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Non-convex perturbation to evolution problems involving Moreau's sweeping process

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Abstract Along this paper, we study an evolution inclusion governed by the so-called sweeping process. The right side of the inclusion contains a set-valued perturbation, supposed to be the external forces exercised on the system. We prove existence and relaxation results under weak assumptions on the perturbation by taking a truncated Lipschitz condition. These perturbations have non-convex and unbounded values without any compactness condition; we just assume a linear growth assumption on the element of minimal norm. The approach is based on the construction of approximate solutions. The relaxation is obtained by proving the density of the solution set of the original problem in a closure of the solution set of the relaxed one.

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Keywords Relaxation; set-valued perturbation; truncation; normal cone; unboundedness.

1 Introduction

In this article we study the first-order sweeping process with perturbation. That evolution inclusion can be expressed in the form

$$(\mathcal{P}_{H,\mathcal{J}}) \begin{cases} -\dot{v}(\tau) \in \mathcal{N}_{V(v(\tau))}(v(\tau)) + H(\tau, v(\tau)), \text{ a.e. } \tau \in \mathcal{J}; \\ v(\tau) \in V(v(\tau)), \ \forall \tau \in \mathcal{J}; \\ v(0) = v_0 \in V(v_0), \end{cases}$$

where $\mathcal{N}_{V(v(\tau))}(v(\tau))$ is the normal cone to $V(v(\tau))$ at $v(\tau)$ and H is a set-valued mapping claimed to be the perturbation to the problem. Such problems were introduced by J. J.

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Moreau (see [24]) for time-dependent sets $V(\tau)$ and $H \equiv \{0\}$. The classical theory of the sweeping process establishes the existence and uniqueness of a Lipschitz solution for a given moving set. Although the theory was derived from the mechanics, nowadays, this problem remains the subject of mathematical research, which is essential also in economics, electrical engineering, biology, etc. However, many applications of the sweeping process have become decisive in many branches of science, particularly in nonsmooth mechanics, convex optimization, modeling of crowd motion, mathematical economics, dynamic networks, switched electrical circuits, etc, see the works [16, 18, 23] and the references therein. Generalization of the sweeping process has been the object of many studies, we can refer to the state-dependent sweeping process, the perturbed problem, the second-order case, the degenerate sweeping process, the implicit sweeping process and the non-convex sweeping process (prox-regular sets, subsmooth sets, positively α-far sets), see [2, 3, 6, 7, 9, 13, 17, 21, 22, 32]. The existence of solution is established via a constructive approach using Moreau's catching-up algorithm. Note that in the investigation of the existence of solutions for differential inclusion the utilization of the convexity and the upper-semi continuous assumptions on the perturbation is widely known, see for instance [1, 4, 5, 12]. To deal with the non-convex case, one way is the use of theory of relaxation that has received great interest from researchers for different problems, where they prove density theorems between the original problem and the convexified one, we cite for instance [14, 20, 26, 29, 30].

The paper is especially intended to study the existence of solutions and relaxation of the perturbed sweeping process under different assumptions weaker than those given in the references cited above. Based on the approach given in [7], we aim to show that the existence of solution of $(\mathcal{P}_{H,\mathcal{I}})$ holds when the convexity and boundeness of the values of the perturbation are not required as well as the usual upper semi-continuity assumption, therefore we replace them by a weaker Hausdorff Lipschitz truncated condition. On the other hand, by using techniques as in [31], we deduce the existence of solutions for the convexified problem

$$(\mathcal{P}_{\overline{co}H,\mathcal{I}}) \begin{cases} -\dot{v}(\tau) \in \mathcal{N}_{V(v(\tau))}(v(\tau)) + \overline{co}H(\tau, v(\tau)), \text{ a.e. } \tau \in \mathcal{I}; \\ v(\tau) \in V(v(\tau)), \ \forall \tau \in \mathcal{I}; \\ v(0) = v_0 \in V(v_0), \end{cases}$$

and we address the relations between solutions of the original problem $(\mathcal{P}_{H,\mathcal{I}})$ and those of the convexified one $(\mathcal{P}_{\overline{co}H,\mathcal{I}})$. The plan of this paper goes as follows. In section 2, we introduce some preliminaries that will be required later in our proof. Section 3 is devoted to the study of the existence of solutions and in the last section, we prove the relaxation theorems.

2 Preliminaries and background

Throughout this paper, $\mathcal{I} = [0, \mathcal{T}]$ $(\mathcal{T} > 0)$ is an interval of $\mathbf{R}$ and $\mathbf{R}^d$ is the d-dimensional euclidean space equipped with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. We denote by:

- $\overline{B}$ the closed unit ball of $\mathbf{R}^d$ centered at the origin, ϱB (resp. $\varrho\overline{B}$) the open (resp. the closed) ball of radius $\varrho > 0$, $\overline{B}(z,\eta)$ the closed ball of center z and radius $\eta > 0$;
- $L^p_{\mathbf{R}^d}(\mathcal{I})\,(1 \leq p < \infty)$ the space of measurable mappings $g : \mathcal{I} \to \mathbf{R}^d$ such that $\int_0^{\mathcal{T}} \|g(\tau)\|^p d\tau < \infty$, with the norm $\|g\|_p = (\int_0^{\mathcal{T}} \|g(\tau)\|^p d\tau)^{\frac{1}{p}}$;
- $\mathcal{C}_{\mathbf{R}^d}(\mathcal{I})$ (resp. $\mathcal{AC}_{\mathbf{R}^d}(\mathcal{I})$) the Banach space of all continuous (resp. absolutely continuous) mappings from $\mathcal{I}$ to $\mathbf{R}^d$ endowed with the norm $\|\cdot\|_{\mathcal{C}}$.

We say that $v \in \mathcal{AC}_{\mathbf{R}^d}(\mathcal{I})$ if there is $g \in L^1_{\mathbf{R}^d}(\mathcal{I})$ such that $v(\tau) = v(0) + \int_0^\tau g(s)ds$, for all $\tau \in \mathcal{I}$.
Let W and W' be two nonempty closed subsets of $\mathbf{R}^d$. We denote by:

- $\mathbf{1}_W$ the characteristic function of W;
- $d(\cdot, W)$ the usual distance function associated with W, i.e. $d(x,W) = \inf\limits_{v\in W} \|x - v\|$ for $x \in \mathbf{R}^d$, assuming that $d(x,W) = +\infty$ if $W = \emptyset$;
- $\delta^*(z', W) = \sup\limits_{v\in W} \langle z', v\rangle$ the support function of W at $z' \in \mathbf{R}^d$;
- $\overline{co}(W)$ the closed convex hull of W, characterized by

$$\overline{co}(W) = \{y \in \mathbf{R}^d \ : \ \forall y' \in \mathbf{R}^d,\ \langle y', y\rangle \leq \delta^*(y', W)\}; \tag{2.1}$$

- $e(W,W') = \sup\{d(a,W'),\ a \in W\}$ the excess of W over W', with the natural convention that $e(W,W') = 0$ whenever $W = \emptyset$. Note that the definition implies $e(W,W') = +\infty$ if W is nonempty and W' is empty;
- $haus(W,W') = \max\{e(W,W'), e(W',W)\} = \sup\limits_{x\in\mathbf{R}^d} |d(x,W) - d(x,W')|$ the Hausdorff distance;
- $haus_\varrho(W,W') = \max\{e(W_\varrho, W'), e(W'_\varrho, W)\}$ the ϱ-Hausdorff distance between W and W' (see [8]), where $\varrho \geq 0$ and $W_\varrho = W \cap \varrho\overline{B}$. According to [25], if $\varrho = +\infty$, $haus_\infty(W,W') = haus(W,W')$.

The following proposition provides a summary of the relation between the Hausdorff distance and the ϱ-Hausdorff distance.

Proposition 2.1. *[8] Consider W and W' two nonempty closed subsets of $\mathbf{R}^d$, such that for some $\varrho_0 \geq 0$, W_{ϱ_0} and W'_{ϱ_0} are nonempty. Then, for all $\varrho > \varrho_0$, we have*

$$haus_\varrho(W,W') \leq haus(W_\varrho, W'_\varrho).$$

Moreover, if W and W' are convex, then for all $\varrho > \varrho_0$,

$$haus(W_\varrho, W'_\varrho) \leq \frac{2\varrho}{\varrho - \varrho_0} haus_\varrho(W,W').$$

In other respects, let a set-valued mapping $G : \mathcal{J} \times \mathbf{R}^d \rightrightarrows \mathbf{R}^d$ for $\mathcal{J} \subset \mathbf{R}$. We denote by:

- $Proj_{G(\tau,v)}(0) = \{m(\tau,v) \in G(\tau,v) : d(0,G(\tau,v)) = \|m(\tau,v)\|\}$ the elements of $G(\tau,v)$ with minimal norm, it exists (resp. unique, measurable) whenever $G(\tau,v)$ is closed (resp. convex, measurable);
- $S(G) = \{f \in L^2_{\mathbf{R}^d}(\mathcal{J}) : f(\tau,v) \in G(\tau,v)\}$ the set of all measurable selectors in $L^2_{\mathbf{R}^d}(\mathcal{J})$ of G;
- $\mathcal{R}(\mathcal{P}_{G,\mathcal{J}})$ the solution set of the problem $(\mathcal{P}_{G,\mathcal{J}})$.

Let $g : \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$ be a proper convex continuous function on $\mathbf{R}^d$ and $z \in \mathbf{R}^d$ such that g is finite,

$$\partial g(z) = \{x \in \mathbf{R}^d : \langle x, w - z\rangle \leq g(w) - g(z),\ \forall w \in \mathbf{R}^d\}$$

is the subdifferential of g, it is a closed convex set if g is convex and $\partial g(z) = \emptyset$ when $g(z) = \infty$. For a subset $V \subset \mathbf{R}^d$ and $v \in V$, the normal cone $\mathcal{N}_V(v)$ to V at v

$$\mathcal{N}_V(v) = \{\zeta \in \mathbf{R}^d \ :\ \langle \zeta, u - v\rangle \leq 0,\ \forall u \in V\},$$

satisfies [15]

$$z \in \mathcal{N}_V(v) \Leftrightarrow v \in V \text{ and } \langle z, v\rangle = \delta^*(z, V),$$
$$z \in Proj_V(v) \Leftrightarrow v - z \in \mathcal{N}_V(z),$$
$$\partial d(v, V) = \mathcal{N}_V(v) \cap \overline{B}.$$

3 Existence results

Let us start by presenting the following assumption:
$(\mathcal{H}_V)$ Let $V : \mathbf{R}^d \rightrightarrows \mathbf{R}^d$ be a set valued mapping satisfying:

$(\mathcal{A}_1)$ for each $v \in \mathbf{R}^d$, the sets $V(v)$ are nonempty closed and convex;

$(\mathcal{A}_2)$ there is a constant $L \in [0, 1[$ such that, for all $u, w, x \in \mathbf{R}^d$, one has

$$|d(x, V(u)) - d(x, V(w))| \leq L\|u - w\|.$$

In the next existence result, we suppose the perturbation of $(\mathcal{P}_{H,\mathcal{J}})$ with unbounded non-convex values.

Theorem 3.1. *Suppose that $(\mathcal{H}_V)$ holds. Let $H : \mathcal{I} \times b\overline{B} \rightrightarrows \mathbf{R}^d$ for $b > 0$ be a set-valued mapping with nonempty closed values satisfying the following assumptions:*

$(\mathcal{H}_1)$ *$H(\cdot, y(\cdot))$ is measurable for every function $y \in \mathcal{C}_{\mathbf{R}^d}(\mathcal{I})$ with $\|y(\tau)\| \leq b$, for all $\tau \in \mathcal{I}$;*

$(\mathcal{H}_2)$ *there exist $\xi, \lambda \in L^2_{\mathbf{R}^+}(\mathcal{I})$, such that, for all $(\tau, y) \in \mathcal{I} \times b\overline{B}$,*

$$d(0, H(\tau, y)) \leq \xi(\tau) + \lambda(\tau)\|y\| \quad \textit{and} \quad d(0, H(\tau, 0)) = 0 \ \textit{ for } \ \xi(\tau) = 0;$$

$(\mathcal{H}_3)$ *for all* (τ, y_1), (τ, y_2) *in* $\mathcal{I} \times b\overline{B}$,

$$haus_{\varrho}(H(\tau, y_1), H(\tau, y_2)) < \lambda(\tau)\|y_1 - y_2\|.$$

Then, for $v_0 \in V(v_0)$, *the problem* $(\mathcal{P}_{H,\mathcal{J}})$ *admits an absolutely continuous solution* $v : \mathcal{I} \to \mathbf{R}^d$ *satisfying*

$$\|\dot{v}(\tau)\| \leq \tilde{\beta}(\tau), \text{ a.e. } \tau \in \mathcal{I},$$

with

$$\tilde{\beta}(\tau) = \frac{2}{1-L}\big(\xi(\tau) + \lambda(\tau)\kappa\big) \quad \text{and} \quad \kappa = 2\Big(\|v_0\| + \frac{2}{1-L}\|\xi\|_1\Big).$$

Proof. For every $n \in \mathbb{N}$, we consider a partition of $\mathcal{I}$ by the points

$$\tau_i^n = i\varepsilon_n \ , \ \varepsilon_n = \frac{\mathcal{T}}{n}, 0 \leq i \leq n,$$

and let us set for each $n \in \mathbb{N}$ and each $0 \leq i \leq n-1$, $\theta_n = \max\limits_{0\leq i\leq n-1} \int\limits_{\tau_i^n}^{\tau_{i+1}^n} \tilde{\beta}(s)ds$. Note that

$$\theta_n \to 0 \text{ as } n \to \infty. \tag{3.1}$$

We mean by $m_H(\tau, y)$ the element of minimal norm of the closed set $H(\tau, y)$ thus by $(\mathcal{H}_2)$ and for all $(\tau, y) \in \mathcal{I} \times b\overline{B}$, one obtains

$$\|m_H(\tau, y)\| \leq \xi(\tau) + \lambda(\tau)\|y\|. \tag{3.2}$$

We suppose that

$$\int_0^{\mathcal{T}} \lambda(s)ds \leq \frac{1-L}{4}. \tag{3.3}$$

Step 1. Construction of approximate solutions.
For $v_0 \in V(v_0)$, let us deal with the following problem on the interval $[0, \tau_1^n]$:

$$\begin{cases} -\dot{v}(\tau) \in \mathcal{N}_{V(v(\tau))}(v(\tau)) + m_H(\tau, v_0), \text{ a.e. } \tau \in [0, \tau_1^n]; \\ v(0) = v_0 \in V(v_0), \end{cases}$$

where $m_H(\cdot, v_0) \in L^2_{\mathbf{R}^d}([0, \tau_1^n])$ depends only on τ. By Proposition 3.2 in [7], the last problem admits an absolutely continuous solution $v_0^n : [0, \tau_1^n] \to \mathbf{R}^d$, such that

$$\|\dot{v}_0^n(\tau)\| \leq \frac{2}{1-L}\|m_H(\tau, v_0)\| \quad \text{a.e. } \tau \in [0, \tau_1^n].$$

Since $v_0^n(\tau_1^n) \in V(v_0^n(\tau_1^n))$ is well defined, take into account in $[\tau_1^n, \tau_2^n]$ the problem

$$\begin{cases} -\dot{v}(\tau) \in \mathcal{N}_{V(v(\tau))}(v(\tau)) + m_H(\tau, v_0^n(\tau_1^n)), \quad \text{a.e. } \tau \in [\tau_1^n, \tau_2^n]; \\ v(\tau_1^n) = v_0^n(\tau_1^n) \in V(v_0^n(\tau_1^n)), \end{cases}$$

which admits a solution denoted by $v_1^n : [\tau_1^n, \tau_2^n] \to \mathbf{R}^d$ with $v_1^n(\tau_1^n) = v_0^n(\tau_1^n)$ and satisfying

$$\|\dot{v}_1^n(\tau)\| \leq \frac{2}{1-L}\|m_H(\tau, v_0^n(\tau_1^n))\| \quad \text{a.e. } \tau \in [\tau_1^n, \tau_2^n].$$

For each n, there exists a finite sequence of absolutely continuous mappings $v_i^n : [\tau_i^n, \tau_{i+1}^n] \to \mathbf{R}^d$, $(0 \leq i \leq n-1)$ such that, for each $0 \leq i \leq n-1$,

$$\begin{cases} -\dot{v}_i^n(\tau) \in \mathcal{N}_{V(v_i^n(\tau))}(v_i^n(\tau)) + m_H(\tau, v_{i-1}^n(\tau_i^n)), \text{ a.e. } \tau \in [\tau_i^n, \tau_{i+1}^n]; \\ v_i^n(\tau_i^n) = v_{i-1}^n(\tau_i^n) \in V(v_{i-1}^n(\tau_i^n)), \end{cases}$$

where $v_{-1}^n(0) = v_0$ and

$$\|\dot{v}_i^n(\tau)\| \leq \frac{2}{1-L}\|m_H(\tau, v_{i-1}^n(\tau_i^n))\|, \text{ a.e. } \tau \in [\tau_i^n, \tau_{i+1}^n].$$

We define the function $v_n : \mathcal{I} \to \mathbf{R}^d$ by

$$v_n(\tau) = v_i^n(\tau); \ \forall \tau \in [\tau_i^n, \tau_{i+1}^n], \ 0 \leq i \leq n-1$$

and $\rho_n : \mathcal{I} \to \mathcal{I}$ by

$$\rho_n(\tau) = \tau_i^n, \ \forall \tau \in [\tau_i^n, \tau_{i+1}^n[, \ 0 \leq i \leq n-1, \ \rho_n(0) = 0.$$

So, $v_n(\cdot)$ is an absolutely continuous solution of

$$\begin{cases} -\dot{v}_n(\tau) \in \mathcal{N}_{V(v_n(\tau))}(v_n(\tau)) + m_H(\tau, v_n(\rho_n(\tau))), \quad \text{a.e. } \tau \in \mathcal{I}; \\ v_n(0) = v_0, \end{cases} \tag{3.4}$$

with

$$\|\dot{v}_n(\tau)\| \leq \frac{2}{1-L}\|m_H(\tau, v_n(\rho_n(\tau)))\|, \text{ a.e. } \tau \in \mathcal{I}. \tag{3.5}$$

By integration on $[\tau_i^n, \tau_{i+1}^n]$, we get

$$\|v_n(\tau_{i+1}^n)\| \leq \|v_n(\tau_i^n)\| + \frac{2}{1-L}\int_{\tau_i^n}^{\tau_{i+1}^n} \|m_H(s, v_n(\tau_i^n))\|ds.$$

So, by iteration we have for each $0 \leq i \leq n-1$

$$\begin{aligned} \|v_n(\tau_{i+1}^n)\| &\leq \|v_0\| + \frac{2}{1-L}\sum_{k=0}^{i}\int_{\tau_k^n}^{\tau_{k+1}^n} \|m_H(s, v_n(\tau_k^n))\|ds \\ &\leq \|v_0\| + \frac{2}{1-L}\sum_{k=0}^{i}\Big\{\int_{\tau_k^n}^{\tau_{k+1}^n} \xi(s)ds + \|v_n(\tau_k^n)\|\int_{\tau_k^n}^{\tau_{k+1}^n} \lambda(s)ds\Big\} \\ &\leq \|v_0\| + \frac{2}{1-L}\Big\{\int_0^{\tau_{i+1}^n} \xi(s)ds + \max_{0\leq k\leq n}\|v_n(\tau_k^n)\|\int_0^{\tau_{i+1}^n} \lambda(s)ds\Big\}, \end{aligned}$$

thus,

$$\max_{0\leq k\leq n}\|v_n(\tau_k^n)\| \leq \|v_0\| + \frac{2}{1-L}\Big\{\int_0^{\mathcal{T}} \xi(s)ds + \max_{0\leq k\leq n}\|v_n(\tau_k^n)\|\int_0^{\mathcal{T}} \lambda(s)ds\Big\}.$$

Using (3.3), we get

$$\max_{0\leq k\leq n}\|v_n(\tau_k^n)\| \leq 2\Big(\|v_0\| + \frac{2}{1-L}\|\xi\|_1\Big),$$

then, one has

$$\|v_n(\rho_n(\tau))\| \leq 2\Big(\|v_0\| + \frac{2}{1-L}\|\xi\|_1\Big) := \kappa. \tag{3.6}$$

By (3.2), (3.5) and (3.6), we get

$$\|m_H(\tau, v_n(\rho_n(\tau)))\| \leq \xi(\tau) + \lambda(\tau)\kappa := \kappa_1(\tau),$$

and

$$\|\dot{v}_n(\tau)\| \leq \frac{2}{1-L}\big(\xi(\tau) + \lambda(\tau)\kappa\big) := \tilde{\beta}(\tau), \text{ a.e. } \tau \in \mathcal{I}. \tag{3.7}$$

Step 2. The convergence of the sequences.
Observe that, $|\rho_n(\tau) - \tau| = |\tau_i^n - \tau| \leq \varepsilon_n$, so $\rho_n(\tau) \to \tau$ as $n \to \infty$. By using (3.7), we find for all $\tau \in \mathcal{I}$, $\|v_n(\tau)\| \leq \|\tilde{\beta}\|_1 + \kappa$, and

$$\|v_n(\rho_n(\tau)) - v_n(\tau)\| \leq \int_{\rho_n(\tau)}^{\tau} \|\dot{v}_n(s)\|ds \leq \int_{\rho_n(\tau)}^{\tau} \tilde{\beta}(s)ds,$$

then,

$$\lim_{n\to\infty} \|v_n(\rho_n(\tau)) - v_n(\tau)\| = 0, \tag{3.8}$$

and $(v_n(\tau))_{n\geq 1}$ is relatively compact, since $\tilde{\beta} \in L^2_{\mathbf{R}^+}(\mathcal{I}) \subset L^1_{\mathbf{R}^+}(\mathcal{I})$. On the other hand $(v_n(\cdot))_{n\geq 1}$ is equi-continous according to (3.7). Therefore, by Ascoli's theorem we conclude that $(v_n)_{n\geq 1}$ is relatively compact in $\mathcal{C}_{\mathbf{R}^d}(\mathcal{I})$, hence we can extract a subsequence (again denoted by) $(v_n)_{n\geq 1}$ converging uniformly to some mapping v. By (3.7), $(\dot{v}_n)_{n\geq 1}$ converges weakly in $L^2_{\mathbf{R}^d}(\mathcal{I})$ to z with $\|z(\tau)\| \leq \tilde{\beta}(\tau)$, a.e. $\tau \in \mathcal{I}$. Fixing $\tau \in \mathcal{I}$ and taking any $\epsilon \in \mathbf{R}^d$, the above weak convergence in $L^2_{\mathbf{R}^d}(\mathcal{I})$ yields

$$\lim_{n\to+\infty} \int_0^{\mathcal{T}} \langle \mathbf{1}_{[0,\tau]}(s)\epsilon, \dot{v}_n(s)\rangle ds = \int_0^{\mathcal{T}} \langle \mathbf{1}_{[0,\tau]}(s)\epsilon, z(s)\rangle ds,$$

or equivalently

$$\lim_{n\to+\infty} \langle \epsilon, v_0 + \int_0^{\tau} \dot{v}_n(s)ds\rangle = \langle \epsilon, v_0 + \int_0^{\tau} z(s)ds\rangle.$$

Then, $\lim_{n\to+\infty} \int_0^{\tau} \dot{v}_n(s)ds = \int_0^{\tau} z(s)ds$, since $v_n(\cdot)$ is an absolutely continuous mapping, we get $v(\tau) = v_0 + \int_0^{\tau} z(s)ds$ and $z = \dot{v}$.
We have that, for all $\tau \in \mathcal{I}$

$$\|v_n(\rho_n(\tau)) - v(\tau)\| \leq \|v_n(\rho_n(\tau)) - v_n(\tau)\| + \|v_n(\tau) - v(\tau)\|,$$

and using (3.8), we obtain

$$\lim_{n\to\infty} \|v_n(\rho_n(\tau)) - v(\tau)\| = 0. \tag{3.9}$$

Put $(m_H(\cdot, v_n(\rho_n(\cdot))))_n = (f_n(\cdot))_n$ for all $n \geq 1$. Since $\|f_n(\tau)\| \leq \kappa_1(\tau)$, we have $f_n(\tau) \in H(\tau, v_n(\rho_n(\tau))) \cap \varrho(\tau)\overline{B}$ with $\varrho(\tau) = \kappa_1(\tau)$ for all $\tau \in \mathcal{I}$ and $n \geq 1$, then, in view of Lemma 3.2 in [33], there exists a measurable sequence $f_m(\tau) \in H(\tau, v_m(\rho_m(\tau)))$ for any $m \in \mathbb{N}$ $(n < m)$, such that

$$\|f_n(\tau) - f_m(\tau)\| \leq d(f_n(\tau), H(\tau, v_m(\rho_m(\tau)))) + \frac{1}{m^2}, \text{ a.e. } \tau \in \mathcal{I}. \tag{3.10}$$

According to (3.10), we can write for all $n \geq 1$ and almost all $\tau \in \mathcal{I}$

$$\begin{aligned}\|f_n(\tau) - f_m(\tau)\| &\leq e(H_{\varrho(\tau)}(\tau, v_n(\rho_n(\tau))), H(\tau, v_m(\rho_m(\tau)))) + \frac{1}{m^2} \\ &\leq haus_{\varrho(\tau)}(H(\tau, v_n(\rho_n(\tau))), H(\tau, v_m(\rho_m(\tau)))) + \frac{1}{m^2},\end{aligned}$$

by $(\mathcal{H}_3)$, we obtain for all $n \geq 1$ and almost all $\tau \in \mathcal{I}$

$$\|f_n(\tau) - f_m(\tau)\| < \lambda(\tau)\|v_n(\rho_n(\tau)) - v_m(\rho_m(\tau))\| + \frac{1}{m^2}.$$

Observe that for any $\tau \in \mathcal{I}$

$$\begin{aligned}&\|v_n(\rho_n(\tau)) - v_m(\rho_m(\tau))\| \\ &\quad \leq \|v_n(\rho_n(\tau)) - v_n(\tau)\| + \|v_n(\tau) - v_m(\tau)\| + \|v_m(\tau) - v_m(\rho_m(\tau))\|.\end{aligned}$$

Next, making use of the absolute continuity of $v_n(\cdot)$, one has for any $\tau \in \mathcal{I}$ and $n \geq 1$

$$\|v_n(\rho_n(\tau)) - v_n(\tau)\| \leq \int_{\rho_n(\tau)}^{\tau} \|\dot{v}_n(s)\| ds \leq \int_{\rho_n(\tau)}^{\tau} \tilde{\beta}(s) ds \leq \int_{\tau_i^n}^{\tau_{i+1}^n} \tilde{\beta}(s) ds \leq \theta_n,$$

similarly, we get

$$\|v_m(\rho_m(\tau)) - v_m(\tau)\| \leq \theta_m.$$

Then, we may deduce that

$$\|f_n(\tau) - f_m(\tau)\| \leq \lambda(\tau)(\theta_n + \theta_m + \|v_n(\tau) - v_m(\tau)\|) + \frac{1}{m^2}.$$

Since $(v_n)_{n\geq 1}$ converges uniformly to v, $(v_n)_{n\geq 1}$ is a Cauchy sequence, i.e.

$$\lim_{n,m\to\infty} \|v_n(\tau) - v_m(\tau)\| = 0,$$

by (3.1) the right-hand side of the previous inequality tends to zero when $n, m \to \infty$. Hence, the sequence $(f_n(\tau))_{n\geq 1}$ is a Cauchy sequence in $\mathbf{R}^d$, then it converges almost everywhere to a mapping $f \in L^2_{\mathbf{R}^d}(\mathcal{I})$. Moreover, by $(\mathcal{H}_3)$ again, we infer that

$$\begin{aligned}d(f(\tau), H(\tau, v(\tau))) &\leq \|f(\tau) - f_n(\tau)\| + d(f_n(\tau), H(\tau, v(\tau))) \\ &\leq \|f(\tau) - f_n(\tau)\| + haus_{\varrho(\tau)}(H(\tau, v_n(\rho_n(\tau))), H(\tau, v(\tau))) \\ &< \|f(\tau) - f_n(\tau)\| + \lambda(\tau)\|v_n(\rho_n(\tau)) - v(\tau)\|.\end{aligned}$$

Thanks to (3.9), the almost everywhere convergence of the sequence $(f_n(\cdot))_{n\geq 1}$ to $f(\cdot)$ and by passing to the limit in the last inequality, we obtain

$$d(f(\tau), H(\tau, v(\tau))) = 0, \text{ a.e. } \tau \in \mathcal{I}.$$

The closedness of $H(\tau, v(\tau))$ gives $f(\tau) \in H(\tau, v(\tau))$, a.e. $\tau \in \mathcal{I}$.
Step 3. We prove that $v \in \mathcal{R}(\mathcal{P}_{H,\mathcal{I}})$.
For every $\tau \in \mathcal{I}$, and for $n \geq 1$, by $(\mathcal{A}_2)$

$$\begin{aligned} &d(v_n(\tau), V(v(\tau))) \\ &\leq \|v_n(\tau) - v_n(\rho_n(\tau)\| + d(v_n(\rho_n(\tau)), V(v(\tau))) - d(v_n(\rho_n(\tau)), V(v_n(\rho_n(\tau)))) \\ &\leq \|v_n(\tau) - v_n(\rho_n(\tau)\| + L\|v_n(\rho_n(\tau)) - v(\tau)\|. \end{aligned}$$

Using (3.8), (3.9), applying the limit to the previous inequality and according to the closedness of $V(v(\tau))$, we get $v(\tau) \in V(v(\tau))$, for all $\tau \in \mathcal{I}$.
Otherwise, we have

$$\| - \dot{v}_n(\tau) + f_n(\tau)\| \leq \|\dot{v}_n(\tau)\| + \|f_n(\tau)\| \leq \tilde{\beta}(\tau) + \kappa_1(\tau) := \tilde{M}(\tau). \tag{3.11}$$

By (3.11) and (3.4), we get for almost all $\tau \in \mathcal{I}$

$$-\dot{v}_n(\tau) + f_n(\tau) \in \mathcal{N}_{V(v_n(\tau))}(v_n(\tau)) \cap \tilde{M}(\tau)\overline{B} = \tilde{M}(\tau)\partial d(v_n(\tau), V(v_n(\tau)).$$

As $(-\dot{v}_n + f_n)$ converges weakly in $L^2_{\mathbf{R}^d}(\mathcal{I})$ to $-\dot{v} + f$, an application of Mazur's theorem to $(-\dot{v}_n + f_n)$ provides a sequence $(\Lambda_n)_{n\geq 1}$ with

$$\Lambda_n \in co\{-\dot{v}_m + f_m : m \geq n\},$$

in a way that $(\Lambda_n)_{n\geq 1}$ converges strongly in $L^2_{\mathbf{R}^d}(\mathcal{I})$ to $-\dot{v} + f$. We can extract from $(\Lambda_n(\cdot))_{n\geq 1}$ a subsequence that converges a.e. to $-\dot{v}(\cdot) + f(\cdot)$. Then, there is a Lebesgue negligible set $N \subset \mathcal{I}$ such that for every $\tau \in \mathcal{I}\backslash N$

$$-\dot{v}(\tau) + f(\tau) \in \bigcap_{n\geq 0} \overline{\{\Lambda_k(\tau) : k \geq n\}} \subset \bigcap_{n\geq 0} \overline{co}\{-\dot{v}_k(\tau) + f_k(\tau) : k \geq n\}.$$

Fix any $\tau \in \mathcal{I}\backslash N$ and $\nu \in \mathbf{R}^d$, from the previous inclusion and by (2.1), we obtain

$$\begin{aligned} \langle -\dot{v}(\tau) + f(\tau), \nu\rangle &\leq \tilde{M}(\tau) \limsup_{n\to\infty} \delta^*(\partial d(v_n(\tau), V(v_n(\tau))), \nu) \\ &\leq \tilde{M}(\tau)\delta^*(\partial d(v(\tau), V(v(\tau))), \nu), \end{aligned}$$

where Proposition 3.1 in [10] entails the second inequality. As $\partial d(v(\tau), V(v(\tau)))$ is closed convex, we get

$$-\dot{v}(\tau) + f(\tau) \in \tilde{M}(\tau)\partial d(v(\tau), V(v(\tau))) \subset \mathcal{N}_{V(v(\tau))}(v(\tau)).$$

Lastly, when $\int_0^{\mathcal{T}} \lambda(s)ds > \frac{1-L}{4}$, we subdivide $\mathcal{I}$ into intervals satisfying (3.3) and, as a result of the foregoing, we construct an absolutely continuous solution in each subinterval, then the problem $(\mathcal{P}_{H,\mathcal{I}})$ has a solution on $\mathcal{I}$. □

For the proof of the next results, we need the following proposition where the perturbation of

$$(\mathcal{P}_{f,\mathcal{I}}) \begin{cases} -\dot{v}(\tau) \in \mathcal{N}_{V(v(\tau))}(v(\tau)) + f(\tau), \text{ a.e. } \tau \in \mathcal{I}; \\ v(\tau) \in V(v(\tau)), \ \forall \tau \in \mathcal{I}; \\ v(0) = v_0 \in V(v_0), \end{cases}$$

is a single valued mapping such that:

$$\exists \xi \in L^2_{\mathbf{R}^+}(\mathcal{I}) : \|f(\tau)\| \leq \xi(\tau), \text{ for all } \tau \in \mathcal{I}. \tag{3.12}$$

Proposition 3.2. *Assume that* $(\mathcal{H}_V)$ *holds.*

(1) Let $f : \mathcal{I} \to \mathbf{R}^d$ *verify* (3.12). *Then,* $(\mathcal{P}_{f,\mathcal{I}})$ *admits a unique absolutely continuous solution* $\tilde{u} : \mathcal{I} \to \mathbf{R}^d$, *with*

$$\|\dot{\tilde{u}}(\tau)\| \leq \frac{2\xi(\tau)}{1-L}, \text{ a.e. } \tau \in \mathcal{I},$$

and for all $\tau \in \mathcal{I}$

$$\|\tilde{u}(\tau)\| \leq \|v_0\| + \frac{2\|\xi\|_1}{1-L}.$$

(2) Let $(f_n)_n : \mathcal{I} \to \mathbf{R}^d$ *be a sequence of mappings which verifies* (3.12), *converges weakly to* f *in* $L^2_{\mathbf{R}^d}(\mathcal{I})$ *and* $\tilde{u}_n(0) = v_0$ *for all* $n \in \mathbb{N}$. *Then, the sequence* $(\tilde{u}_n)_{n\in\mathbb{N}} \subset \mathcal{R}(\mathcal{P}_{f_n,\mathcal{I}})$ *converges uniformly to* $\tilde{u} \in \mathcal{R}(\mathcal{P}_{f,\mathcal{I}})$.

(3) If $\tilde{u}_n \in \mathcal{R}(\mathcal{P}_{f_n,\mathcal{I}})$ *(resp.* $\tilde{u}_m \in \mathcal{R}(\mathcal{P}_{f_m,\mathcal{I}})$*), one has, for all* $\tau \in \mathcal{I}$

$$\|\tilde{u}_n(\tau) - \tilde{u}_m(\tau)\| \leq \|v_0^n - v_0^m\| + \int_0^\tau \|f_n(s) - f_m(s)\| ds, \tag{3.13}$$

where $\tilde{u}_n(0) = v_0^n$ *(resp.* $\tilde{u}_m(0) = v_0^m$*).*

Proof. 1) The existence of solution is a direct consequence of Theorem 3.1, it is enough to show the uniqueness. Let $\tilde{u}_1$ and $\tilde{u}_2$ be two solutions of $(\mathcal{P}_{f_n,\mathcal{I}})$ with $\tilde{u}_1(0) = \tilde{u}_2(0) = v_0$. The monotonicity of the normal cone yields, for almost $\tau \in \mathcal{I}$,

$$\frac{1}{2}\frac{d}{d\tau}(\|\tilde{u}_1(\tau) - \tilde{u}_2(\tau)\|^2) = \langle \dot{\tilde{u}}_1(\tau) - \dot{\tilde{u}}_2(\tau), \tilde{u}_1(\tau) - \tilde{u}_2(\tau)\rangle \leq 0.$$

By integration, we obtain $\|\tilde{u}_1(\tau) - \tilde{u}_2(\tau)\|^2 \leq \|\tilde{u}_1(0) - \tilde{u}_2(0)\|^2 = 0$.

2) Since for all $n \geq 1$, $\tilde{u}_n \in \mathcal{R}(\mathcal{P}_{f_n,\mathcal{I}})$ and

$$\|\dot{\tilde{u}}_n(\tau)\| \leq \frac{2\xi(\tau)}{1-L}, \text{ a.e. } \tau \in \mathcal{I},$$

as $\xi \in L^2_{\mathbf{R}^+}(\mathcal{I}) \subset L^1_{\mathbf{R}^+}(\mathcal{I})$, we get the relatively compact of $(\tilde{u}_n(\tau))_{n\geq 1}$, for all $\tau \in \mathcal{I}$ and the equi-continuity of $(\tilde{u}_n(\cdot))_{n\geq 1}$. By Ascoli's theorem $(\tilde{u}_n)_{n\geq 1}$ is relatively compact in $\mathcal{C}_{\mathbf{R}^d}(\mathcal{I})$, so we can extract a subsequence of $(\tilde{u}_n)_{n\geq 1}$ (that we do not relabel) which converges uniformly to some mapping z.

As $\tilde{u}$ is the solution of $(\mathcal{P}_{f,\mathcal{I}})$, by the monotonicity property we can write for all $\tau \in \mathcal{I}$

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\|\tilde{u}(\tau) - \tilde{u}_n(\tau)\|^2 &\leq \langle f_n(\tau) - f(\tau), \tilde{u}(\tau) - \tilde{u}_n(\tau)\rangle \\ &= \langle f_n(\tau) - f(\tau), \tilde{u}(\tau) - z(\tau)\rangle + \langle f_n(\tau) - f(\tau), z(\tau) - \tilde{u}_n(\tau)\rangle,\end{aligned}$$

so,

$$\begin{aligned}\frac{1}{2}\|\tilde{u}(\tau) - \tilde{u}_n(\tau)\|^2 &\leq \int_0^\tau \langle f_n(s) - f(s), \tilde{u}(s) - z(s)\rangle ds + \int_0^\tau \langle f_n(s) - f(s), z(s) - \tilde{u}_n(s)\rangle ds \\ &\leq \int_0^\tau \langle f_n(s) - f(s), \tilde{u}(s) - z(s)\rangle ds + 2\|\xi\|_1 \|z - \tilde{u}_n\|_{\mathcal{E}}.\end{aligned}$$

By passing to the limit and since $(f_n - f)$ converges weakly to 0 in $L^2_{\mathbf{R}^d}(\mathcal{I})$ and $(\tilde{u}_n)$ converges uniformly to z, we obtain $\tilde{u} = z$.

3) By the monotonicity of the normal cone again, we get

$$\langle -(\dot{\tilde{u}}_n(\tau) + f_n(\tau)) + (\dot{\tilde{u}}_m(\tau) + f_m(\tau)), \tilde{u}_n(\tau) - \tilde{u}_m(\tau)\rangle \geq 0, \text{ a.e. } \tau \in \mathcal{I},$$

thus,

$$\langle \dot{\tilde{u}}_n(\tau) - \dot{\tilde{u}}_m(\tau), \tilde{u}_n(\tau) - \tilde{u}_m(\tau)\rangle \leq \langle f_n(\tau) - f_m(\tau), \tilde{u}_n(\tau) - \tilde{u}_m(\tau)\rangle, \text{ a.e. } \tau \in \mathcal{I},$$

or equivalently

$$\frac{1}{2}\frac{d}{d\tau}\|\tilde{u}_n(\tau) - \tilde{u}_m(\tau)\|^2 \leq \langle f_n(\tau) - f_m(\tau), \tilde{u}_n(\tau) - \tilde{u}_m(\tau)\rangle.$$

By integration, we get

$$\frac{1}{2}\|\tilde{u}_n(\tau) - \tilde{u}_m(\tau)\|^2 \leq \frac{1}{2}\|v_0^n - v_0^m\|^2 + \int_0^\tau \|f_n(s) - f_m(s)\|\|\tilde{u}_n(s) - \tilde{u}_m(s)\|ds,$$

and using Lemma $A.5$ in [11], we find

$$\|\tilde{u}_n(\tau) - \tilde{u}_m(\tau)\| \leq \|v_0^n - v_0^m\| + \int_0^\tau \|f_n(s) - f_m(s)\|ds.$$

□

In the following proposition we treat the boundedness of the solution by adding a condition on ϱ. This result will take a major part in the proof of the relaxation theorem. We denote by $\omega : \mathcal{I} \to \mathbf{R}^+$ the solution of

$$\begin{cases} \dot{\omega}(\tau) = \xi(\tau) + \lambda(\tau)\omega(\tau), \ \forall \tau \in \mathcal{I}; \\ \omega(0) = \omega_0 \geq 0, \end{cases} \tag{3.14}$$

where

$$\omega(\tau) = \omega_0 e^{\tilde{m}(\tau)} + \int_0^\tau e^{\tilde{m}(\tau)-\tilde{m}(s)}\xi(s)ds, \ \tilde{m}(\tau) = \int_0^\tau \lambda(s)ds, \ \forall \tau \in \mathcal{I}.$$

Proposition 3.3. *Suppose that all hypotheses of Theorem 3.1 hold, ω satisfies (3.14) with $\|v_0\| \leq \omega_0 < b$ and $0 \leq \varrho \leq \dot{\omega}(\tau)$. Then, $(\mathcal{P}_{H,\mathcal{I}})$ admits an absolutely continuous solution $u : \mathcal{I} \to \mathbf{R}^d$ such that for all $\tau \in \mathcal{I}$, $\|v(\tau)\| \leq \omega(\tau) \leq b$.*

Proof. Using (3.14), the mapping $\omega : \mathcal{I} \to \mathbf{R}^d$ is nondecreasing. As $\|v_0\| \leq \omega_0 < b$, either $\omega(\tau) \leq b$, for all $\tau \in \mathcal{I}$ or there exists a unique $d \in]0, \mathcal{T}]$ such that $\omega(\tau) > b$, for $\tau \in]d, \mathcal{T}]$ and $\omega(d) = b$.
Put $\mathcal{K} = [0, d]$. By $(\mathcal{H}_2)$ and (3.14), we obtain for $y \in \omega(\tau)\overline{B}$,

$$H(\tau, y) \cap \dot{\omega}(\tau)\overline{B} \neq \emptyset, \text{ a.e. } \tau \in \mathcal{K}.$$

Let $(\tau, y_1), (\tau, y_2)$ in $\mathcal{K} \times \omega(\tau)\overline{B}$ and $\varrho(\tau) = \dot{\omega}(\tau)$, a.e. $\tau \in \mathcal{K}$, there exists $p(\tau) \in H(\tau, y_1)_{\varrho(\tau)}$, such that

$$d(p(\tau), H(\tau, y_2)) \leq e(H(\tau, y_1)_{\varrho(\tau)}, H(\tau, y_2)) \leq haus_{\varrho(\tau)}(H(\tau, y_1), H(\tau, y_2)).$$

Using $(\mathcal{H}_3)$, we obtain

$$d(p(\tau), H(\tau, y_2)) < \lambda(\tau)\|y_1 - y_2\|,$$

then,

$$p(\tau) \in H(\tau, y_2) + \lambda(\tau)\|y_1 - y_2\|\overline{B}, \text{ a.e. } \tau \in \mathcal{K}. \tag{3.15}$$

We claim that one can define by induction a sequence $(h_n, u_n) \in L^2_{\mathbf{R}^d}(\mathcal{K}) \times \mathcal{C}_{\mathbf{R}^d}(\mathcal{K})$ such that for all $n \geq 1$ the following hold

$$h_0(\tau) = u_0(\tau) = 0, \ \|u_n(\tau)\| \leq \|v_0\| + \int_0^\tau \|h_n(s)\|ds, \ \forall \tau \in \mathcal{K}; \tag{3.16}$$

$$h_n(\tau) \in H(\tau, u_{n-1}(\tau)), \text{ a.e. } \tau \in \mathcal{K}; \tag{3.17}$$

$$u_n \text{ the corresponding solution of } (\mathcal{P}_{h_n,\mathcal{K}}); \tag{3.18}$$

$$(u_n(\tau))_{n\geq 0} \text{ and } (h_n(\tau))_{n\geq 1} \text{ are Cauchy sequences for all } \tau \in \mathcal{K}.$$

Indeed, put $\mathcal{K}_0 = \{\tau \in \mathcal{K} : \xi(\tau) = 0\}$, hence $(\mathcal{H}_2)$ entails

$$\tilde{\xi}_0(\tau) = d(0, H(\tau, u_0(\tau))) \leq \xi(\tau), \text{ a.e. } \tau \in \mathcal{K}\backslash\mathcal{K}_0. \tag{3.19}$$

Using $(\mathcal{H}_1)$, the mapping $\tilde{\xi}_0$ is measurable and

$$\tilde{\xi}_0(\tau) < \frac{\tilde{\xi}_0(\tau) + \xi(\tau)}{2} < \xi(\tau), \text{ a.e. } \tau \in \mathcal{K}\backslash\mathcal{K}_0.$$

By (3.19), $H(\cdot, u_0(\cdot)) \cap \frac{\tilde{\xi}_0(\cdot)+\xi(\cdot)}{2}\overline{B}$, is measurable with nonempty closed values. Then, there exists a measurable selection w_0 of this mapping. Set

$$h_1(\tau) = \begin{cases} w_0(\tau) & \text{if } \tau \in \mathcal{K}\backslash\mathcal{K}_0; \\ h_0(\tau) & \text{if } \tau \in \mathcal{K}_0, \end{cases}$$

and

$$\|h_1(\tau)\| \leq \xi(\tau), \text{ a.e. } \tau \in \mathcal{K}, \tag{3.20}$$

$$h_1(\tau) \in H(\tau, u_0(\tau)), \text{ a.e. } \tau \in \mathcal{K}.$$

By Proposition 3.2, let u_1 be the corresponding solution of $(\mathcal{P}_{h_1,\mathcal{K}})$. Along with (3.13), (3.14) and (3.20), it follows that

$$\|u_1(\tau)\| \leq \|v_0\| + \int_0^\tau \|h_1(s)\|ds \leq \omega_0 + \int_0^\tau \dot{\omega}(s)ds = \omega(\tau).$$

Since for $\tau \in \mathcal{K}$, $h_1(\tau) \in H(\tau, u_0(\tau)) \cap \dot{\omega}(\tau)\overline{B}$, a.e. $\tau \in \mathcal{K}$ and $\|u_0(\tau)\| = 0 \leq \omega(\tau)$, (3.15) gives

$$h_1(\tau) \in H(\tau, u_1(\tau)) + \lambda(\tau)\|u_1(\tau)\|\overline{B}, \text{ a.e. } \tau \in \mathcal{K},$$

and

$$d(h_1(\tau), H(\tau, u_1(\tau))) \leq \lambda(\tau)\|u_1(\tau)\|, \text{ a.e. } \tau \in \mathcal{K},$$

when $\|u_1(\tau)\| \neq 0$. Let $\mathcal{K}_1 = \{\tau \in \mathcal{K} : \|u_1(\tau) - u_0(\tau)\| = 0\}$.
Since $\tilde{\xi}_1(\tau) = d(h_1(\tau), H(\tau, u_1(\tau)))$ is measurable and

$$\tilde{\xi}_1(\tau) < \frac{\tilde{\xi}_1(\tau) + \lambda(\tau)\|u_1(\tau)\|}{2} < \lambda(\tau)\|u_1(\tau)\|, \ \tau \notin \mathcal{K}_1,$$

as the set $\mathcal{K}_1 \subset \mathcal{K}$ is closed, the set-valued mapping

$$H(\cdot, u_1(\cdot)) \cap \Big(h_1(\cdot) + \frac{\tilde{\xi}_1(\cdot) + \lambda(\cdot)\|u_1(\cdot)\|}{2}\overline{B}\Big) \text{ on } \mathcal{K}\backslash\mathcal{K}_1$$

is measurable with nonempty closed values. Then, on $\mathcal{K}\backslash\mathcal{K}_1$ there exists a measurable selection w_1 of this mapping. Set

$$h_2(\tau) = \begin{cases} w_1(\tau) & \text{if } \tau \in \mathcal{K}\backslash\mathcal{K}_1; \\ h_1(\tau) & \text{if } \tau \in \mathcal{K}_1, \end{cases}$$

and

$$h_2(\tau) \in H(\tau, u_1(\tau)), \text{ a.e. } \tau \in \mathcal{K},$$
$$\|h_1(\tau) - h_2(\tau)\| \leq \lambda(\tau)\|u_1(\tau)\|, \text{ a.e. } \tau \in \mathcal{K}. \tag{3.21}$$

Therefore, for a.e. $\tau \in \mathcal{K}$

$$\|h_2(\tau)\| \leq \|h_1(\tau)\| + \lambda(\tau)\|u_1(\tau)\| \leq \xi(\tau) + \lambda(\tau)\omega(\tau) = \dot{\omega}(\tau).$$

By Proposition 3.2, put u_2 the corresponding solution of $(\mathcal{P}_{h_2,\mathcal{K}})$. Then, from (3.13), we get $\|u_2(\tau)\| \leq \|v_0\| + \int_0^\tau \|h_2(s)\|ds$, thus, $\|u_2(\tau)\| \leq \omega(\tau)$, and by (3.21),

$$\|h_1(\tau) - h_2(\tau)\| \leq \lambda(\tau)\|u_1(\tau)\| \leq \lambda(\tau)\Big(\|v_0\| + \int_0^\tau \|h_1(s)\|ds\Big),$$

so,

$$\|h_1(\tau) - h_2(\tau)\| \leq \lambda(\tau)\Big(\omega_0 + \int_0^\tau \xi(s)ds\Big), \text{ a.e. } \tau \in \mathcal{K}. \tag{3.22}$$

In view of (3.13) and (3.22), we get

$$\|u_2(\tau) - u_1(\tau)\| \leq \int_0^\tau \|h_1(s) - h_2(s)\|ds \leq \omega_0 \int_0^\tau \lambda(s)ds + \int_0^\tau \xi(\varsigma)\int_\varsigma^\tau \lambda(s)dsd\varsigma$$
$$= \omega_0\tilde{m}(\tau) + \int_0^\tau (\tilde{m}(\tau) - \tilde{m}(\varsigma))\xi(\varsigma)d\varsigma.$$

By induction, assume that one has defined $(h_1, u_1), \cdots, (h_n, u_n)$ satisfying (3.16), (3.17), (3.18) and

$$\|h_{n+1}(\tau) - h_n(\tau)\| \leq \lambda(\tau)\|u_n(\tau) - u_{n-1}(\tau)\|, \text{ a.e. } \tau \in \mathcal{K},$$
$$\|u_n(\tau)\| \leq \omega(\tau), \ \forall \tau \in \mathcal{K}, \ \|h_n(\tau)\| \leq \dot{\omega}(\tau), \text{ a.e. } \tau \in \mathcal{K}. \tag{3.23}$$

Reasoning as above, for $n \geq 1$, let $\mathcal{K}_n = \{\tau \in \mathcal{K} : \|u_n(\tau) - u_{n-1}(\tau)\| = 0\}$. Since $\tilde{\xi}_n(\tau) = d(h_n(\tau)), H(\tau, u_n(\tau))$ is measurable and

$$\tilde{\xi}_n(\tau) < \frac{\tilde{\xi}_n(\tau) + \lambda(\tau)\|u_n(\tau) - u_{n-1}(\tau)\|}{2} < \lambda(\tau)\|u_n(\tau) - u_{n-1}(\tau)\|, \ \tau \notin \mathcal{K}_n,$$

as $\mathcal{K}_n \subset \mathcal{K}$ is closed, for $\tau \in \mathcal{K}\backslash\mathcal{K}_n$

$$H(\cdot, u_n(\cdot)) \cap \Big(h_n(\cdot) + \frac{\tilde{\xi}_n(\cdot) + \lambda(\cdot)\|u_n(\cdot) - u_{n-1}(\cdot)\|}{2}\overline{B}\Big),$$

is measurable with nonempty closed values. Thus, there exists a measurable selection w_n of this mapping on $\mathcal{K}\backslash\mathcal{K}_n$. Put

$$h_{n+1}(\tau) = \begin{cases} w_n(\tau) & \text{if } \tau \in \mathcal{K}\backslash\mathcal{K}_n; \\ h_n(\tau) & \text{if } \tau \in \mathcal{K}_n. \end{cases}$$

Hence, we obtain for a.e. $\tau \in \mathcal{K}$, $h_{n+1}(\tau) \in H(\tau, u_n(\tau))$, $\|h_{n+1}(\tau) - h_n(\tau)\| \leq \lambda(\tau)\|u_{n-1}(\tau) - u_n(\tau)\|$ and

$$\|h_{n+1}(\tau) - h_n(\tau)\| \leq \lambda(\tau)\Big\{\omega_0\frac{(\tilde{m}(\tau))^{n-1}}{(n-1)!} + \int_0^\tau \frac{(\tilde{m}(\tau) - \tilde{m}(s))^{n-1}}{(n-1)!}\xi(s)ds\Big\}, \text{ a.e. } \tau \in \mathcal{K}.$$

Proposition 3.2 provides the corresponding solution u_{n+1} of $(\mathcal{P}_{h_{n+1},\mathcal{K}})$. On the other hand,

$$\|h_{n+1}(\tau)\| - \|h_1(\tau)\| \leq \sum_{j=1}^{i}\|h_{j+i}(\tau) - h_j(\tau)\| \leq \lambda(\tau)\sum_{j=1}^{i}\|u_j(\tau) - u_{j-1}(\tau)\|,$$

so,

$$\begin{aligned}
\|h_{n+1}(\tau)\| &\leq \lambda(\tau)\sum_{j=1}^{i}\|u_j(\tau) - u_{j-1}(\tau)\| + \|h_1(\tau)\| \\
&\leq \lambda(t)\Big(\omega_0\sum_{j=0}^{i-1}\frac{(\tilde{m}(\tau))^j}{j!} + \sum_{j=0}^{i-1}\int_0^\tau \frac{(\tilde{m}(\tau) - \tilde{m}(s))^j}{j!}\xi(s)ds\Big) + \xi(\tau) \\
&\leq \lambda(\tau)\Big(\omega_0 e^{\tilde{m}(\tau)} + \int_0^\tau e^{\tilde{m}(\tau)-\tilde{m}(s)}\xi(s)ds\Big) + \xi(\tau) \\
&= \lambda(\tau)\omega(\tau) + \xi(\tau) = \dot{\omega}(\tau).
\end{aligned}$$

Thanks to (3.13), we get $\|u_{n+1}(\tau)\| \leq \|v_0\| + \int_0^\tau \|h_{n+1}(s)\|ds$, and $\|u_{n+1}(\tau)\| \leq \omega(\tau)$. Thus,

$$\begin{aligned}
\|u_{n+1}(\tau) - u_n(\tau)\| &\leq \int_0^\tau \|h_{n+1}(s) - h_n(s)\|ds \\
&\leq \int_0^\tau \lambda(s)\Big\{\omega_0\frac{(\tilde{m}(s))^{n-1}}{(n-1)!} + \int_0^s \frac{(\tilde{m}(s) - \tilde{m}(\varsigma))^{n-1}}{(n-1)!}\xi(\varsigma)d\varsigma\Big\}ds \\
&\leq \omega_0\int_0^\tau \lambda(s)\frac{(\tilde{m}(s))^{n-1}}{(n-1)!}ds + \int_0^\tau \lambda(s)\Big(\int_0^s \frac{(\tilde{m}(s) - \tilde{m}(\varsigma))^{n-1}}{(n-1)!}\xi(\varsigma)d\varsigma\Big)ds \\
&\leq \omega_0\int_0^\tau \lambda(s)\frac{(\tilde{m}(s))^{n-1}}{(n-1)!}ds + \int_0^s \xi(\varsigma)\Big(\int_0^\tau \lambda(s)\frac{(\tilde{m}(s) - \tilde{m}(\varsigma))^{n-1}}{(n-1)!}ds\Big)d\varsigma
\end{aligned}$$

$$
\begin{aligned}
&\leq \omega_0 \int_0^\tau \lambda(s)\frac{(\tilde{m}(s))^{n-1}}{(n-1)!}ds + \int_0^\tau \xi(\varsigma)\Big(\int_\varsigma^\tau \lambda(s)\frac{(\tilde{m}(s)-\tilde{m}(\varsigma))^{n-1}}{(n-1)!}ds\Big)d\varsigma\\
&= \frac{\omega_0}{n(n-1)!}\int_0^\tau \lambda(s)n(\tilde{m}(s))^{n-1}ds + \frac{1}{n(n-1)!}\int_0^\tau \xi(\varsigma)\Big(\int_\varsigma^\tau \lambda(s)n(\tilde{m}(s)\\
&- \tilde{m}(\varsigma))^{n-1}ds\Big)d\varsigma\\
&= \frac{\omega_0}{n(n-1)!}\int_0^\tau n(\tilde{m}(s))^{n-1}\frac{d}{ds}(\tilde{m}(s))ds + \frac{1}{n(n-1)!}\int_0^\tau \xi(\varsigma)\Big(\int_\varsigma^\tau n(\tilde{m}(s)\\
&- \tilde{m}(\varsigma))^{n-1}\frac{d}{ds}(\tilde{m}(s)-\tilde{m}(\varsigma))ds\Big)d\varsigma\\
&= \frac{\omega_0}{n(n-1)!}\int_0^\tau \frac{d}{ds}(\tilde{m}(s))^n ds + \frac{1}{n(n-1)!}\int_0^\tau \xi(\varsigma)\Big(\int_\varsigma^\tau \frac{d}{ds}(\tilde{m}(s)-\tilde{m}(\varsigma))^n ds\Big)d\varsigma\\
&= \omega_0\frac{(\tilde{m}(\tau))^n}{n!} + \int_0^\tau \frac{(\tilde{m}(\tau)-\tilde{m}(\varsigma))^n}{n!}\xi(\varsigma)d\varsigma.
\end{aligned}
$$

Then, we can write

$$
\begin{aligned}
\sum_{n=0}^{\infty}\|u_{n+1}(\tau)-u_n(\tau)\| &\leq \omega_0\sum_{n=0}^{\infty}\frac{(\tilde{m}(\tau))^n}{n!} + \int_0^\tau\sum_{n=0}^{\infty}\frac{(\tilde{m}(\tau)-\tilde{m}(s))^n}{n!}\xi(s)ds\\
&= \omega_0 e^{\tilde{m}(\tau)} + \int_0^\tau e^{\tilde{m}(\tau)-\tilde{m}(s)}\xi(s)ds = \omega(\tau).
\end{aligned}
$$

Hence, $\sum\limits_{n=0}^{\infty}\|u_{n+1}(\tau)-u_n(\tau)\|$ converges for all $\tau\in\mathcal{K}$, therefore, $(u_n(\tau))_{n\geq 1}$ is a Cauchy sequence, so it converges to the mapping $u(\tau)$, and we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\|h_{n+1}(\tau)-h_n(\tau)\| &\leq \lambda(\tau)\Big\{\omega_0\sum_{n=1}^{\infty}\frac{(\tilde{m}(\tau))^{n-1}}{(n-1)!} + \int_0^\tau\sum_{n=1}^{\infty}\frac{(\tilde{m}(\tau)-\tilde{m}(s))^{n-1}}{(n-1)!}\xi(s)ds\Big\}\\
&= \lambda(\tau)\Big\{\omega_0 e^{\tilde{m}(\tau)} + \int_0^\tau e^{\tilde{m}(\tau)-\tilde{m}(s)}\xi(s)ds\Big\} = \lambda(\tau)\omega(\tau),
\end{aligned}
$$

then, $\sum\limits_{n=1}^{\infty}\|h_{n+1}(\tau)-h_n(\tau)\|$ converges for almost all $\tau\in\mathcal{K}$, and $(h_n(\tau))_{n\geq 1}$ is a Cauchy sequence, hence it converges to some mapping $h(\tau)$.
For all $\tau\in\mathcal{K}$, since $\|\dot{u}_n(\tau)\|\leq\dot{\omega}(\tau)$ and $\dot{\omega}\in L^2_{\mathbf{R}^+}(\mathcal{K})\subset L^1_{\mathbf{R}^+}(\mathcal{K})$, $(\dot{u}_n)_{n\geq 1}$ converges weakly in $L^1_{\mathbf{R}^d}(\mathcal{K})$ to y with $\|y(\tau)\|\leq\dot{\omega}(\tau)$. Fixing $\tau\in\mathcal{K}$ and taking any $\epsilon\in\mathbf{R}^d$, the above weak convergence in $L^1_{\mathbf{R}^d}(\mathcal{K})$ yields

$$
\lim_{n\to+\infty}\int_0^d\langle\mathbf{1}_{[0,\tau]}(s)\epsilon,\dot{u}_n(s)\rangle ds = \int_0^d\langle\mathbf{1}_{[0,\tau]}(s)\epsilon,y(s)\rangle ds,
$$

or equivalently

$$
\lim_{n\to+\infty}\langle\epsilon,v_0+\int_0^\tau\dot{u}_n(s)ds\rangle = \langle\epsilon,v_0+\int_0^\tau y(s)ds\rangle.
$$

Then, $\lim_{n\to+\infty}\int_0^\tau \dot{u}_n(s)ds = \int_0^\tau y(s)ds$, since $u_n(\cdot)$ is an absolutely continuous mapping, we get $u(\tau) = v_0 + \int_0^\tau y(s)ds$ and $y = \dot{u}$. According to Mazur's theorem and using the above arguments in proof of Theorem 3.1, one gets

$$\dot{u}(\tau) \in -\mathcal{N}_{V(u(\tau))}(u(\tau)) + h(\tau).$$

By $(\mathcal{H}_3)$, (3.15), (3.17) and (3.23), we obtain

$$d(h_{n+1}(\tau), H(\tau, u(\tau))) \leq \lambda(\tau)\|u_n(\tau) - u(\tau)\|, \text{ a.e. } \tau \in \mathcal{K},$$

and by passing to the limit in the preceding inequality, we get

$$d(h(\tau), H(\tau, u(\tau))) = 0, \text{ a.e. } \tau \in \mathcal{K},$$

then, the closedness of $H(\tau, u(\tau))$ gives $h(\tau) \in H(\tau, u(\tau))$, a.e. $\tau \in \mathcal{K}$. □

The following proposition gives conditions under which the convexified problem admits a solution.

Proposition 3.4. *Assume that $(\mathcal{H}_V)$ holds and $\|v_0\| \leq \omega_0 < b$. Let $H : \mathcal{I} \times b\overline{B} \rightrightarrows \mathbf{R}^d$ be a set-valued mapping with nonempty closed values satisfying the following assumptions:*

$(\mathcal{H}_1^{\overline{co}})$ *$\overline{co}H(\cdot, y(\cdot))$ is measurable for every $y \in \mathcal{C}_{\mathbf{R}^d}(\mathcal{I})$ with $\|y(\tau)\| \leq b$, for all $\tau \in \mathcal{I}$;*

$(\mathcal{H}_2^{\overline{co}})$ *there exist $\mu \in L^2_{\mathbf{R}^+}(\mathcal{I})$ and $\zeta > 0$ such that for all $\varrho > 0$ and all y_1, y_2 in $b\overline{B}$, $y_1 \neq y_2$*

$$haus_\varrho(\overline{co}H(\tau, y_1), \overline{co}H(\tau, y_2)) < (\mu(\tau) + \zeta\varrho)\|y_1 - y_2\|;$$

$(\mathcal{H}_3^{\overline{co}})$ *for all $\varrho_0 \in L^2_{\mathbf{R}^+}(\mathcal{I})$ and all $(\tau, y) \in \mathcal{I} \times b\overline{B}$, $d(0, \overline{co}H(\tau, y)) \leq \varrho_0(\tau)$.*

Then, there exists a mapping $v \in \mathcal{R}(\mathcal{P}_{\overline{co}H,\mathcal{I}})$ satisfying

$$\|v(\tau)\| \leq \breve{\omega}(\tau), \ \forall \tau \in \mathcal{I},$$

with $\breve{\omega}(\tau) \leq b$ and $\breve{\omega}$ the solution of

$$\begin{cases} \dot{\breve{\omega}}(\tau) = \varrho(\tau) + 4(\mu(\tau) + \zeta\varrho(\tau))\breve{\omega}(\tau), \ \forall \tau \in \mathcal{I}; \\ \breve{\omega}(0) = \omega_0 \geq 0. \end{cases}$$

Proof. Put $\varrho(\tau) = 2\varrho_0(\tau)$ for all $\tau \in \mathcal{I}$. It follows from $(\mathcal{H}_3^{\overline{co}})$ that $\overline{co}H(\tau, y)) \cap \varrho_0(\tau)\overline{B} \neq \emptyset$ and $\overline{co}H(\tau, y)) \cap \varrho(\tau)\overline{B} \neq \emptyset$, hence we can define a measurable set-valued mapping $U : \mathcal{I} \times b\overline{B} \rightrightarrows \mathbf{R}^d$ with nonempty closed values by

$$U(\tau, y) = \overline{co}H(\tau, y)) \cap \varrho(\tau)\overline{B}, \text{ for all } \tau \in \mathcal{I}.$$

Proposition 2.1 implies that for all $u, w \in b\overline{B}$ and all $\tau \in \mathcal{I}$

$$\begin{aligned} haus(U(\tau, u), U(\tau, w)) &= haus(\overline{co}H(\tau, u)_{\varrho(\tau)}, \overline{co}H(\tau, w)_{\varrho(\tau)}) \\ &\leq \frac{2\varrho(\tau)}{\varrho(\tau) - \varrho_0(\tau)} haus_{\varrho(\tau)}(\overline{co}H(\tau, u), \overline{co}H(\tau, w)), \end{aligned}$$

by $(\mathcal{H}_2^{\overline{co}})$, we can write

$$\begin{aligned} haus(U(\tau,u),U(\tau,w)) &< \frac{2\varrho(\tau)}{\varrho(\tau)-\varrho_0(\tau)}(\mu(\tau)+\zeta\varrho(\tau))\|u-w\| \\ &= 4(\mu(\tau)+\zeta\varrho(\tau))\|u-w\|. \end{aligned}$$

For $x(\tau)\in U(\tau,0)$ and $y\in b\overline{B}$, we obtain

$$\begin{aligned} d(0,U(\tau,y)) &\le d(0,x(\tau))+d(x(\tau),U(\tau,y)) \\ &\le \|x(\tau)\|+\sup_{x(\tau)\in U(\tau,0)} d(x(\tau),U(\tau,y)) \\ &\le \varrho(\tau)+haus(U(\tau,y),U(\tau,0)) \\ &< \varrho(\tau)+4(\mu(\tau)+\zeta\varrho(\tau))\|y\|. \end{aligned}$$

By Theorem 3.1, there exists a mapping v solution of $(\mathcal{P}_{U,\mathcal{I}})$, since $U(\tau,v(\tau))\subset \overline{co}H(\tau,v(\tau))$, we conclude that $v\in\mathcal{R}(\mathcal{P}_{\overline{co}H,\mathcal{I}})$ and

$$\|\dot v(\tau)\|\le \frac{2}{1-L}(\varrho(\tau)+4m_1(\mu(\tau)+\zeta\varrho(\tau))),\ \text{a.e. } \tau\in\mathcal{I}\quad\text{with}\quad m_1=2\Big(\|v_0\|+\frac{2}{1-L}\|\varrho\|_1\Big).$$

□

Now, we present another version of Theorem 3.1 needed in the study of the relaxation property.
Let $\psi:\mathcal{I}\to\mathbf{R}^d$ be a Lipschitz mapping with $\psi(0)=v_0$ and $Q:\mathbf{R}^d\rightrightarrows\mathcal{I}\times\mathbf{R}^d$ be a set-valued mapping defined by

$$Q(\psi)=\left\{(\tau,y)\in\mathcal{I}\times\mathbf{R}^d:\|\psi(\tau)-y\|\le b\right\},\ b>0.$$

Theorem 3.5. *Assume that $(\mathcal{H}_V)$ holds. Let $H:Q(\psi)\rightrightarrows\mathbf{R}^d$ be a set-valued mapping with nonempty closed values such that*

$(\mathcal{H}_4)$ *$H(\cdot,y(\cdot))$ is measurable for every $y\in\mathcal{C}_{\mathbf{R}^d}(\mathcal{I})$ with $\|y(\tau)-\psi(\tau)\|\le b$, for all $\tau\in\mathcal{I}$;*

$(\mathcal{H}_5)$ *for all $(\tau,y)\in Q(\psi)$, $d(\dot\psi(t),H(\tau,y))\le\xi(\tau)+\lambda(\tau)\|y-\psi(\tau)\|$;*

$(\mathcal{H}_6)$ *for all $\varrho\ge 0$ and $y_1\ne y_2$ such that $(\tau,y_1),(\tau,y_2)$ are in $Q(\psi)$, one has*

$$haus_\varrho(-\dot\psi(\tau)+H(\tau,y_1),-\dot\psi(\tau)+H(\tau,y_2))<\lambda(\tau)\|y_1-y_2\|.$$

Then, the problem $(\mathcal{P}_{H,\mathcal{I}})$ admits a solution.

Proof. Let $D_\psi:\mathbf{R}^d\rightrightarrows\mathbf{R}^d$, be a set-valued mapping defined by:

$$D_\psi(u)=V(u+\psi(\tau))-\psi(\tau).$$

The set $D_\psi(\cdot)$ satisfies $(\mathcal{A}_1)$ and by $(\mathcal{A}_2)$, we get for all $\nu,z_1,z_2\in\mathbf{R}^d$

$$|d(\nu,D_\psi(z_1))-d(\nu,D_\psi(z_2))|\le L\|z_1-z_2\|.$$

Let $G : \mathcal{I} \times b\overline{B} \rightrightarrows \mathbf{R}^d$ be a measurable set-valued mapping with nonempty closed values defined by $G(\tau, z) = -\dot{\psi}(\tau) + H(\tau, z + \psi(\tau))$. For all $x(\tau) \in G(\tau, z)$ and all $\tau \in \mathcal{I}$,

$$d(0, G(\tau, z)) = \inf_{x(\tau) \in (-\dot{\psi}(\tau) + H(\tau, z+\psi(\tau)))} \|x(\tau)\|.$$

Setting $w(\tau) = x(\tau) + \dot{\psi}(\tau)$, one obtains

$$\begin{aligned} d(0, G(\tau, z)) &= \inf_{w(\tau) \in H(\tau, z+\psi(\tau))} \| - \dot{\psi}(\tau) + w(\tau)\| \\ &= d(\dot{\psi}(\tau), H(\tau, z + \psi(\tau))), \end{aligned}$$

and using $(\mathcal{H}_5)$, we get

$$d(0, G(\tau, z)) \leq \xi(\tau) + \lambda(\tau)\|z\|.$$

For all $\tau \in \mathcal{I}$ and $z_1, z_2 \in b\overline{B}$ such that $z_1 \neq z_2$, we have

$$haus_{\varrho}(G(\tau, z_1), G(\tau, z_2)) = haus_{\varrho}(-\dot{\psi}(\tau) + H(\tau, z_1 + \psi(\tau)), -\dot{\psi}(\tau) + H(\tau, z_2 + \psi(\tau))).$$

Combining with $(\mathcal{H}_6)$, it follows

$$haus_{\varrho}(G(\tau, z_1), G(\tau, z_2)) < \lambda(\tau)\|z_1 - z_2\|.$$

By Theorem 3.1, the problem

$$\begin{cases} -\dot{y}(\tau) \in \mathcal{N}_{D_\psi(y(\tau))}(y(\tau)) + G(\tau, y(\tau)), \text{ a.e. } \tau \in \mathcal{I}; \\ y(\tau) \in D_\psi(y(\tau)), \ \forall \tau \in \mathcal{I}; \\ y(0) = 0 \in D_\psi(0), \end{cases}$$

admits a solution $y : \mathcal{I} \to \mathbf{R}^d$ such that $\|\dot{y}(\tau)\| \leq \tilde{\beta}(\tau)$, a.e. $\tau \in \mathcal{I}$, and $y(\tau) = v(\tau) - \psi(\tau)$ for all $\tau \in \mathcal{I}$. Then, $v \in \mathcal{R}(\mathcal{P}_{H,\mathcal{I}})$ and

$$\|\dot{v}(\tau) - \dot{\psi}(\tau)\| \leq \tilde{\beta}(\tau), \text{ a.e. } \tau \in \mathcal{I}.$$

□

4 Relaxation

Along this section, we will set up the approximation of the solution sets $\mathcal{R}(\mathcal{P}_{\overline{co}H,\mathcal{I}})$ by the solution sets $\mathcal{R}(\mathcal{P}_{H,\mathcal{I}})$. For this purpose assume $\gamma \in L^2_{\mathbf{R}^+}(\mathcal{I})$ and v_{co} such that

$$\begin{cases} -\dot{v}_{co}(\tau) \in \mathcal{N}_{V(v_{co}(\tau))}(v_{co}(\tau)) + m(\tau, v_{co}(\tau)), \text{ a.e. } \tau \in \mathcal{I}; \\ v_{co}(\tau) \in V(v_{co}(\tau)), \ \forall \tau \in \mathcal{I} \text{ and } v_{co}(0) = v_0, \end{cases}$$

where $m(\tau, v_{co}(\tau)) = Proj_{\overline{co}(H(\tau, v_{co}(\tau)) \cap \gamma(\tau)\overline{B})}(0)$ and $v_{co} \in \mathcal{R}(\mathcal{P}_{\overline{co}H,\mathcal{I}})$. We define a set-valued mapping $Q(v_{co}) : \mathbf{R}^d \rightrightarrows \mathcal{I} \times \mathbf{R}^d$ as follows:

$$Q(v_{co}) = \left\{(\tau, y) \in \mathcal{I} \times \mathbf{R}^d : \|v_{co}(\tau) - y\| \leq b\right\}, \ b > 0.$$

Theorem 4.1. *Suppose that* $(\mathcal{H}_V)$ *holds. Let* $H : Q(v_{co}) \rightrightarrows \mathbf{R}^d$ *justifies* $(\mathcal{H}_4)$ *and the following assumptions:*

$(\mathcal{H}_7)$ *there exists* $\xi \in L^2_{\mathbf{R}^+}(\mathcal{I})$, *such that for all* $(\tau, y) \in Q(v_{co})$, $H(\tau, y) \cap \xi(\tau)\overline{B} \neq \emptyset$;

$(\mathcal{H}_8)$ *there exist* $\zeta > 0$ *and* $\phi \in L^2_{\mathbf{R}^+}(\mathcal{I})$ *such that for all* $(\tau, y_1), (\tau, y_2)$ *in* $Q(v_{co})$ *with* $\varrho \geq 0$,

$$haus_\varrho(H(\tau, y_1), H(\tau, y_2)) < (\phi(\tau) + \zeta\varrho)\|y_1 - y_2\|;$$

$(\mathcal{H}_9)$ *there exists* $\sigma \in L^2_{\mathbf{R}^+}(\mathcal{I})$, *such that for all* $(\tau, y) \in Q(v_{co})$ *with* $y \neq v_{co}$, *we have*

$$d(\dot{y}(\tau), H(\tau, y)) \leq \sigma(\tau)\|v_{co}(\tau) - y\|.$$

Then, $\mathcal{R}(\mathcal{P}_{H,\mathcal{I}})$ *is dense in* $\mathcal{R}(\mathcal{P}_{\overline{co}H,\mathcal{I}})$ *with respect to the topology of uniform convergence.*

Proof. Let $\Pi : \mathcal{I} \rightrightarrows \mathbf{R}^d$ be a measurable, closed and nonempty set-valued mapping defined by

$$\Pi(\tau) = H(\tau, v_{co}(\tau)) \cap \gamma(\tau)\overline{B}.$$

Using Theorem 1.5 in [19], we get $S(\overline{co}\Pi) = \overline{co}S(\Pi)$. Then $m(\tau, v_{co}(\tau)) \in \overline{co}S(\Pi)$ and

$$\forall \epsilon > 0, \exists h_\epsilon \in coS(\Pi) : \|m - h_\epsilon\|_2 \leq \epsilon.$$

By Proposition 3.2, the problem $(\mathcal{P}_{h_\epsilon,\mathcal{I}})$ admits a solution $v_{h_\epsilon} : \mathcal{I} \to \mathbf{R}^d$, such that v_{h_ϵ} converges uniformly to v_{co} as $\epsilon \to 0$.
Since $h_\epsilon \in coS(\Pi)$ there exists a finite collection of elements $f_i \in S(\Pi)$, $1 \leq i \leq \bar{n}$, and positive constants α_i, such that $\sum_{i=1}^{\bar{n}} \alpha_i = 1$, and $h_\epsilon = \sum_{i=1}^{\bar{n}} \alpha_i f_i$.
Let $F : \mathcal{I} \rightrightarrows \mathbf{R}^d$ be a set-valued mapping defined by

$$F(\tau) = \{f_i(\tau) : 1 \leq i \leq \bar{n}\}, \ \forall \tau \in \mathcal{I}.$$

Then, $h_\epsilon \in coF(\tau)$, since $\tau \to \|F(\tau)\|$ is square integrable by using Theorem 2.2 in [27] there exists a sequence $(r_n)_{n\in N^*} \subset L^2_{\mathbf{R}^d}(\mathcal{I})$ such that

$$r_n(\tau) \in F(\tau) \subset \Pi(\tau),$$

and

$$\lim_{n\to\infty} \max_{0\leq s\leq\tau\leq\mathcal{I}} \| \int_s^\tau (r_n(\varsigma) - h_\epsilon(\varsigma))d\varsigma\| = 0.$$

By Lemma 2.3 in [28], we conclude that $(r_n)_n$ converges weakly to h_ϵ in $L^2_{\mathbf{R}^d}(\mathcal{I})$. By Proposition 3.2 $(\mathcal{P}_{r_n,\mathcal{I}})$ admits a solution $g_n : \mathcal{I} \to \mathbf{R}^d$ such that g_n converges uniformly to v_{h_ϵ} when $n \to \infty$.
Since

$$\|g_n - v_{co}\|_{\mathcal{C}} \leq \|g_n - v_{h_\epsilon}\|_{\mathcal{C}} + \|v_{h_\epsilon} - v_{co}\|_{\mathcal{C}},$$

we deduce that g_n converges uniformly to v_{co} when $n \to \infty$. Then, we can write

$$\exists n_0 \in \mathbb{N}, \ \forall n \geq n_0 \ : \ \|g_n - v_{co}\|_{\mathcal{C}} \leq b/2. \tag{4.1}$$

Set

$$\vartheta(\tau) = \max(\xi(\tau), \phi(\tau), \gamma(\tau), \sigma(\tau)),\ \forall \tau \in \mathcal{I}. \tag{4.2}$$

Using $(\mathcal{H}_7)$, (4.2) and (4.1), one has for all $n \geq n_0$ and all $(\tau, y(\tau)) \in Q(v_{co})$

$$H(\tau, y(\tau)) \cap \vartheta(\tau)\overline{B} \neq \emptyset. \tag{4.3}$$

By $(\mathcal{H}_9)$, (4.1) and (4.2)

$$d(\dot{g}_n(\tau), H(\tau, g_n(\tau))) \leq \vartheta(\tau)\|v_{co}(\tau) - g_n(\tau)\|. \tag{4.4}$$

Furthermore, for all $n \geq n_0$ and all $(\tau, z) \in \mathcal{I} \times (b/2)\overline{B}$

$$\begin{aligned} -\dot{g}_n(\tau) + H(\tau, z + g_n(\tau)) \cap \vartheta(\tau)\overline{B} &\subset (-\dot{g}_n(\tau) + H(\tau, z + g_n(\tau))) \cap (-\dot{g}_n(\tau) + \vartheta(\tau)\overline{B}) \\ &\subset (-\dot{g}_n(\tau) + H(\tau, z + g_n(\tau))) \cap \Big(\frac{(3-L)\vartheta(\tau)}{1-L}\Big)\overline{B}. \end{aligned} \tag{4.5}$$

Taking into account (4.3), for $n \geq n_0$ and all $(\tau, z) \in \mathcal{I} \times (b/2)\overline{B}$, there exists a mapping $\acute{p}_n(\tau) \in H(\tau, z + g_n(\tau)) \cap \vartheta(\tau)\overline{B}$. Set $\acute{y}_n(\tau) = \acute{p}_n(\tau) - \dot{g}_n(\tau)$, so

$$\acute{y}_n(\tau) \in -\dot{g}_n(\tau) + H(\tau, z + g_n(\tau)) \cap \vartheta(\tau)\overline{B},$$

from (4.5), we conclude that for all $(\tau, z) \in \mathcal{I} \times (b/2)\overline{B}$, $n \geq n_0$

$$(-\dot{g}_n(\tau) + H(\tau, z + g_n(\tau))) \cap \Big(\frac{(3-L)\vartheta(\tau)}{1-L}\Big)\overline{B} \neq \emptyset.$$

Set $\varrho(\tau) = \frac{(5-L)\vartheta(\tau)}{1-L}$ and $\mu_n(\tau) = \acute{y}_n(\tau) + \dot{g}_n(\tau)$, then, $\mu_n(\tau) \in H_{\varrho(\tau)}(\tau, z + g_n(\tau))$, and $(\mathcal{H}_8)$ gives for all $z, \acute{z} \in (b/2)\overline{B}$,

$$\begin{aligned} d(\mu_n(\tau), H(\tau, \acute{z} + g_n(\tau))) &\leq e(H_{\varrho(\tau)}(\tau, z + g_n(\tau)), H(\tau, \acute{z} + g_n(\tau))) \\ &\leq haus_{\varrho(\tau)}(H(\tau, z + g_n(\tau)), H(\tau, \acute{z} + g_n(\tau))) \\ &< (\phi(\tau) + \zeta\varrho(\tau))\|z - \acute{z}\| \\ &\leq \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|. \end{aligned}$$

So, there exists $\acute{n}(\tau) \in H(\tau, \acute{z} + g_n(\tau))$, such that

$$\mu_n(\tau) - \acute{n}(\tau) \in \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|\overline{B},$$

then,

$$\begin{aligned} \acute{y}_n(\tau) &\in -\dot{g}_n(\tau) + \acute{n}(\tau) + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|\overline{B} \\ &\subset -\dot{g}_n(\tau) + H(\tau, \acute{z} + g_n(\tau)) + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|\overline{B}. \end{aligned} \tag{4.6}$$

Let $G_n : \mathcal{I} \times (b/2)\overline{B} \rightrightarrows \mathbf{R}^d$ be a measurable set-valued mapping defined by

$$G_n(\tau, z) = -\dot{g}_n(\tau) + H(\tau, z + g_n(\tau)),\ n \geq n_0. \tag{4.7}$$

From (4.6) and (4.7), we can write for all $z, \acute{z} \in (b/2)\overline{B}$ and all $n \geq n_0$,

$$G_n(\tau, z) \cap \Big(\frac{(3-L)\vartheta(\tau)}{1-L}\Big)\overline{B} \subset G_n(\tau, \acute{z}) + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|\overline{B}. \tag{4.8}$$

If $\varrho(\tau) = \frac{(3-L)\vartheta(\tau)}{1-L}$, this last inclusion entails that

$$e(G_{n,\varrho(\tau)}(\tau, z), G_n(\tau, \acute{z})) \leq \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|, \tag{4.9}$$

and by alternating the roles between z and $\acute{z}$, we obtain

$$e(G_{n,\varrho(\tau)}(\tau, \acute{z}), G_n(\tau, z)) \leq \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|. \tag{4.10}$$

From (4.9) and (4.10), we may write

$$haus_{\varrho(\tau)}(G_n(\tau, z), G_n(\tau, \acute{z})) \leq \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z - \acute{z}\|. \tag{4.11}$$

Coming back to (4.8), by interchanging the roles between z and $\acute{z}$ and for $\acute{z} = 0$, we get for all $(\tau, z) \in \mathcal{I} \times (b/2)\overline{B}$ and all $n \geq n_0$

$$G_n(\tau, 0) \cap \Big(\frac{(3-L)\vartheta(\tau)}{1-L}\Big)\overline{B} \subset G_n(\tau, z) + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z\|\overline{B}.$$

Thus, for $n \geq n_0$ with $\|z\| \leq b/2$ and $\acute{y}_n(\tau) \in G_n(\tau, 0) \cap \Big(\frac{(3-L)\vartheta(\tau)}{1-L}\Big)\overline{B}$ there exists $\acute{w}_n(\tau) \in G_n(\tau, z)$, such that

$$\|\acute{y}_n(\tau) - \acute{w}_n(\tau)\| \leq \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z\|,$$

or equivalently

$$d(\acute{y}_n(\tau), G_n(\tau, z)) \leq \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z\|. \tag{4.12}$$

Thus,

$$d(0, G_n(\tau, z)) \leq \|\acute{y}_n(\tau)\| + d(\acute{y}_n(\tau), G_n(\tau, z)).$$

By (4.12), we can write

$$d(0, G_n(\tau, z)) \leq \|\acute{y}_n(\tau)\| + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z\|. \tag{4.13}$$

Furthermore, (4.4) and (4.7), give

$$d(0, G_n(\tau, 0)) \leq \vartheta(\tau)\|v_{co}(\tau) - g_n(\tau)\|. \tag{4.14}$$

Since $\acute{y}_n(\tau) \in G_n(\tau,0) \cap \Big(\frac{(3-L)\vartheta(\tau)}{1-L}\Big)\overline{B}$, the inequality (4.13) implies that

$$\begin{aligned} d(0,G_n(\tau,z)) &\leq \inf_{\acute{y}_n(\tau)\in G_n(\tau,0)} \|\acute{y}_n(\tau)\| + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z\| \\ &= d(0,G_n(\tau,0)) + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z\|, \end{aligned}$$

for all $z \in (b/2)\overline{B}$ and $n \geq n_0$, thus it follows from (4.14)

$$d(0,G_n(\tau,z)) \leq \vartheta(\tau)\|v_{co} - g_n\|_{\mathfrak{C}} + \Big(\vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big)\Big)\|z\|. \tag{4.15}$$

Put

$$\tilde{a}_n(\tau) = \vartheta(\tau)\|v_{co} - g_n\|_{\mathfrak{C}},$$

and

$$\tilde{k}(\tau) = \vartheta(\tau) + \zeta\Big(\frac{(5-L)\vartheta(\tau)}{1-L}\Big).$$

With this notation, (4.11) and (4.15) become

$$d(0,G_n(\tau,z)) \leq \tilde{a}_n(\tau) + \tilde{k}(\tau)\|z\|,$$

$$haus_{\varrho(\tau)}(G_n(\tau,z),G_n(\tau,\acute{z})) \leq \tilde{k}(\tau)\|z-\acute{z}\|.$$

Let us consider the differential equation below

$$\begin{cases} \dot{\omega}_n(\tau) = \tilde{a}_n(\tau) + \tilde{k}(\tau)\omega_n(\tau), \ \forall \tau \in \mathcal{I}; \\ \omega_n(0) = 0, \end{cases}$$

with the solution $\omega_n(\tau) = \int_0^\tau e^{\eta(\tau)-\eta(s)}\tilde{a}_n(s)ds$ and $\eta(\tau) = \int_0^\tau \tilde{k}(s)ds$, $\forall \tau \in \mathcal{I}$. Note that for all $\tau \in \mathcal{I}$, $\omega_n(\tau)$ converges to 0, when n tends to ∞, hence, for all $n \geq n_0$ and all $\tau \in \mathcal{I}$, $0 \leq \omega_n(\tau) \leq b/2$. By Theorem 3.5,

$$\begin{cases} \dot{v}_n(\tau) \in -\mathcal{N}_{V(v_n(\tau))}(v_n(\tau)) + H(\tau,v_n(\tau)), \text{ a.e. } \tau \in \mathcal{I}; \\ v_n(\tau) \in V(v_n(\tau)), \ \forall \tau \in \mathcal{I}; \\ v_n(0) = v_0 \in V(v_0), \end{cases}$$

admits a solution $v_n : \mathcal{I} \to \mathbf{R}^d$ satisfying

$$\|v_n(\tau) - g_n(\tau)\| \leq \omega_n(\tau), \ \forall \tau \in \mathcal{I}.$$

Then,

$$\|v_n(\tau) - v_{co}(\tau)\| \leq \omega_n(\tau) + \|g_n(\tau) - v_{co}(\tau)\|,$$

by passing to the limit, then $(v_n)_{n\geq 1}$ converges uniformly to v_{co}, with $\|v_n(\tau) - v_{co}(\tau)\| \leq b$. The proof is complete.

□

Conclusion

The paper investigates the existence and relaxation of solutions to a first order perturbed sweeping process in a finite dimensional setting without use of the convexity and any boundeness assumption and by applying a weaker Hausdorff Lipschitz truncated condition on the perturbation. Our method is based on the approach given in [7] and the relaxation result in [31]. The existence result, stated for convex sets, can be extended to more large class of sets like uniformly r-prox regular or subsmooth sets. The study of the relaxation for an iterative inclusion with non-convex right hand side will be the subject of forthcoming works.

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