of the underlying graph X.

4. SIGNALS OVER SIMPLICIAL COMPLEXES

Graph signal processing models many situations, but it is limited in the sense that it only allows for at most two signals to interact, with the interaction represented by an edge between two vertices. Many signal systems interact in more complex ways and cannot be represented by simple flow models that graphs provide. Instead, there might be three or more signals interacting, in which case a natural construct to model such interactions are simplicial complexes. In this way, an n-simplex represents a signal defined over an n-tuple of points. Some examples are neural networks [7], [13], collaboration networks [16], [19], and discourse networks [9]. Many more examples are given in [3].

Much of the theory of signals over simplicial complexes extends the one over graphs [3], [4], [12]. Namely, let $X(n)$ be the set of n-simplices of a complex X (i.e. the set of elements of X of cardinality $n + 1$; this is not to be confused with the n-skeleton X^n which contains the n-simplices but also their faces).

Definition 4.1. A signal over $X(n)$ is a function

$$s^n : X(n) \rightarrow \mathbb{R} \tag{16}$$

The collection $\{s^0, s^1, \ldots, s^d\}$, where d is as usual the dimension of X, is a *signal over the complex X*.

Note that s^0 is precisely the signal from (12), with $X(0) = X$.

In analogy with (14), we can consider the eigendecomposition of the higher Laplacians (2):

$$\mathbf{L}_n = \mathbf{A}_n \mathbf{A}_n^T + \mathbf{A}_{n+1} \mathbf{A}_{n+1}^T = \mathbf{U}_n \Lambda_n \mathbf{U}_n^T$$

Then, for each $0 \leq n \leq d$, define the *(complex) Fourier transform of order n* to be

$$\mathbf{X}_n = \mathbf{U}_n^T s^n . \tag{17}$$

This of course generalizes the graph Fourier transform (15) and represents the projection of the signal s^n onto the eigenvectors of the nth Laplacian $\mathbf{L}_n$. A signal over the n-simplices can then be decomposed in terms of those vectors as

$$s^n = \mathbf{U}_n \mathbf{X}_n .$$

The interaction of the Laplacians of various orders is of interest in the spectral analysis of the signal; more can be found in [3, III.A].

Now recall the vector space of cochains $C^n(X;\mathbb{R})$ from (9). This is a real vector space of dimension equal to the cardinality of the set $X(n)$ (as it is generated over $\mathbb{R}$ by the n-simplices).

Just like a graph signal s^0 can be identified with a 0-cochain (13), a signal over the n-simplices can be identified with an n-cochain, i.e.

$$s_n \in C^n(X;\mathbb{R})$$

Since $\mathbf{A}_n$ is a linear transformation of this space, by standard linear algebra we can use it to decompose $C^n(X;\mathbb{R})$ as

$$C^n(X;\mathbb{R}) \cong \ker(\mathbf{A}_n) \oplus \ker(\mathbf{A}_n)^{\perp} \tag{18}$$

Furthermore, we also have a well-known isomorphism

$$\ker(\mathbf{A}_n)^{\perp} \cong \ker(\mathbf{A}_n^T)$$

Now recall from (6) that $\mathbf{A}_n \mathbf{A}_{n+1} = 0$. In other words, we have that $\mathrm{im}(\mathbf{A}_{n+1})$ is a subspace of $\ker(\mathbf{A}_n)$. This in turn means that a vector in $\ker(\mathbf{A}_n)$ can be decomposed into its projection onto $\mathrm{im}(\mathbf{A}_{n+1})$ and onto its orthogonal complement $\mathrm{im}(\mathbf{A}_{n+1})^{\perp}$. Putting this together ultimately implies

Proposition 4.2. The space of n-cochains decomposes as

$$C^n(X;\mathbb{R}) \cong \mathrm{im}(\mathbf{A}_{n+1}) \oplus \ker(\mathbf{L}_n) \oplus \mathrm{im}(\mathbf{A}_n^T). \tag{19}$$

One way to see this is the following: For an $a \times k$ matrix $\mathbf{A}$ and an $k \times b$ matrix $\mathbf{B}$ satisfying $\mathbf{AB} = 0$, we in general have

$$\mathbb{R}^k = \mathrm{im}(\mathbf{A}^T) \oplus \ker(\mathbf{A}^T \mathbf{A} + \mathbf{B}\mathbf{B}^T) \oplus \mathrm{im}(\mathbf{B}) \tag{20}$$

For details, see for example [15]. In our case, where $\mathbf{A} = \mathbf{A}_n$ and $\mathbf{B} = \mathbf{A}_{n+1}$, $\mathbf{A}^T\mathbf{A}+\mathbf{B}\mathbf{B}^T$ is precisely $\mathbf{L}_n$ by (2).

Equation (19) is called the Hodge decomposition of the real vector space $C^n(X;\mathbb{R})$. (The original Hodge decomposition concerns the deRham cohomology of differential forms on a Riemannian manifold, and what we have here can be regarded as an adaptation to simplicial complexes.)

But basic linear algebra then shows that the middle term in (20) is precisely the nth cohomology group of X (see (10)). We thus have

$$\ker(\mathbf{L}_n) \cong H^n(X;\mathbb{R}). \tag{21}$$

This is consistent with the familiar and useful notion that $\mathbf{L}_n$ has something to do with clustering, which in terms of zeroth cohomology has to do with the number of connected components of a space. The above can be understood as as generalization of that observation, providing insight into the structure of high-order "holes" in the space underlying the signal.

The relationship (21) exemplifies the potential usefulness of interactively employing the spectral analysis approach and the algebraic topology approach. As the isomorphism makes evident, the eigenvalue analysis of the multi-order signal can be understood in terms of the topology of the underlying complex as captured by its cohomology.

For example, if certain Betti numbers are zero, which means that X does not have holes of certain dimensions, then the middle term in (19) goes away. This gives information about higher-order curl and divergence of the signal s^n. Topology can also be useful in reconstructing the signal from some of its samples or reconstrucing the underlying simplicial complex X from the signals. For more on questions like these, see [3].

5. SIGNALS AND SHEAF THEORY

Tools of algebraic topology allow for an even further generalization than was considered in the previous section. Namely, consider the situation where the signal is not just a single piece of information, but a collection of measurements. Examples of this abound; in fact, this happens any time a collection of parameters has to be combined into a single, comprehensive picture, like wireless networks, image processing, meteorological data, intelligence data, wave propagation, queues, etc. It also provides a more general framework for studying some of the basic ideas of signal processing like filtering and sampling.

Therefore what changes in this setup is that the signal function s_n from (16) may not take values in $\mathbb{R}$ but rather in some higher-dimensional vector space $\mathbb{R}^k$.

The study of simplicial complexes endowed with a collection of vector spaces over their simplices is the subject of *sheaf theory*. The basic idea behind sheaves is that an underlying structure like a simplicial complex has some signals associated to it *locally* in the sense that the measurements are only valid in some neighborhoods of the signal source. Sheaf theory, and in particular *sheaf cohomology* attempts to answer the question whether the data can be coherently and compatibly put together into useful information about the entire system and, if not, what the obstructions for doing so might be.

Applications of sheaves to signal processing are relatively recent [6], [21], [22], [23], [24]. Developing sheaf theory requires quite a bit of machinery from algebraic topology and category theory, so we will contend ourselves with just some of the highlights. For further reading, the most comprehensive source on the subject is the book [23]. One classic reference for sheaf theory from the topology and geometry points of view is [5].

Suppose as usual that X is a simplicial complex.

Definition 5.1. *A sheaf $\mathcal{F}$ of vector spaces over X* is an assignment of

- A vector space $V(\sigma)$ to each simplex σ of X, called the stalk at σ;
- For each inclusion (attachment) of a face σ to a face τ, a linear map

$$\mathcal{F}(\sigma \to \tau) : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$$

called the *restriction of σ to τ* satisfying

- $\mathcal{F}(\sigma \to \sigma)$ is the identity map, and
- if σ is a face of τ and τ is a face of ρ, then

$$\mathcal{F}(\tau \to \rho) \circ \mathcal{F}(\sigma \to \tau) = \mathcal{F}(\sigma \to \rho)$$

A schematic example of a sheaf over a 1-complex (graph) is provided in Figure 4; the example was taken from [9].

As mentioned above, the language of sheaves is meant to help us understand the transition from local to global information. Let $\bigoplus_{\sigma \in X} \mathcal{F}$ be the direct sum of all the stalks of $\mathcal{F}$. Define a *(global) section* of a sheaf $\mathcal{F}$ over X to be the function

$$\begin{aligned} s &: X \to \bigoplus_{\sigma \in X} \mathcal{F}(\sigma) \\ \tau &\to s(\tau) \end{aligned} \tag{22}$$

where $s(\tau) \in \mathcal{F}(\tau)$ and such that, if $\tau \to \tau'$ is an inclusion of a face, then $\mathcal{F}(\tau \to \tau')(s(\tau)) = s(\tau')$.

Thus a section is a choice of a vector in each vector space over each simplex, but these choices have to be coherent, i.e. they have to be compatible with the linear transformations that "sit over" each (inclusion of a) face. But this is *precisely a signal defined over all of X in a compatible way*. Signal processing over complexes can, said somewhat simplistically, thus be regarded as the study of sections of sheaves over X.

It is not hard to see that the set of sections of a sheaf $\mathcal{F}$ over X itself forms a vector space.

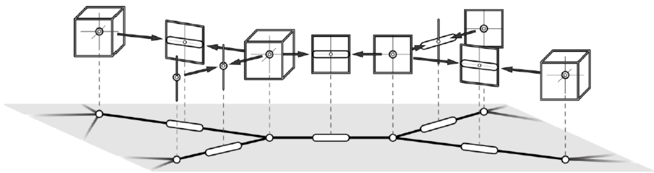


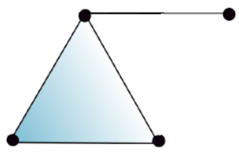
Figure 4: A diagrammatic example of a sheaf of vector spaces over a graph [9].

Example 5.2. The real line $\mathbb{R}$ can be regarded as a simplicial complex X by declaring the integers to be the 0-simplices and intervals $[n, n+1]$, $n \in \mathbb{Z}$, to be the 1-simplices.

- The *constant sheaf* over X assigns $\mathbb{R}^k$ for some k to each 0-simplex and the identity map to each 1-simplex.
- If the vector spaces alternate between some $\mathbb{R}^k$ and the trivial vector space $\{0\}$, with maps the zero-map an the inclusion of the zero-vector, one gets the *$\mathbb{R}^k$-sampling sheaf supported* on $\mathbb{Z}$.
- More generally, instead of alternating between $\mathbb{R}^0 = \{0\}$ and $\mathbb{R}^k$, one can take the stalks to be $\mathbb{R}^{k-1}$ and $\mathbb{R}^k$.

Example 5.3. If $X = \mathbb{R}^2$ with the lattice $\mathbb{Z}^2$ determining the 0-, 1-, and 2-simplices in the obvious way (each square can be divided diagonally to make two 2-simplices), then one can regard the integer lattice as pixels and the collection of all images as a sheaf over X. An image is a section of this sheaf. Or, if one has fragments of an image associated to the vertices of the lattice, then the existence of a section says that the pieces can be patched together into a single image. This is of course very useful in image analysis.

Example 5.4. [1]Let X be the complex



and let $\mathcal{F}$ be the sheaf

[1]This example was adapted from http://www.drmichaelrobinson.net/sheaftutorial/20150825_tutorial_3.pdf.

$$\begin{tikzcd}
 & & \mathbb{R}^3 \arrow[r, "f_1"] \arrow[dl, "f_3"'] \arrow[dr, "f_4"] & \mathbb{R}^2 & \mathbb{R}^3 \arrow[l, "f_2"'] \\
 & \mathbb{R}^2 \arrow[r, "f_5"] & \mathbb{R} & \mathbb{R}^2 \arrow[l, "f_6"'] & \\
\mathbb{R} \arrow[ur, "f_7"] \arrow[rr, "f_{10}"] & & \mathbb{R}^2 \arrow[u, "f_8"] & & \mathbb{R}^3 \arrow[ll, "f_{11}"'] \arrow[ul, "f_9"']
\end{tikzcd}$$

over X. The linear transformations are as follows, represented by matrices (so the map is given by multiplication by the matrix):

$$f_1 = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix},$$
$$f_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$
$$f_5 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad f_7 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$f_8 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad f_9 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix},$$
$$f_{10} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad f_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The reader might check that these maps are indeed compatible, e.g. $f_5 \circ f_3 = f_6 \circ f_4$.

An example of a section of $\mathcal{F}$ is

$$\begin{tikzcd}
 & & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \arrow[r] \arrow[dl] \arrow[dr] & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \arrow[l] \\
 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \arrow[r] & 1 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \arrow[l] & \\
1 \arrow[ur] \arrow[rr] & & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \arrow[u] & & \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

Define the *n-cochains of $\mathcal{F}$ over X* to be

$$C^n(X : \mathcal{F}) = \bigoplus_{\sigma \in X} \mathcal{F}(\sigma), \tag{23}$$

i.e. the direct sum of all the stalks over the n-simplices of X. The coboundary operator

$$\partial^n : C^n(X;\mathcal{F}) \to C^{n+1}(X;\mathcal{F})$$

is dened by sending the function $f \in C^n(X;\mathcal{F})$ to (the linear extension of)

$$\partial^n : C_{n+1}(X : \mathcal{F}) \to \mathbb{R}$$
$$\tau \to \sum_{\sigma \in X(k)} \varepsilon(\sigma,\tau)\mathcal{F}(\sigma \to \tau)(f(\sigma))$$

Here $\epsilon(\sigma,\tau)$ is the sign that depends on whether the orientations of σ and τ agree or disagree.

It can then be seen [23, Lemma 4.2] that, as usual, $\partial^2 = 0$ so that we indeed get a cochain complex.

Definition 5.5. Define the *sheaf cohomology groups of* $\mathcal{F}$, denoted by $H^n(X;\mathcal{F})$, to be the cohomology groups of the complex described above.

It does not take much work to show that $H^0(X;\mathcal{F})$ is exactly the vector space of sections, i.e. the space of signals dened compatibly over all of X. One observes from equations (22) and (23) that an element of $C^0(X;\mathcal{F})$ is precisely an assignment of values of stalks over the vertices of X. It then remains to understand ker ∂^0; for details, see for example [23, Theorem 4.3].

In the usual spirit of the first (co)homology describing "circular holes", it can be seen that $H^1(X;\mathcal{F})$ has to do with signal loops. Higher cohomology groups capture higher-order global features of the signal.

As an example of the kind of results that are accessible through the language of sheaves, we state a theorem about recovering the signal from its samples over the simplicial complex. Sampling is of course one of the main topics in signal processing, and the following result generalizes one of the cornerstones of the theory, the Nyquist-Shannon Theorem.

First, suppose $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X.

Definition 5.6. A sheaf morphism $f : \mathcal{F} \to \mathcal{G}$ is a linear map

$$f_\sigma : \mathcal{F}(\sigma) \to \mathcal{G}(\sigma)$$

for each $\sigma \in X$ such that, for every inclusion $\sigma \to \tau$, the diagram

$$\begin{tikzcd}
\mathcal{F}(\sigma) \arrow[r, "f_\sigma"] \arrow[d, "\mathcal{F}(\sigma \hookrightarrow \tau)"'] & \mathcal{G}(\sigma) \arrow[d, "\mathcal{G}(\sigma \hookrightarrow \tau)"] \\
\mathcal{F}(\tau) \arrow[r, "f_\tau"] & \mathcal{G}(\tau)
\end{tikzcd}$$

commutes.

Now suppose Y is a subcomplex of X. Then, given a sheaf $\mathcal{G}$ on X, it can be restricted to Y, and we thus obtain a sheaf $\mathcal{F}$ on Y. The inclusion $Y \to X$ induces a sheaf morphism $f : \mathcal{F} \to \mathcal{G}$. One then also has a sheaf $\mathcal{A}$ on Y, called the *ambiguity sheaf* whose stalks are dened as kernels of the f restricted to the stalks of $\mathcal{F}$ and $\mathcal{G}$.

Theorem 5.7. *Under the above setup, if $H^0(X;\mathcal{A}) = 0$, then sections (signals) of $\mathcal{F}$ can be uniquely recovered from sections (signals) of $\mathcal{G}$. If furthermore $H^1(X;\mathcal{A}) = 0$, then there is a correspondence between sections of $\mathcal{F}$ and sections of $\mathcal{G}$.*

For the proof, as well as the discussion why this theorem generalizes the Nyquist-Shannon Theorem, see [23, Section 4.5].

Another direction of investigation and generalizations of what we have seen so far is the spectral theory for sheaves over complexes. It would take us too far afield to discuss this here; interested reader should look at [10].

Yet another usefulness of the sheaf theory approach to signal processing is the issue of signal filtering. Sheaves admit the notion of a *topological filter* which generalizes the standard notions of linear, translation-invariant filters [23, Section 4.5.3].

Sheaf theory is a vast field in mathematics, appearing in algebraic topology, algebraic geometry, homological algebra, and category theory (in fact, the language of *categorications* is a very useful point of view in sheaf theory). Sheaves can also be defined over arbitrary topological spaces by letting the open sets play the role of simplices and their intersections the role of faces. Sheaves can also be defined in settings other than just vector spaces, such as abelian groups, modules over a ring, and so on. Many of these directions and points of view are still waiting for applications in signal processing.

6. CONCLUSION

We have given an overview of the role of algebraic topology in signal processing. The main invariants of spaces, homology and cohomology, were presented, and their role in graph signal processing, simplicial complex signal processing, and sheaf-theoretic signal processing was discussed.

One of the main themes was that various common methods in signal processing, such as spectral analysis, can be regarded as topological phenomena. The real usefulness might be in that the topological approach also lends itself to generalizations, and this is most evident in the language of sheaf theory. As an example, we stated a sampling therem that extends the standard Nyquist-Shannon Theorem. One could also talk about filtering, detection, and noise in the language of sheaves.

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