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Fixed Point Results in Incomplete Metric and Dislocated Metric Spaces

Hakima Bouhadjera

Abstract This paper deals with the existence and uniqueness of common fixed points in metric and dislocated metric spaces. In addition, an application and some illustrative examples are given in order to justify the validity and credibility of our results, also, their superiority over some similar theorems existing in unique common fixed points theory's domain.

AMS Subject Classification (2020) 47H10; 54H25
Keywords Metric space; dislocated metric space; occasionally weakly biased maps of type $(\mathcal{A})$; unique common fixed point theorems

1 Introduction

The study of unique fixed point and unique common fixed points for a single map or pairs of maps satisfying different and various contractive conditions is a modern and extremely active axis during recent years. Several authors have proved the existence and uniqueness of common fixed points for pairs of single and multi-valued maps satisfying diverse contractive conditions in different spaces. In particular, Matthews introduced the notions of dislocated [7] and partial [8] metric spaces so as to show that in addition of metric spaces, fixed points can exist in other spaces under various contractive conditions. For this end, in a few months we gave a new type of compatibility called occasionally weakly biased maps of type $(\mathcal{A})$ [3] for generalizing weakly compatible [4], weakly biased [5], occasionally weakly compatible [2] and weakly biased of type $(\mathcal{A})$ maps [9].

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2 Preliminary Notes

In this section, we only give the following definitions:

Definition 2.1. *([7]) A* ***metric domain (dislocated metric space)*** *is a pair* $< \mathcal{D}, \sigma >$ *where* $\mathcal{D}$ *is a non-empty set, and* σ *is a function from* $\mathcal{D} \times \mathcal{D}$ *to* $\mathbb{R}_+$ *such that*

1. $\forall\ a,\ b \in \mathcal{D}$: $\sigma(a,b) = 0 \Rightarrow a = b$
2. $\forall\ a,\ b \in \mathcal{D}$: $\sigma(a,b) = \sigma(b,a)$
3. $\forall\ a,\ b,\ c \in \mathcal{D}$: $\sigma(a,b) \leq \sigma(a,c) + \sigma(c,b)$.

Definition 2.2. *([8]) A* ***partial metric (pmetric)*** *is a function* $p : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, *such that*

$(P1)$ $\forall\ \alpha,\ \beta \in \mathcal{A}$: $\alpha = \beta \Leftrightarrow p(\alpha,\alpha) = p(\alpha,\beta) = p(\beta,\beta)$

$(P2)$ $\forall\ \alpha,\ \beta \in \mathcal{A}$: $p(\alpha,\alpha) \leq p(\alpha,\beta)$

$(P3)$ $\forall\ \alpha,\ \beta \in \mathcal{A}$: $p(\alpha,\beta) = p(\beta,\alpha)$

$(P4)$ $\forall\ \alpha,\ \beta,\ \gamma \in \mathcal{A}$: $p(\alpha,\gamma) \leq p(\alpha,\beta) + p(\beta,\gamma) - p(\beta,\beta)$.

Definition 2.3. *([3]) Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be self-maps of a non-empty set* $\mathcal{X}$. *The pair* $(\mathcal{S},\mathcal{T})$ *is said to be* ***occasionally weakly*** $\mathcal{S}$***-biased of type*** $(\mathcal{A})$ *and* ***occasionally weakly*** $\mathcal{T}$***-biased of type*** $(\mathcal{A})$, *respectively, if and only if, there exists a point* ι *in* $\mathcal{X}$ *such that* $\mathcal{S}\iota = \mathcal{T}\iota$ *implies*

$$d(\mathcal{S}\mathcal{S}\iota, \mathcal{T}\iota) \leq d(\mathcal{T}\mathcal{S}\iota, \mathcal{S}\iota),$$
$$d(\mathcal{T}\mathcal{T}\iota, \mathcal{S}\iota) \leq d(\mathcal{S}\mathcal{T}\iota, \mathcal{T}\iota),$$

respectively.

3 Mahendra Singh et al. Results

In their paper [6], Y. Mahendra Singh et al. proved the next results:

Theorem 3.1. *Let* f *and* g *be two self maps of a metric space* $(\mathcal{X}, d)$ *and* $g\mathcal{X}$ *is closed in* $\mathcal{X}$ *such that*

1. f *and* g *satisfy property* $(E.A)$;
2. *for every* $x \neq y \in \mathcal{X}$

$$d(fx,fy) < \max\left\{d(gx,gy), \frac{1}{2}\left[d(fx,gx) + d(fy,gy)\right], \frac{1}{2}\left[d(fy,gx) + d(fx,gy)\right]\right\}.$$

Then, $C(f,g) \neq \emptyset$. *If* f *and* g *are occasionally weakly* g*-biased then,* f *and* g *have a unique common fixed point.*

Remark 3.1. Note that

1. $C(f,g) = \{u \in \mathcal{X} : fu = gu\}$.
2. Maps f and g are said to satisfy property $(E.A)$ if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in \mathcal{X}$ ([1]).

Proposition 3.2. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{T}$ be self maps of a metric space $(\mathcal{X}, d)$ satisfying the following conditions*

1. *$\mathcal{B}\mathcal{X} \subseteq \mathcal{S}\mathcal{X}$ and $\mathcal{T}\mathcal{X}$ is closed in $\mathcal{X}$,*
2. *the pair $(\mathcal{B}, \mathcal{T})$ satisfies property $(E.A)$ and*
3. *for every x, $y \in \mathcal{X}$*

$$d(\mathcal{A}x, \mathcal{B}y) \leq F\left(\max\left\{d(\mathcal{S}x, \mathcal{T}y), \frac{1}{2}\left[d(\mathcal{A}x, \mathcal{S}x) + d(\mathcal{B}y, \mathcal{T}y)\right], \frac{1}{2}\left[d(\mathcal{T}y, \mathcal{A}x) + d(\mathcal{S}x, \mathcal{B}y)\right]\right\}\right),$$

where $F : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing and $0 < F(t) < t$, $\forall t > 0$. Then, $C(\mathcal{A}, \mathcal{S}) \neq \emptyset$ and $C(\mathcal{B}, \mathcal{T}) \neq \emptyset$.

Theorem 3.3. *In addition to all the hypothesis of the above proposition on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$, if the maps pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are occasionally weakly $\mathcal{S}$ and $\mathcal{T}$-biased, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in $\mathcal{X}$.*

In this contribution, we will improve and extend Theorem 3.1, Proposition 3.2 and Theorem 3.3 above and some similar results in metric, partial metric and dislocated metric spaces.

4 Unique Common Fixed Points in Metric Spaces

Theorem 4.1. *Let $\mathcal{B}$, $\mathcal{F}$, $\mathcal{J}$ and $\mathcal{M}$ be self-maps of a metric space $(\mathcal{X}, d)$ such that*

$$d(\mathcal{B}x, \mathcal{F}y) < \max\left\{d(\mathcal{J}x, \mathcal{M}y), \frac{1}{2}[d(\mathcal{B}x, \mathcal{J}x) + d(\mathcal{F}y, \mathcal{M}y)], \frac{1}{2}[d(\mathcal{J}x, \mathcal{F}y) + d(\mathcal{B}x, \mathcal{M}y)]\right\} \tag{4.1}$$

for all x, $y \in \mathcal{X}$ with $x \neq y$. If $\mathcal{B}$ and $\mathcal{J}$ are occasionally weakly $\mathcal{J}$-biased of type $(\mathcal{A})$ and $\mathcal{F}$ and $\mathcal{M}$ are occasionally weakly $\mathcal{M}$-biased of type $(\mathcal{A})$, then, the four maps have a unique common fixed point.

Proof. Since $\mathcal{B}$ and $\mathcal{J}$ are occasionally weakly $\mathcal{J}$-biased of type $(\mathcal{A})$, then, there is a point $\theta \in \mathcal{X}$ such that $\mathcal{B}\theta = \mathcal{J}\theta$ implies $d(\mathcal{J}\mathcal{J}\theta, \mathcal{B}\theta) \leq d(\mathcal{B}\mathcal{J}\theta, \mathcal{J}\theta)$. The same, as $\mathcal{F}$ and $\mathcal{M}$ are occasionally weakly $\mathcal{M}$-biased of type $(\mathcal{A})$, then, there exists an element $\vartheta \in \mathcal{X}$ such that $\mathcal{F}\vartheta = \mathcal{M}\vartheta$ implies $d(\mathcal{M}\mathcal{M}\vartheta, \mathcal{F}\vartheta) \leq d(\mathcal{F}\mathcal{M}\vartheta, \mathcal{M}\vartheta)$. For the existence and uniqueness of the common fixed point we need four steps.

First step: We claim that $\mathcal{B}\theta = \mathcal{F}\vartheta$. Suppose that $d(\mathcal{B}\theta, \mathcal{F}\vartheta) > 0$. Using inequality (4.1) we get

$$\begin{aligned} d(\mathcal{B}\theta, \mathcal{F}\vartheta) &< \max\left\{ d(\mathcal{J}\theta, \mathcal{M}\vartheta), \frac{1}{2}[d(\mathcal{B}\theta, \mathcal{J}\theta) + d(\mathcal{F}\vartheta, \mathcal{M}\vartheta)], \right. \\ &\qquad \left. \frac{1}{2}[d(\mathcal{J}\theta, \mathcal{F}\vartheta) + d(\mathcal{B}\theta, \mathcal{M}\vartheta)] \right\} \\ &= \max\{d(\mathcal{B}\theta, \mathcal{F}\vartheta), 0, d(\mathcal{B}\theta, \mathcal{F}\vartheta)\} \\ &= d(\mathcal{B}\theta, \mathcal{F}\vartheta) \end{aligned}$$

which is a contradiction and hence $\mathcal{B}\theta = \mathcal{F}\vartheta$.

Second step: Assume that $d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta) > 0$. The use of condition (4.1) gives

$$\begin{aligned} d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta) &= d(\mathcal{B}\mathcal{B}\theta, \mathcal{F}\vartheta) \\ &< \max\left\{ d(\mathcal{J}\mathcal{B}\theta, \mathcal{M}\vartheta), \frac{1}{2}[d(\mathcal{B}\mathcal{B}\theta, \mathcal{J}\mathcal{B}\theta) + d(\mathcal{F}\vartheta, \mathcal{M}\vartheta)], \right. \\ &\qquad \left. \frac{1}{2}[d(\mathcal{J}\mathcal{B}\theta, \mathcal{F}\vartheta) + d(\mathcal{B}\mathcal{B}\theta, \mathcal{M}\vartheta)] \right\} \\ &= \max\left\{ d(\mathcal{J}\mathcal{J}\theta, \mathcal{B}\theta), \frac{1}{2}d(\mathcal{B}\mathcal{B}\theta, \mathcal{J}\mathcal{J}\theta), \frac{1}{2}[d(\mathcal{J}\mathcal{J}\theta, \mathcal{B}\theta) + d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta)] \right\} \\ &\leq \max\left\{ d(\mathcal{J}\mathcal{J}\theta, \mathcal{B}\theta), \frac{1}{2}[d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta) + d(\mathcal{B}\theta, \mathcal{J}\mathcal{J}\theta)], \right. \\ &\qquad \left. \frac{1}{2}[d(\mathcal{J}\mathcal{J}\theta, \mathcal{B}\theta) + d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta)] \right\} \\ &\leq \max\left\{ d(\mathcal{B}\mathcal{J}\theta, \mathcal{J}\theta), \frac{1}{2}[d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta) + d(\mathcal{J}\theta, \mathcal{B}\mathcal{J}\theta)], \right. \\ &\qquad \left. \frac{1}{2}[d(\mathcal{B}\mathcal{J}\theta, \mathcal{J}\theta) + d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta)] \right\} \\ &= d(\mathcal{B}\mathcal{B}\theta, \mathcal{B}\theta) \end{aligned}$$

which is a contradiction and thus $\mathcal{B}\mathcal{B}\theta = \mathcal{B}\theta$, hence, $\mathcal{J}\mathcal{B}\theta = \mathcal{B}\theta$.

Third step: Now, suppose that $\mathcal{F}\mathcal{F}\vartheta \neq \mathcal{F}\vartheta$. Utilizing inequality (4.1) we obtain

$$\begin{aligned} d(\mathcal{B}\theta, \mathcal{F}\mathcal{F}\vartheta) &< \max\left\{ d(\mathcal{J}\theta, \mathcal{M}\mathcal{F}\vartheta), \frac{1}{2}[d(\mathcal{B}\theta, \mathcal{J}\theta) + d(\mathcal{F}\mathcal{F}\vartheta, \mathcal{M}\mathcal{F}\vartheta)], \right. \\ &\qquad \left. \frac{1}{2}[d(\mathcal{J}\theta, \mathcal{F}\mathcal{F}\vartheta) + d(\mathcal{B}\theta, \mathcal{M}\mathcal{F}\vartheta)] \right\}, \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathcal{F}\vartheta, \mathcal{F}\mathcal{F}\vartheta) &< \max\left\{d(\mathcal{F}\vartheta, \mathcal{M}\mathcal{M}\vartheta), \frac{1}{2}d(\mathcal{F}\mathcal{F}\vartheta, \mathcal{M}\mathcal{M}\vartheta), \frac{1}{2}[d(\mathcal{F}\vartheta, \mathcal{F}\mathcal{F}\vartheta) + d(\mathcal{F}\vartheta, \mathcal{M}\mathcal{M}\vartheta)]\right\} \\ &\leq \max\left\{d(\mathcal{F}\vartheta, \mathcal{M}\mathcal{M}\vartheta), \frac{1}{2}[d(\mathcal{F}\mathcal{F}\vartheta, \mathcal{F}\vartheta) + d(\mathcal{F}\vartheta, \mathcal{M}\mathcal{M}\vartheta)], \right. \\ &\quad \left. \frac{1}{2}[d(\mathcal{F}\vartheta, \mathcal{F}\mathcal{F}\vartheta) + d(\mathcal{F}\vartheta, \mathcal{M}\mathcal{M}\vartheta)]\right\} \\ &\leq \max\left\{d(\mathcal{M}\vartheta, \mathcal{F}\mathcal{M}\vartheta), \frac{1}{2}[d(\mathcal{F}\mathcal{F}\vartheta, \mathcal{F}\vartheta) + d(\mathcal{M}\vartheta, \mathcal{F}\mathcal{M}\vartheta)], \right. \\ &\quad \left. \frac{1}{2}[d(\mathcal{F}\vartheta, \mathcal{F}\mathcal{F}\vartheta) + d(\mathcal{M}\vartheta, \mathcal{F}\mathcal{M}\vartheta)]\right\} \\ &= d(\mathcal{F}\vartheta, \mathcal{F}\mathcal{F}\vartheta). \end{aligned}$$

This contradiction implies that $\mathcal{F}\mathcal{F}\vartheta = \mathcal{F}\vartheta$ and hence $\mathcal{M}\mathcal{F}\vartheta = \mathcal{F}\vartheta$. Consequently, $\mathcal{B}\theta = \mathcal{J}\theta = \mathcal{F}\vartheta = \mathcal{M}\vartheta = \tau$ is a common fixed point of maps $\mathcal{B}$, $\mathcal{F}$, $\mathcal{J}$ and $\mathcal{M}$.

Fourth step: Assume that there is another common fixed point ρ for the four maps. Then, the use of condition (4.1) gives

$$\begin{aligned} d(\mathcal{B}\tau, \mathcal{F}\rho) &< \max\left\{d(\mathcal{J}\tau, \mathcal{M}\rho), \frac{1}{2}[d(\mathcal{B}\tau, \mathcal{J}\tau) + d(\mathcal{F}\rho, \mathcal{M}\rho)], \right. \\ &\quad \left. \frac{1}{2}[d(\mathcal{J}\tau, \mathcal{F}\rho) + d(\mathcal{B}\tau, \mathcal{M}\rho)]\right\}, \end{aligned}$$

i.e.,

$$\begin{aligned} d(\tau, \rho) &< \max\{d(\tau, \rho), 0, d(\tau, \rho)\} \\ &= d(\tau, \rho) \end{aligned}$$

which is a contradiction, hence the common fixed point is unique. □

The next example supports our result.

Example 4.1. Let $\mathcal{X} = (0, +\infty)$ with the usual metric $d(x, y) = |x - y|$. Define

$$\mathcal{B}x = \begin{cases} 1 & \text{if } x \in (0, 1] \\ \frac{1}{2x} & \text{if } x \in (1, +\infty) \end{cases}, \quad \mathcal{F}x = \begin{cases} 1 & \text{if } x \in (0, 1] \\ \frac{3}{4x} & \text{if } x \in (1, +\infty) \end{cases},$$

$$\mathcal{J}x = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 101 & \text{if } x \in (1, +\infty) \end{cases}, \quad \mathcal{M}x = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 11 & \text{if } x \in (1, +\infty) \end{cases}.$$

First, it is clear to see that $\mathcal{B}$ and $\mathcal{J}$ are occasionally weakly $\mathcal{J}$-biased of type $(\mathcal{A})$ and $\mathcal{F}$ and $\mathcal{M}$ are occasionally weakly $\mathcal{M}$-biased of type $(\mathcal{A})$. We have

1. for x, $y \in (0,1]$ with $x \neq y$, $\mathcal{B}x = 1$, $\mathcal{F}y = 1$, $\mathcal{J}x = \frac{1}{x}$, $\mathcal{M}y = \frac{1}{y}$ and

$$\begin{aligned} d(\mathcal{B}x, \mathcal{F}y) &= 0 \\ &< \max\left\{\left|\frac{1}{x} - \frac{1}{y}\right|, \frac{1}{2}\left[\left|1 - \frac{1}{x}\right| + \left|1 - \frac{1}{y}\right|\right]\right\} \\ &= \max\left\{d(\mathcal{J}x, \mathcal{M}y), \frac{1}{2}[d(\mathcal{B}x, \mathcal{J}x) + d(\mathcal{F}y, \mathcal{M}y)],\right. \\ &\quad \left.\frac{1}{2}[d(\mathcal{J}x, \mathcal{F}y) + d(\mathcal{B}x, \mathcal{M}y)]\right\}, \end{aligned}$$

2. for x, $y \in (1, +\infty)$ with $x \neq y$, $\mathcal{B}x = \frac{1}{2x}$, $\mathcal{F}y = \frac{3}{4y}$, $\mathcal{J}x = 101$, $\mathcal{M}y = 11$ and

$$\begin{aligned} d(\mathcal{B}x, \mathcal{F}y) &= \left|\frac{1}{2x} - \frac{3}{4y}\right| \\ &< \max\left\{90, \frac{1}{2}\left[\left|\frac{1}{2x} - 101\right| + \left|\frac{3}{4y} - 11\right|\right],\right. \\ &\quad \left.\frac{1}{2}\left[\left|101 - \frac{3}{4y}\right| + \left|\frac{1}{2x} - 11\right|\right]\right\} \\ &= \max\left\{d(\mathcal{J}x, \mathcal{M}y), \frac{1}{2}[d(\mathcal{B}x, \mathcal{J}x) + d(\mathcal{F}y, \mathcal{M}y)],\right. \\ &\quad \left.\frac{1}{2}[d(\mathcal{J}x, \mathcal{F}y) + d(\mathcal{B}x, \mathcal{M}y)]\right\}, \end{aligned}$$

3. for $x \in (0,1]$ and $y \in (1, +\infty)$ with $x \neq y$, $\mathcal{B}x = 1$, $\mathcal{F}y = \frac{3}{4y}$, $\mathcal{J}x = \frac{1}{x}$, $\mathcal{M}y = 11$ and

$$\begin{aligned} d(\mathcal{B}x, \mathcal{F}y) &= \left|1 - \frac{3}{4y}\right| \\ &< \max\left\{\left|\frac{1}{x} - 11\right|, \frac{1}{2}\left[\left|1 - \frac{1}{x}\right| + \left|\frac{3}{4y} - 11\right|\right], \frac{1}{2}\left[\left|\frac{1}{x} - \frac{3}{4y}\right| + 10\right]\right\} \\ &= \max\left\{d(\mathcal{J}x, \mathcal{M}y), \frac{1}{2}[d(\mathcal{B}x, \mathcal{J}x) + d(\mathcal{F}y, \mathcal{M}y)],\right. \\ &\quad \left.\frac{1}{2}[d(\mathcal{J}x, \mathcal{F}y) + d(\mathcal{B}x, \mathcal{M}y)]\right\}, \end{aligned}$$

4. for $x \in (1, +\infty)$ and $y \in (0,1]$ with $x \neq y$, $\mathcal{B}x = \frac{1}{2x}$, $\mathcal{F}y = 1$, $\mathcal{J}x = 101$, $\mathcal{M}y = \frac{1}{y}$ and

$$\begin{aligned} d(\mathcal{B}x, \mathcal{F}y) &= \left|\frac{1}{2x} - 1\right| \\ &< \max\left\{\left|101 - \frac{1}{y}\right|, \frac{1}{2}\left[\left|\frac{1}{2x} - 101\right| + \left|1 - \frac{1}{y}\right|\right],\right. \\ &\quad \left.\frac{1}{2}\left[100 + \left|\frac{1}{2x} - \frac{1}{y}\right|\right]\right\} \end{aligned}$$

$$= \max\left\{d(\mathcal{J}x,\mathcal{M}y),\frac{1}{2}[d(\mathcal{B}x,\mathcal{J}x)+d(\mathcal{F}y,\mathcal{M}y)],\right.$$
$$\left.\frac{1}{2}[d(\mathcal{J}x,\mathcal{F}y)+d(\mathcal{B}x,\mathcal{M}y)]\right\},$$

so, all hypotheses of Theorem 4.1 are satisfied and 1 is the unique common fixed point of maps $\mathcal{B}$, $\mathcal{F}$, $\mathcal{J}$ and $\mathcal{M}$.

Remark 4.1. Note that $\mathcal{B}\mathcal{X} = (0,\frac{1}{2})\cup\{1\} \nsubseteq \mathcal{M}\mathcal{X} = [1,+\infty)$ and $\mathcal{F}\mathcal{X} = (0,\frac{3}{4})\cup\{1\} \nsubseteq \mathcal{J}\mathcal{X} = [1,+\infty)$.

Theorem 4.2. *Let $\mathcal{B}$, $\mathcal{F}$, $\mathcal{J}$ and $\mathcal{M}$ be self-maps of a metric space $(\mathcal{X},d)$ such that*

$$d(\mathcal{B}x,\mathcal{F}y) \leq \Omega\left[\max\left\{d(\mathcal{J}x,\mathcal{M}y),\frac{1}{2}[d(\mathcal{B}x,\mathcal{J}x)+d(\mathcal{F}y,\mathcal{M}y)],\right.\right. \tag{4.2}$$
$$\left.\left.\frac{1}{2}[d(\mathcal{J}x,\mathcal{F}y)+d(\mathcal{B}x,\mathcal{M}y)]\right\}\right]$$

for all x, $y\in\mathcal{X}$, where $\Omega:[0,+\infty)\to[0,+\infty)$ is nondecreasing and $0<\Omega(t)<t$ for all $t>0$. If $\mathcal{B}$ and $\mathcal{J}$ are occasionally weakly $\mathcal{J}$-biased of type $(\mathcal{A})$ and $\mathcal{F}$ and $\mathcal{M}$ are occasionally weakly $\mathcal{M}$-biased of type $(\mathcal{A})$, then, the four maps have a unique common fixed point.

Proof. As in the proof of the above theorem, there are two elements θ and $\vartheta\in\mathcal{X}$ such that $\mathcal{B}\theta=\mathcal{J}\theta$ implies $d(\mathcal{J}\mathcal{J}\theta,\mathcal{B}\theta)\leq d(\mathcal{B}\mathcal{J}\theta,\mathcal{J}\theta)$ and $\mathcal{F}\vartheta=\mathcal{M}\vartheta$ implies $d(\mathcal{M}\mathcal{M}\vartheta,\mathcal{F}\vartheta)\leq d(\mathcal{F}\mathcal{M}\vartheta,\mathcal{M}\vartheta)$.

First, we claim that $\mathcal{B}\theta=\mathcal{F}\vartheta$, otherwise by inequality (4.2), we obtain

$$\begin{aligned} d(\mathcal{B}\theta,\mathcal{F}\vartheta) &\leq \Omega\left[\max\left\{d(\mathcal{J}\theta,\mathcal{M}\vartheta),\frac{1}{2}[d(\mathcal{B}\theta,\mathcal{J}\theta)+d(\mathcal{F}\vartheta,\mathcal{M}\vartheta)],\right.\right.\\ &\quad\left.\left.\frac{1}{2}[d(\mathcal{J}\theta,\mathcal{F}\vartheta)+d(\mathcal{B}\theta,\mathcal{M}\vartheta)]\right\}\right]\\ &= \Omega[\max\{d(\mathcal{B}\theta,\mathcal{F}\vartheta),0,d(\mathcal{B}\theta,\mathcal{F}\vartheta)\}]\\ &= \Omega[d(\mathcal{B}\theta,\mathcal{F}\vartheta)]\\ &< d(\mathcal{B}\theta,\mathcal{F}\vartheta)\end{aligned}$$

a contradiction and hence $\mathcal{B}\theta=\mathcal{F}\vartheta$.

Now, we show that $\mathcal{B}\mathcal{B}\theta=\mathcal{B}\theta$. If not, then, by condition (4.2), we find

$$\begin{aligned} d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta) &= d(\mathcal{B}\mathcal{B}\theta,\mathcal{F}\vartheta)\\ &\leq \Omega\left[\max\left\{d(\mathcal{J}\mathcal{B}\theta,\mathcal{M}\vartheta),\frac{1}{2}[d(\mathcal{B}\mathcal{B}\theta,\mathcal{J}\mathcal{B}\theta)+d(\mathcal{F}\vartheta,\mathcal{M}\vartheta)],\right.\right.\\ &\quad\left.\left.\frac{1}{2}[d(\mathcal{J}\mathcal{B}\theta,\mathcal{F}\vartheta),d(\mathcal{B}\mathcal{B}\theta,\mathcal{M}\vartheta)]\right\}\right]\\ &= \Omega\left[\max\left\{d(\mathcal{J}\mathcal{J}\theta,\mathcal{B}\theta),\frac{1}{2}d(\mathcal{B}\mathcal{B}\theta,\mathcal{J}\mathcal{J}\theta),\frac{1}{2}[d(\mathcal{J}\mathcal{J}\theta,\mathcal{B}\theta)+d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta)]\right\}\right]\end{aligned}$$

$$\begin{aligned}
&\leq \Omega\left[\max\left\{d(\mathcal{J}\mathcal{J}\theta,\mathcal{B}\theta),\frac{1}{2}[d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta)+d(\mathcal{B}\theta,\mathcal{J}\mathcal{J}\theta)],\right.\right.\\
&\quad\left.\left.\frac{1}{2}[d(\mathcal{J}\mathcal{J}\theta,\mathcal{B}\theta)+d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta)]\right\}\right]\\
&\leq \Omega\left[\max\left\{d(\mathcal{B}\mathcal{J}\theta,\mathcal{J}\theta),\frac{1}{2}[d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta)+d(\mathcal{J}\theta,\mathcal{B}\mathcal{J}\theta)],\right.\right.\\
&\quad\left.\left.\frac{1}{2}[d(\mathcal{B}\mathcal{J}\theta,\mathcal{J}\theta)+d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta)]\right\}\right]\\
&= \Omega[d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta)]\\
&< d(\mathcal{B}\mathcal{B}\theta,\mathcal{B}\theta)
\end{aligned}$$

which is a contradiction, hence, $\mathcal{B}\mathcal{B}\theta=\mathcal{B}\theta$, consequently, $\mathcal{J}\mathcal{B}\theta=\mathcal{B}\theta$.

In the following we show that $\mathcal{F}\mathcal{F}\vartheta=\mathcal{F}\vartheta$. For this, by inequality (4.2), triangle inequality, properties of Ω and occasionally weakly $\mathcal{M}$-biased of type $(\mathcal{A})$ of pair $(\mathcal{F},\mathcal{M})$, we obtain

$$\begin{aligned}
d(\mathcal{B}\theta,\mathcal{F}\mathcal{F}\vartheta) \leq\ & \Omega\left[\max\left\{d(\mathcal{J}\theta,\mathcal{M}\mathcal{F}\vartheta),\frac{1}{2}[d(\mathcal{B}\theta,\mathcal{J}\theta)+d(\mathcal{F}\mathcal{F}\vartheta,\mathcal{M}\mathcal{F}\vartheta)],\right.\right.\\
&\left.\left.\frac{1}{2}[d(\mathcal{J}\theta,\mathcal{F}\mathcal{F}\vartheta)+d(\mathcal{B}\theta,\mathcal{M}\mathcal{F}\vartheta)]\right\}\right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathcal{F}\vartheta,\mathcal{F}\mathcal{F}\vartheta) &\leq \Omega\left[\max\left\{d(\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta),\frac{1}{2}[d(\mathcal{F}\vartheta,\mathcal{F}\vartheta)+d(\mathcal{F}\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta)],\right.\right.\\
&\quad\left.\left.\frac{1}{2}[d(\mathcal{F}\vartheta,\mathcal{F}\mathcal{F}\vartheta)+d(\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta)]\right\}\right]\\
&= \Omega\left[\max\left\{d(\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta),\frac{1}{2}d(\mathcal{F}\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta),\frac{1}{2}[d(\mathcal{F}\vartheta,\mathcal{F}\mathcal{F}\vartheta)+d(\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta)]\right\}\right]\\
&\leq \Omega\left[\max\left\{d(\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta),\frac{1}{2}[d(\mathcal{F}\mathcal{F}\vartheta,\mathcal{F}\vartheta)+d(\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta)],\right.\right.\\
&\quad\left.\left.\frac{1}{2}[d(\mathcal{F}\vartheta,\mathcal{F}\mathcal{F}\vartheta)+d(\mathcal{F}\vartheta,\mathcal{M}\mathcal{M}\vartheta)]\right\}\right]\\
&\leq \Omega\left[\max\left\{d(\mathcal{M}\vartheta,\mathcal{F}\mathcal{M}\vartheta),\frac{1}{2}[d(\mathcal{F}\mathcal{F}\vartheta,\mathcal{F}\vartheta)+d(\mathcal{M}\vartheta,\mathcal{F}\mathcal{M}\vartheta)],\right.\right.\\
&\quad\left.\left.\frac{1}{2}[d(\mathcal{F}\vartheta,\mathcal{F}\mathcal{F}\vartheta)+d(\mathcal{M}\vartheta,\mathcal{F}\mathcal{M}\vartheta)]\right\}\right]\\
&= \Omega[d(\mathcal{F}\vartheta,\mathcal{F}\mathcal{F}\vartheta)]\\
&< d(\mathcal{F}\vartheta,\mathcal{F}\mathcal{F}\vartheta)
\end{aligned}$$

a contradiction if $\mathcal{F}\mathcal{F}\vartheta\neq\mathcal{F}\vartheta$, hence $\mathcal{F}\mathcal{F}\vartheta=\mathcal{F}\vartheta$ and consequently $\mathcal{M}\mathcal{F}\vartheta=\mathcal{F}\vartheta$. Therefore $\mathcal{B}\theta=\mathcal{J}\theta=\mathcal{F}\vartheta=\mathcal{M}\vartheta=\tau$ is a common fixed point of maps $\mathcal{B}$, $\mathcal{F}$, $\mathcal{J}$ and $\mathcal{M}$.

Finally, assume that there exists another common fixed point ρ for the four maps.

Then, using condition (4.2) we get

$$\begin{aligned} d(\mathcal{B}\tau,\mathcal{F}\rho) &\leq \Omega\left[\max\left\{d(\mathcal{J}\tau,\mathcal{M}\rho),\frac{1}{2}[d(\mathcal{B}\tau,\mathcal{J}\tau)+d(\mathcal{F}\rho,\mathcal{M}\rho)],\right.\right.\\ &\left.\left.\frac{1}{2}[d(\mathcal{J}\tau,\mathcal{F}\rho)+d(\mathcal{B}\tau,\mathcal{M}\rho)]\right\}\right], \end{aligned}$$

i.e.,

$$\begin{aligned} d(\tau,\rho) &\leq \Omega\left[\max\left\{d(\tau,\rho),0,\frac{1}{2}[d(\tau,\rho)+d(\tau,\rho)]\right\}\right]\\ &= \Omega[d(\tau,\rho)]\\ &< d(\tau,\rho) \end{aligned}$$

which is a contradiction, thus, the common fixed point is unique and this completes the proof. □

The following example illustrates the validity of the above result.

Example 4.2. Let $\mathcal{X}=(0,+\infty)$ with the usual metric $d(x,y)=|x-y|$. Define

$$\mathcal{B}x=\begin{cases}1 & \text{if } x\in(0,1]\\ \frac{1}{3x} & \text{if } x\in(1,+\infty)\end{cases},\ \mathcal{F}x=\begin{cases}1 & \text{if } x\in(0,1]\\ \frac{2}{3x} & \text{if } x\in(1,+\infty)\end{cases},$$

$$\mathcal{J}x=\begin{cases}\frac{1}{x} & \text{if } x\in(0,1]\\ 501 & \text{if } x\in(1,+\infty)\end{cases},\ \mathcal{M}x=\begin{cases}\frac{1}{x} & \text{if } x\in(0,1]\\ 1001 & \text{if } x\in(1,+\infty)\end{cases}.$$

First, it is clear to see that $\mathcal{B}$ and $\mathcal{J}$ are occasionally weakly $\mathcal{J}$-biased of type $(\mathcal{A})$ and $\mathcal{F}$ and $\mathcal{M}$ are occasionally weakly $\mathcal{M}$-biased of type $(\mathcal{A})$. Taking $\Omega(t)=\frac{3}{4}t$, we get

1. for $x,\ y\in(0,1]$, $\mathcal{B}x=1$, $\mathcal{F}y=1$, $\mathcal{J}x=\frac{1}{x}$, $\mathcal{M}y=\frac{1}{y}$ and

$$\begin{aligned} d(\mathcal{B}x,\mathcal{F}y) &= 0\\ &\leq \frac{3}{4}\max\left\{\left|\frac{1}{x}-\frac{1}{y}\right|,\frac{1}{2}\left[\left|1-\frac{1}{x}\right|+\left|1-\frac{1}{y}\right|\right]\right\}\\ &= \Omega\left[\max\left\{d(\mathcal{J}x,\mathcal{M}y),\frac{1}{2}[d(\mathcal{B}x,\mathcal{J}x)+d(\mathcal{F}y,\mathcal{M}y)],\right.\right.\\ &\left.\left.\frac{1}{2}[d(\mathcal{J}x,\mathcal{F}y)+d(\mathcal{B}x,\mathcal{M}y)]\right\}\right], \end{aligned}$$

2. for $x,\ y\in(1,+\infty)$, $\mathcal{B}x=\frac{1}{3x}$, $\mathcal{F}y=\frac{2}{3y}$, $\mathcal{J}x=501$, $\mathcal{M}y=1001$ and

$$\begin{aligned} d(\mathcal{B}x,\mathcal{F}y) &= \left|\frac{1}{3x}-\frac{2}{3y}\right|\\ &\leq \frac{3}{4}\max\left\{500,\frac{1}{2}\left[\left|\frac{1}{3x}-501\right|+\left|\frac{2}{3y}-1001\right|\right],\right.\\ &\left.\frac{1}{2}\left[\left|501-\frac{2}{3y}\right|+\left|\frac{1}{3x}-1001\right|\right]\right\} \end{aligned}$$

$$= \Omega\left[\max\left\{d(\mathcal{J}x,\mathcal{M}y),\frac{1}{2}[d(\mathcal{B}x,\mathcal{J}x)+d(\mathcal{F}y,\mathcal{M}y)],\right.\right.$$
$$\left.\left.\frac{1}{2}[d(\mathcal{J}x,\mathcal{F}y)+d(\mathcal{B}x,\mathcal{M}y)]\right\}\right],$$

3. for $x\in(0,1]$ and $y\in(1,+\infty)$, $\mathcal{B}x=1$, $\mathcal{F}y=\frac{2}{3y}$, $\mathcal{J}x=\frac{1}{x}$, $\mathcal{M}y=1001$ and

$$\begin{aligned} d(\mathcal{B}x,\mathcal{F}y) &= \left|1-\frac{2}{3y}\right| \\ &\leq \frac{3}{4}\max\left\{\left|\frac{1}{x}-1001\right|,\frac{1}{2}\left[\left|1-\frac{1}{x}\right|+\left|\frac{2}{3y}-1001\right|\right],\right. \\ &\quad \left.\frac{1}{2}\left[\left|\frac{1}{x}-\frac{2}{3y}\right|+1000\right]\right\} \\ &= \Omega\left[\max\left\{d(\mathcal{J}x,\mathcal{M}y),\frac{1}{2}[d(\mathcal{B}x,\mathcal{J}x)+d(\mathcal{F}y,\mathcal{M}y)],\right.\right. \\ &\quad \left.\left.\frac{1}{2}[d(\mathcal{J}x,\mathcal{F}y)+d(\mathcal{B}x,\mathcal{M}y)]\right\}\right], \end{aligned}$$

4. for $x\in(1,+\infty)$ and $y\in(0,1]$, $\mathcal{B}x=\frac{1}{3x}$, $\mathcal{F}y=1$, $\mathcal{J}x=501$, $\mathcal{M}y=\frac{1}{y}$ and

$$\begin{aligned} d(\mathcal{B}x,\mathcal{F}y) &= \left|\frac{1}{3x}-1\right| \\ &\leq \frac{3}{4}\max\left\{\left|501-\frac{1}{y}\right|,\frac{1}{2}\left[\left|\frac{1}{3x}-501\right|+\left|1-\frac{1}{y}\right|\right],\right. \\ &\quad \left.\frac{1}{2}\left[500+\left|\frac{1}{3x}-\frac{1}{y}\right|\right]\right\} \\ &= \Omega\left[\max\left\{d(\mathcal{J}x,\mathcal{M}y),\frac{1}{2}[d(\mathcal{B}x,\mathcal{J}x)+d(\mathcal{F}y,\mathcal{M}y)],\right.\right. \\ &\quad \left.\left.\frac{1}{2}[d(\mathcal{J}x,\mathcal{F}y)+d(\mathcal{B}x,\mathcal{M}y)]\right\}\right], \end{aligned}$$

so, all hypotheses of Theorem 4.2 are satisfied and 1 is the unique common fixed point of maps $\mathcal{B}$, $\mathcal{F}$, $\mathcal{J}$ and $\mathcal{M}$.

Remark 4.2. Note that $\mathcal{B}\mathcal{X}=(0,\frac{1}{3})\cup\{1\}\nsubseteq\mathcal{M}\mathcal{X}=[1,+\infty)$ and $\mathcal{F}\mathcal{X}=(0,\frac{2}{3})\cup\{1\}\nsubseteq\mathcal{J}\mathcal{X}=[1,+\infty)$.

Remark 4.3. Note that Theorem 3.3 of [6] is not applicable because $\mathcal{F}\mathcal{X}\nsubseteq\mathcal{J}\mathcal{X}$ and $\mathcal{M}\mathcal{X}$ is not closed. Also, $\mathcal{B}\mathcal{X}\nsubseteq\mathcal{M}\mathcal{X}$ and $\mathcal{J}\mathcal{X}$ is not closed.

5 Unique Common Fixed Points in Dislocated Metric Spaces

Theorem 5.1. *Let $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{J}$ and $\mathfrak{M}$ be self-maps of a dislocated metric space $(\mathfrak{X}, d)$ such that*

$$d(\mathfrak{B}x, \mathfrak{F}y) < \max\left\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), d(\mathfrak{B}x, \mathfrak{M}y)\right\} \quad (5.1)$$

for all $x, y \in \mathfrak{X}$ with $x \neq y$. If $\mathfrak{B}$ and $\mathfrak{J}$ are occasionally weakly $\mathfrak{J}$-biased of type $(\mathcal{A})$ and $\mathfrak{F}$ and $\mathfrak{M}$ are occasionally weakly $\mathfrak{M}$-biased of type $(\mathcal{A})$, then, the four maps have a unique common fixed point.

Proof. Since $\mathfrak{B}$ and $\mathfrak{J}$ are occasionally weakly $\mathfrak{J}$-biased of type $(\mathcal{A})$, then, there is a point $\theta \in \mathfrak{X}$ such that $\mathfrak{B}\theta = \mathfrak{J}\theta$ implies $d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta) \leq d(\mathfrak{B}\mathfrak{J}\theta, \mathfrak{J}\theta)$. The same, as $\mathfrak{F}$ and $\mathfrak{M}$ are occasionally weakly $\mathfrak{M}$-biased of type $(\mathcal{A})$, then, there exists an element $\vartheta \in \mathfrak{X}$ such that $\mathfrak{F}\vartheta = \mathfrak{M}\vartheta$ implies $d(\mathfrak{M}\mathfrak{M}\vartheta, \mathfrak{F}\vartheta) \leq d(\mathfrak{F}\mathfrak{M}\vartheta, \mathfrak{M}\vartheta)$. For the existence and uniqueness of the common fixed point we need four steps.

First step: We claim that $\mathfrak{B}\theta = \mathfrak{F}\vartheta$. Suppose that $d(\mathfrak{B}\theta, \mathfrak{F}\vartheta) > 0$. Using inequality (5.1) we get

$$\begin{aligned}
d(\mathfrak{B}\theta, \mathfrak{F}\vartheta) &< \max\left\{d(\mathfrak{J}\theta, \mathfrak{M}\vartheta), \frac{1}{2}d(\mathfrak{B}\theta, \mathfrak{J}\theta), \frac{1}{2}d(\mathfrak{F}\vartheta, \mathfrak{M}\vartheta), d(\mathfrak{J}\theta, \mathfrak{F}\vartheta), d(\mathfrak{B}\theta, \mathfrak{M}\vartheta)\right\} \\
&= \max\left\{d(\mathfrak{B}\theta, \mathfrak{F}\vartheta), \frac{1}{2}d(\mathfrak{B}\theta, \mathfrak{B}\theta), \frac{1}{2}d(\mathfrak{F}\vartheta, \mathfrak{F}\vartheta), d(\mathfrak{B}\theta, \mathfrak{F}\vartheta), d(\mathfrak{B}\theta, \mathfrak{F}\vartheta)\right\} \\
&\leq \max\left\{d(\mathfrak{B}\theta, \mathfrak{F}\vartheta), \frac{1}{2}[d(\mathfrak{B}\theta, \mathfrak{F}\theta) + d(\mathfrak{F}\theta, \mathfrak{B}\theta)],\right. \\
&\quad \left.\frac{1}{2}[d(\mathfrak{F}\vartheta, \mathfrak{B}\vartheta) + d(\mathfrak{B}\theta, \mathfrak{F}\theta)], d(\mathfrak{B}\theta, \mathfrak{F}\vartheta), d(\mathfrak{B}\theta, \mathfrak{F}\vartheta)\right\} \\
&= d(\mathfrak{B}\theta, \mathfrak{F}\vartheta)
\end{aligned}$$

which is a contradiction and hence $\mathfrak{B}\theta = \mathfrak{F}\vartheta$.

Second step: Assume that $d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta) > 0$. The use of condition (5.1) gives

$$\begin{aligned}
d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta) &= d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{F}\vartheta) \\
&< \max\{d(\mathfrak{J}\mathfrak{B}\theta, \mathfrak{M}\vartheta), \frac{1}{2}d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{J}\mathfrak{B}\theta), \frac{1}{2}d(\mathfrak{F}\vartheta, \mathfrak{M}\vartheta), d(\mathfrak{J}\mathfrak{B}\theta, \mathfrak{F}\vartheta), \\
&\quad d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{M}\vartheta)\} \\
&= \max\{d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), \frac{1}{2}d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{J}\mathfrak{J}\theta), \frac{1}{2}d(\mathfrak{B}\theta, \mathfrak{B}\theta), d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), \\
&\quad d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)\} \\
&\leq \max\left\{d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), \frac{1}{2}[d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta) + d(\mathfrak{B}\theta, \mathfrak{J}\mathfrak{J}\theta)],\right. \\
&\quad \left.\frac{1}{2}[d(\mathfrak{B}\theta, \mathfrak{B}\mathfrak{B}\theta) + d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)], d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)\right\}
\end{aligned}$$

$$\begin{aligned} &\leq \max\{d(\mathfrak{B}\mathfrak{J}\theta,\mathfrak{J}\theta),\frac{1}{2}[d(\mathfrak{B}\mathfrak{B}\theta,\mathfrak{B}\theta)+d(\mathfrak{J}\theta,\mathfrak{B}\mathfrak{J}\theta)],d(\mathfrak{B}\theta,\mathfrak{B}\mathfrak{B}\theta),\\ &\quad d(\mathfrak{B}\mathfrak{J}\theta,\mathfrak{J}\theta),d(\mathfrak{B}\mathfrak{B}\theta,\mathfrak{B}\theta)\}\\ &= d(\mathfrak{B}\mathfrak{B}\theta,\mathfrak{B}\theta) \end{aligned}$$

which is a contradiction and thus $\mathfrak{B}\mathfrak{B}\theta = \mathfrak{B}\theta$, hence, $\mathfrak{J}\mathfrak{B}\theta = \mathfrak{B}\theta$.

Third step: Now, suppose that $\mathfrak{F}\mathfrak{F}\vartheta \neq \mathfrak{F}\vartheta$. Utilizing inequality (5.1) we obtain

$$\begin{aligned} d(\mathfrak{B}\theta,\mathfrak{F}\mathfrak{F}\vartheta) &< \max\{d(\mathfrak{J}\theta,\mathfrak{M}\mathfrak{F}\vartheta),\frac{1}{2}d(\mathfrak{B}\theta,\mathfrak{J}\theta),\frac{1}{2}d(\mathfrak{F}\mathfrak{F}\vartheta,\mathfrak{M}\mathfrak{F}\vartheta),d(\mathfrak{J}\theta,\mathfrak{F}\mathfrak{F}\vartheta),\\ &\quad d(\mathfrak{B}\theta,\mathfrak{M}\mathfrak{F}\vartheta)\}, \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta) &< \max\{d(\mathfrak{F}\vartheta,\mathfrak{M}\mathfrak{M}\vartheta),\frac{1}{2}d(\mathfrak{F}\vartheta,\mathfrak{F}\vartheta),\frac{1}{2}d(\mathfrak{F}\mathfrak{F}\vartheta,\mathfrak{M}\mathfrak{M}\vartheta),d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta),\\ &\quad d(\mathfrak{F}\vartheta,\mathfrak{M}\mathfrak{M}\vartheta)\}\\ &\leq \max\left\{d(\mathfrak{F}\vartheta,\mathfrak{M}\mathfrak{M}\vartheta),\frac{1}{2}[d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta)+d(\mathfrak{F}\mathfrak{F}\vartheta,\mathfrak{F}\vartheta)],\right.\\ &\quad \left.\frac{1}{2}[d(\mathfrak{M}\mathfrak{M}\vartheta,\mathfrak{F}\vartheta)+d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta)],d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta),d(\mathfrak{F}\vartheta,\mathfrak{M}\mathfrak{M}\vartheta)\right\}\\ &\leq \max\{d(\mathfrak{M}\vartheta,\mathfrak{F}\mathfrak{M}\vartheta),d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta),\frac{1}{2}[d(\mathfrak{F}\mathfrak{M}\vartheta,\mathfrak{M}\vartheta)+d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta)],\\ &\quad d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta),d(\mathfrak{M}\vartheta,\mathfrak{F}\mathfrak{M}\vartheta)\}\\ &= d(\mathfrak{F}\vartheta,\mathfrak{F}\mathfrak{F}\vartheta). \end{aligned}$$

This contradiction implies that $\mathfrak{F}\mathfrak{F}\vartheta = \mathfrak{F}\vartheta$ and hence $\mathfrak{M}\mathfrak{F}\vartheta = \mathfrak{F}\vartheta$. Consequently, $\mathfrak{B}\theta = \mathfrak{J}\theta = \mathfrak{F}\vartheta = \mathfrak{M}\vartheta = \tau$ is a common fixed point of maps $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{J}$ and $\mathfrak{M}$.

Fourth step: Assume that there is another common fixed point ρ for the four maps. Then, the use of condition (5.1) gives

$$d(\mathfrak{B}\tau,\mathfrak{F}\rho) < \max\left\{d(\mathfrak{J}\tau,\mathfrak{M}\rho),\frac{1}{2}d(\mathfrak{B}\tau,\mathfrak{J}\tau),\frac{1}{2}d(\mathfrak{F}\rho,\mathfrak{M}\rho),d(\mathfrak{J}\tau,\mathfrak{F}\rho),d(\mathfrak{B}\tau,\mathfrak{M}\rho)\right\},$$

i.e.,

$$\begin{aligned} d(\tau,\rho) &< \max\left\{d(\tau,\rho),\frac{1}{2}d(\tau,\tau),\frac{1}{2}d(\rho,\rho),d(\tau,\rho),d(\tau,\rho)\right\}\\ &\leq \max\{d(\tau,\rho),\frac{1}{2}[d(\tau,\rho)+d(\rho,\tau)],\frac{1}{2}[d(\rho,\tau)+d(\tau,\rho)],\\ &\quad d(\tau,\rho),d(\tau,\rho)\}\\ &= d(\tau,\rho) \end{aligned}$$

which is a contradiction, hence the common fixed point is unique. □

Now, we give an illustrative example which highlights our result.

Example 5.1. Let $\mathcal{X} = (0, +\infty)$ with the dislocated metric $d(x, y) = x + y$. Define

$$\mathfrak{B}x = \begin{cases} x & \text{if } x \in (0,1] \\ \frac{1}{10} & \text{if } x \in (1,+\infty) \end{cases}, \ \mathfrak{F}x = \begin{cases} x & \text{if } x \in (0,1] \\ \frac{1}{20} & \text{if } x \in (1,+\infty) \end{cases},$$

$$\mathfrak{J}x = \begin{cases} \frac{1}{x} & \text{if } x \in (0,1] \\ 100 & \text{if } x \in (1,+\infty) \end{cases}, \ \mathfrak{M}x = \begin{cases} \frac{1}{x} & \text{if } x \in (0,1] \\ 200 & \text{if } x \in (1,+\infty) \end{cases}.$$

First, it is clear to see that $\mathfrak{B}$ and $\mathfrak{J}$ are occasionally weakly $\mathfrak{J}$-biased of type $(\mathcal{A})$ and $\mathfrak{F}$ and $\mathfrak{M}$ are occasionally weakly $\mathfrak{M}$-biased of type $(\mathcal{A})$. We have

1. for x, $y \in (0,1]$ with $x \neq y$, $\mathfrak{B}x = x$, $\mathfrak{F}y = y$, $\mathfrak{J}x = \frac{1}{x}$, $\mathfrak{M}y = \frac{1}{y}$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= x + y \\ &< \max\left\{\frac{1}{x} + \frac{1}{y}, \frac{1}{2}\left(x + \frac{1}{x}\right), \frac{1}{2}\left(y + \frac{1}{y}\right), \frac{1}{x} + y, x + \frac{1}{y}\right\} \\ &= \max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}, \end{aligned}$$

2. for x, $y \in (1, +\infty)$ with $x \neq y$, $\mathfrak{B}x = \frac{1}{10}$, $\mathfrak{F}y = \frac{1}{20}$, $\mathfrak{J}x = 100$, $\mathfrak{M}y = 200$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= \frac{3}{20} \\ &< \max\left\{300, \frac{1001}{20}, \frac{4001}{40}, \frac{2001}{20}, \frac{2001}{10}\right\} \\ &= \max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}, \end{aligned}$$

3. for $x \in (0,1]$ and $y \in (1, +\infty)$ with $x \neq y$, $\mathfrak{B}x = x$, $\mathfrak{F}y = \frac{1}{20}$, $\mathfrak{J}x = \frac{1}{x}$, $\mathfrak{M}y = 200$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= x + \frac{1}{20} \\ &< \max\left\{\frac{1}{x} + 200, \frac{1}{2}\left(x + \frac{1}{x}\right), \frac{4001}{40}, \frac{1}{x} + \frac{1}{20}, x + 200\right\} \\ &= \max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}, \end{aligned}$$

4. for $x \in (1, +\infty)$ and $y \in (0,1]$ with $x \neq y$, $\mathfrak{B}x = \frac{1}{10}$, $\mathfrak{F}y = y$, $\mathfrak{J}x = 100$, $\mathfrak{M}y = \frac{1}{y}$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= \frac{1}{10} + y \\ &< \max\left\{100 + \frac{1}{y}, \frac{1001}{20}, \frac{1}{2}\left(y + \frac{1}{y}\right), 100 + y, \frac{1}{10} + \frac{1}{y}\right\} \\ &= \max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}, \end{aligned}$$

so, all hypotheses of Theorem 5.1 are satisfied and 1 is the unique common fixed point of maps $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{J}$ and $\mathfrak{M}$.

Remark 5.1. Note that $\mathfrak{B}\mathcal{X} = (0,1] \not\subseteq \mathfrak{M}\mathcal{X} = [1,+\infty)$ and $\mathfrak{F}\mathcal{X} = (0,1] \not\subseteq \mathfrak{J}\mathcal{X} = [1,+\infty)$.

Theorem 5.2. *Let $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{J}$ and $\mathfrak{M}$ be self-maps of a dislocated metric space $(\mathcal{X}, d)$ such that*

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) \quad \leq \quad & \Omega[\max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ & d(\mathfrak{B}x, \mathfrak{M}y)\}] \end{aligned} \tag{5.2}$$

for all x, $y \in \mathcal{X}$, where $\Omega : [0,+\infty) \to [0,+\infty)$ is nondecreasing and $0 < \Omega(t) < t$ for all $t > 0$. If $\mathfrak{B}$ and $\mathfrak{J}$ are occasionally weakly $\mathfrak{J}$-biased of type $(\mathcal{A})$ and $\mathfrak{F}$ and $\mathfrak{M}$ are occasionally weakly $\mathfrak{M}$-biased of type $(\mathcal{A})$, then, the four maps have a unique common fixed point.

Proof. As in the proof of the above theorem, there are two elements θ and $\vartheta \in \mathcal{X}$ such that $\mathfrak{B}\theta = \mathfrak{J}\theta$ implies $d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta) \leq d(\mathfrak{B}\mathfrak{J}\theta, \mathfrak{J}\theta)$ and $\mathfrak{F}\vartheta = \mathfrak{M}\vartheta$ implies $d(\mathfrak{M}\mathfrak{M}\vartheta, \mathfrak{F}\vartheta) \leq d(\mathfrak{F}\mathfrak{M}\vartheta, \mathfrak{M}\vartheta)$.

First, we claim that $\mathfrak{B}\theta = \mathfrak{F}\vartheta$ otherwise by inequality (5.2), we obtain

$$\begin{aligned} d(\mathfrak{B}\theta, \mathfrak{F}\vartheta) \quad &\leq \quad \Omega\left[\max\left\{d(\mathfrak{J}\theta, \mathfrak{M}\vartheta), \frac{1}{2}d(\mathfrak{B}\theta, \mathfrak{J}\theta), \frac{1}{2}d(\mathfrak{F}\vartheta, \mathfrak{M}\vartheta), d(\mathfrak{J}\theta, \mathfrak{F}\vartheta), d(\mathfrak{B}\theta, \mathfrak{M}\vartheta)\right\}\right] \\ &= \quad \Omega\left[\max\left\{d(\mathfrak{B}\theta, \mathfrak{F}\vartheta), \frac{1}{2}d(\mathfrak{B}\theta, \mathfrak{B}\theta), \frac{1}{2}d(\mathfrak{F}\vartheta, \mathfrak{F}\vartheta), d(\mathfrak{B}\theta, \mathfrak{F}\vartheta), d(\mathfrak{B}\theta, \mathfrak{F}\vartheta)\right\}\right]. \end{aligned}$$

Since Ω is nondecreasing, we get

$$d(\mathfrak{B}\theta, \mathfrak{F}\vartheta) \leq \Omega[d(\mathfrak{B}\theta, \mathfrak{F}\vartheta)] < d(\mathfrak{B}\theta, \mathfrak{F}\vartheta)$$

a contradiction and hence $\mathfrak{B}\theta = \mathfrak{F}\vartheta$.

Now, we show that $\mathfrak{B}\mathfrak{B}\theta = \mathfrak{B}\theta$. If not, then, by condition (5.2), we find

$$\begin{aligned} d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta) \quad &= \quad d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{F}\vartheta) \\ &\leq \quad \Omega[\max\{d(\mathfrak{J}\mathfrak{B}\theta, \mathfrak{M}\vartheta), \frac{1}{2}d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{J}\mathfrak{B}\theta), \frac{1}{2}d(\mathfrak{F}\vartheta, \mathfrak{M}\vartheta), d(\mathfrak{J}\mathfrak{B}\theta, \mathfrak{F}\vartheta), \\ &\qquad d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{M}\vartheta)\}] \\ &= \quad \Omega[\max\{d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), \frac{1}{2}d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{J}\mathfrak{J}\theta), \frac{1}{2}d(\mathfrak{B}\theta, \mathfrak{B}\theta), d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), \\ &\qquad d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)\}] \\ &\leq \quad \Omega\left[\max\left\{d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), \frac{1}{2}[d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta) + d(\mathfrak{B}\theta, \mathfrak{J}\mathfrak{J}\theta)], \right.\right. \\ &\qquad \left.\left.\frac{1}{2}[d(\mathfrak{B}\theta, \mathfrak{B}\mathfrak{B}\theta) + d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)], d(\mathfrak{J}\mathfrak{J}\theta, \mathfrak{B}\theta), d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)\right\}\right] \\ &\leq \quad \Omega[\max\{d(\mathfrak{B}\mathfrak{J}\theta, \mathfrak{J}\theta), \frac{1}{2}[d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta) + d(\mathfrak{J}\theta, \mathfrak{B}\mathfrak{J}\theta)], d(\mathfrak{B}\theta, \mathfrak{B}\mathfrak{B}\theta), \\ &\qquad d(\mathfrak{B}\mathfrak{J}\theta, \mathfrak{J}\theta), d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)\} \\ &= \quad \Omega[d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta)] \\ &< \quad d(\mathfrak{B}\mathfrak{B}\theta, \mathfrak{B}\theta) \end{aligned}$$

which is a contradiction, hence, $\mathfrak{B}\mathfrak{B}\theta = \mathfrak{B}\theta$, consequently, $\mathfrak{J}\mathfrak{B}\theta = \mathfrak{B}\theta$.

In the following we show that $\mathfrak{F}\mathfrak{F}\vartheta = \mathfrak{F}\vartheta$. For this, by inequality (5.2), triangle inequality, properties of Ω and occasionally weakly $\mathfrak{M}$-biased of type $(\mathcal{A})$ of pair $(\mathfrak{F}, \mathfrak{M})$, we obtain

$$\begin{aligned} d(\mathfrak{B}\theta, \mathfrak{F}\mathfrak{F}\vartheta) &\leq \Omega[\max\{d(\mathfrak{J}\theta, \mathfrak{M}\mathfrak{F}\vartheta), \frac{1}{2}d(\mathfrak{B}\theta, \mathfrak{J}\theta), \frac{1}{2}d(\mathfrak{F}\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{F}\vartheta), d(\mathfrak{J}\theta, \mathfrak{F}\mathfrak{F}\vartheta), \\ &\quad d(\mathfrak{B}\theta, \mathfrak{M}\mathfrak{F}\vartheta)\}], \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta) &\leq \Omega[\max\{d(\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{M}\vartheta), \frac{1}{2}d(\mathfrak{F}\vartheta, \mathfrak{F}\vartheta), \frac{1}{2}d(\mathfrak{F}\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{M}\vartheta), d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta), \\ &\quad d(\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{M}\vartheta)\}] \\ &\leq \Omega\left[\max\left\{d(\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{M}\vartheta), \frac{1}{2}[d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta) + d(\mathfrak{F}\mathfrak{F}\vartheta, \mathfrak{F}\vartheta)], \right.\right. \\ &\quad \left.\left.\frac{1}{2}[d(\mathfrak{M}\mathfrak{M}\vartheta, \mathfrak{F}\vartheta) + d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta)], d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta), d(\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{M}\vartheta)\right\}\right] \\ &= \Omega[\max\{d(\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{M}\vartheta), d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta), \frac{1}{2}[d(\mathfrak{M}\mathfrak{M}\vartheta, \mathfrak{F}\vartheta) + d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta)], \\ &\quad d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta), d(\mathfrak{F}\vartheta, \mathfrak{M}\mathfrak{M}\vartheta)\}] \\ &\leq \Omega[\max\{d(\mathfrak{M}\vartheta, \mathfrak{F}\mathfrak{M}\vartheta), d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta), \frac{1}{2}[d(\mathfrak{F}\mathfrak{M}\vartheta, \mathfrak{M}\vartheta) + d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta)], \\ &\quad d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta), d(\mathfrak{M}\vartheta, \mathfrak{F}\mathfrak{M}\vartheta)\}] \\ &= \Omega[d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta)] \\ &< d(\mathfrak{F}\vartheta, \mathfrak{F}\mathfrak{F}\vartheta) \end{aligned}$$

a contradiction if $\mathfrak{F}\mathfrak{F}\vartheta \neq \mathfrak{F}\vartheta$, hence $\mathfrak{F}\mathfrak{F}\vartheta = \mathfrak{F}\vartheta$ and consequently $\mathfrak{M}\mathfrak{F}\vartheta = \mathfrak{F}\vartheta$. Therefore $\mathfrak{B}\theta = \mathfrak{J}\theta = \mathfrak{F}\vartheta = \mathfrak{M}\vartheta = \tau$ is a common fixed point of maps $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{J}$ and $\mathfrak{M}$.

Finally, assume that there exists another common fixed point ρ for the four maps. Then, using condition (5.2) we get

$$d(\mathfrak{B}\tau, \mathfrak{F}\rho) \leq \Omega\left[\max\left\{d(\mathfrak{J}\tau, \mathfrak{M}\rho), \frac{1}{2}d(\mathfrak{B}\tau, \mathfrak{J}\tau), \frac{1}{2}d(\mathfrak{F}\rho, \mathfrak{M}\rho), d(\mathfrak{J}\tau, \mathfrak{F}\rho), d(\mathfrak{B}\tau, \mathfrak{M}\rho)\right\}\right],$$

i.e.,

$$\begin{aligned} d(\tau, \rho) &\leq \Omega\left[\max\left\{d(\tau, \rho), \frac{1}{2}d(\tau, \tau), \frac{1}{2}d(\rho, \rho), d(\tau, \rho), d(\tau, \rho)\right\}\right] \\ &\leq \Omega\left[\max\left\{d(\tau, \rho), \frac{1}{2}[d(\tau, \rho) + d(\rho, \tau)], \frac{1}{2}[d(\rho, \tau) + d(\tau, \rho)], d(\tau, \rho), d(\tau, \rho)\right\}\right] \\ &= \Omega[d(\tau, \rho)] \\ &< d(\tau, \rho) \end{aligned}$$

which is a contradiction, thus, the common fixed point is unique and this completes the proof. □

The following example illustrates the validity of the above result.

Example 5.2. Let $\mathfrak{X} = (-100, 100)$ with the dislocated metric $d(x, y) = \max\{|x|, |y|\}$. Define

$$\mathfrak{B}x = \begin{cases} -\frac{x}{100} & \text{if } x \in (-100, 0] \\ -\frac{1}{100} & \text{if } x \in (0, 100) \end{cases}, \ \mathfrak{F}x = \begin{cases} -\frac{x}{200} & \text{if } x \in (-100, 0] \\ -\frac{1}{200} & \text{if } x \in (0, 100) \end{cases},$$

$$\mathfrak{J}x = \begin{cases} -x & \text{if } x \in (-100, 0] \\ 90 & \text{if } x \in (0, 100) \end{cases}, \ \mathfrak{M}x = \begin{cases} -x & \text{if } x \in (-100, 0] \\ 80 & \text{if } x \in (0, 100) \end{cases}.$$

First, it is clear to see that $\mathfrak{B}$ and $\mathfrak{J}$ are occasionally weakly $\mathfrak{J}$-biased of type $(\mathcal{A})$ and $\mathfrak{F}$ and $\mathfrak{M}$ are occasionally weakly $\mathfrak{M}$-biased of type $(\mathcal{A})$. Taking $\Omega(t) = \frac{3}{4}t$, we get

1. for x, $y \in (-100, 0]$, $\mathfrak{B}x = -\frac{x}{100}$, $\mathfrak{F}y = -\frac{y}{200}$, $\mathfrak{J}x = -x$, $\mathfrak{M}y = -y$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= \max\left\{-\frac{x}{100}, -\frac{y}{200}\right\} \\ &\leq \frac{3}{4}\max\left\{\max\{-x, -y\}, -\frac{x}{2}, -\frac{y}{2}, \max\left\{-x, -\frac{y}{200}\right\}, \right. \\ &\quad \left. \max\left\{-\frac{x}{100}, -y\right\}\right\} \\ &= \Omega[\max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}], \end{aligned}$$

2. for x, $y \in (0, 100)$, $\mathfrak{B}x = -\frac{1}{100}$, $\mathfrak{F}y = -\frac{1}{200}$, $\mathfrak{J}x = 90$, $\mathfrak{M}y = 80$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= \frac{1}{100} \\ &\leq \frac{3}{4}\max\{90, 45, 40, 90, 80\} \\ &= \frac{135}{2} \\ &= \Omega[\max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}], \end{aligned}$$

3. for $x \in (-100, 0]$ and $y \in (0, 100)$, $\mathfrak{B}x = -\frac{x}{100}$, $\mathfrak{F}y = -\frac{1}{200}$, $\mathfrak{J}x = -x$, $\mathfrak{M}y = 80$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= \max\left\{-\frac{x}{100}, \frac{1}{200}\right\} \\ &\leq \frac{3}{4}\max\left\{\max\{-x, 80\}, -\frac{x}{2}, 40, \max\left\{-x, \frac{1}{200}\right\}, 80\right\} \\ &= \Omega[\max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}], \end{aligned}$$

4. for $x \in (0, 100)$ and $y \in (-100, 0]$, $\mathfrak{B}x = -\frac{1}{100}$, $\mathfrak{F}y = -\frac{y}{200}$, $\mathfrak{J}x = 90$, $\mathfrak{M}y = -y$ and

$$\begin{aligned} d(\mathfrak{B}x, \mathfrak{F}y) &= \max\left\{\frac{1}{100}, -\frac{y}{200}\right\} \\ &\leq \frac{3}{4}\max\left\{\max\{90, -y\}, 45, -\frac{y}{2}, 90, \max\left\{\frac{1}{100}, -y\right\}\right\} \\ &= \Omega[\max\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), \\ &\quad d(\mathfrak{B}x, \mathfrak{M}y)\}], \end{aligned}$$

so, all hypotheses of Theorem 5.2 are satisfied and 0 is the unique common fixed point of maps $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{J}$ and $\mathfrak{M}$.

Remark 5.2. Note that $\mathfrak{B}\mathcal{X} = [0, 1) \cup \left\{-\frac{1}{100}\right\} \nsubseteq \mathfrak{M}\mathcal{X} = [0, 100)$ and $\mathfrak{F}\mathcal{X} = [0, 1) \cup \left\{-\frac{1}{200}\right\} \nsubseteq \mathfrak{J}\mathcal{X} = [0, 100)$.

Remark 5.3. We mention that our two results remain valid if we replace

$$\max\left\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{2}d(\mathfrak{B}x, \mathfrak{J}x), \frac{1}{2}d(\mathfrak{F}y, \mathfrak{M}y), d(\mathfrak{J}x, \mathfrak{F}y), d(\mathfrak{B}x, \mathfrak{M}y)\right\}$$

by

$$\max\left\{d(\mathfrak{J}x, \mathfrak{M}y), \frac{1}{4}[d(\mathfrak{B}x, \mathfrak{J}x) + d(\mathfrak{F}y, \mathfrak{M}y)], d(\mathfrak{J}x, \mathfrak{F}y), d(\mathfrak{B}x, \mathfrak{M}y)\right\}.$$

6 Application

Consider the following integral equation

$$u(x) = \int_0^x m(x, t)p_i(t, u(t))dt + \int_0^1 n(x, t)q_i(t, u(t))dt, \text{ for all } x \in [0, 1] \tag{6.1}$$

where $m(x, t)$, $n(x, t) : [0, 1] \times [0, 1] \to \mathbb{R}_+$ are continuous functions and p_i, $q_i : [0, 1] \times \mathbb{R} \to [0, +\infty)$, $i = 1, 2$ are non-negative continuous functions.

Let $\mathcal{X} = C([0, 1])$ be the set of all non-negative real continuous functions on $[0, 1]$, endowed with the dislocated metric $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ such that for all $u, v \in \mathcal{X}$

$$\begin{aligned} d(u, v) &= \|u\|_\infty + \|v\|_\infty \\ &= \max_{x\in[0,1]} u(x) + \max_{x\in[0,1]} v(x). \end{aligned}$$

Clearly, $(\mathcal{X}, d)$ is a dislocated metric space.

Theorem 6.1. *The integral equation (6.1) has a unique solution if the following conditions hold:*

1. $\int_0^1 \max_{x\in[0,1]} |m(x, t)|dt \leq \frac{1}{\sqrt{2}}$,

2. $\int_0^1 \max_{x\in[0,1]} |n(x,t)|dt \leq \frac{1}{2}$,

3. $|p_i(t,u(t))| \leq \frac{1}{\sqrt{2}}|u(t)|$, $i = 1,2$,

4. $|q_i(t,u(t))| \leq \frac{1}{2}|u(t)|$, $i = 1,2$,

5. *the functions commute at their each coincidence points,*

Proof. Define maps $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{J}$ and $\mathfrak{M} : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned}
\mathfrak{B}u(x) &= \int_0^x m(x,t)p_1(t,u(t))dt \\
\mathfrak{F}u(x) &= \int_0^x m(x,t)p_2(t,u(t))dt \\
\mathfrak{C}u(x) &= \int_0^1 n(x,t)q_1(t,u(t))dt \\
\mathfrak{D}u(x) &= \int_0^1 n(x,t)q_2(t,u(t))dt \\
\mathfrak{J}u(x) &= (\mathfrak{I} - \mathfrak{C})u(x) \\
\mathfrak{M}u(x) &= (\mathfrak{I} - \mathfrak{D})u(x)
\end{aligned}$$

where $\mathfrak{I}$ is the identity function on $\mathcal{X}$.

By the virtue of the fifth condition, we can see that pairs of maps $(\mathfrak{J},\mathfrak{B})$ and $(\mathfrak{M},\mathfrak{F})$ are occasionally weakly $\mathfrak{J}$-biased (respectively $\mathfrak{M}$-biased) of type $(\mathcal{A})$.

Now, we prove that condition (5.1) of Theorem 5.1 is satisfied. Let $u(x)$, $v(x) \in \mathcal{X}$. Then, for all $x \in [0,1]$, we have

$$\begin{aligned}
|\mathfrak{B}u(x)| &= \left|\int_0^x m(x,t)p_1(t,u(t))dt\right| \\
&\leq \int_0^x |m(x,t)||p_1(t,u(t))|dt \\
&\leq \frac{1}{\sqrt{2}}\int_0^x |m(x,t)||u(t)|dt \\
&\leq \frac{1}{\sqrt{2}}\int_0^1 |m(x,t)| \max_{t\in[0,1]} |u(t)|dt \\
&\leq \frac{1}{\sqrt{2}}\|u\|_\infty \int_0^1 \max_{x\in[0,1]} |m(x,t)|dt,
\end{aligned}$$

implying that

$$\|\mathfrak{B}u\|_\infty \leq \frac{1}{2}\|u\|_\infty.$$

It follows that, for all u, $v \in \mathcal{X}$,

$$d(\mathfrak{B}u,\mathfrak{F}v) \leq \frac{1}{2}d(u,v). \tag{6.2}$$

Similarly, we have

$$\begin{aligned}
|\mathfrak{C}u(x)| &= \left|\int_0^1 n(x,t)q_1(t,u(t))dt\right| \\
&\leq \int_0^1 |n(x,t)||q_1(t,u(t))|dt \\
&\leq \frac{1}{2}\int_0^1 |n(x,t)||u(t)|dt \\
&\leq \frac{1}{2}\int_0^1 |n(x,t)| \max_{t\in[0,1]} |u(t)|dt \\
&\leq \frac{1}{2}\|u\|_\infty \int_0^1 \max_{x\in[0,1]} |n(x,t)|dt,
\end{aligned}$$

which implies that

$$\|\mathfrak{C}u\|_\infty \leq \frac{1}{4}\|u\|_\infty.$$

It follows that, for all $u,\ v \in \mathfrak{X}$,

$$d(\mathfrak{C}u, \mathfrak{D}v) \leq \frac{1}{4}d(u,v). \tag{6.3}$$

Hence, we have

$$\begin{aligned}
d(\mathfrak{J}u, \mathfrak{M}v) &= \|\mathfrak{J}u\|_\infty + \|\mathfrak{M}v\|_\infty \\
&= \max_{x\in[0,1]} \mathfrak{J}u(x) + \max_{x\in[0,1]} \mathfrak{M}v(x) \\
&= \max_{x\in[0,1]} [(\mathfrak{I}-\mathfrak{C})u(x) + (\mathfrak{I}-\mathfrak{D})v(x)] \\
&= [\max_{x\in[0,1]} u(x) + \max_{x\in[0,1]} v(x)] - [\max_{x\in[0,1]} \mathfrak{C}u(x) + \max_{x\in[0,1]} \mathfrak{D}v(x)] \\
&= d(u,v) - d(\mathfrak{C}u, \mathfrak{D}v) \\
&\geq d(u,v) - \frac{1}{4}d(u,v) \\
&= \frac{3}{4}d(u,v),
\end{aligned}$$

which implies that

$$d(u,v) \leq \frac{4}{3}d(\mathfrak{J}u, \mathfrak{M}v). \tag{6.4}$$

From (6.2) and (6.4), we obtain

$$\begin{aligned}
d(\mathfrak{B}u, \mathfrak{F}v) &\leq \frac{1}{2}d(u,v) \\
&\leq \frac{1}{2}\cdot\frac{4}{3}d(\mathfrak{J}u, \mathfrak{M}v) \\
&= \frac{2}{3}d(\mathfrak{J}u, \mathfrak{M}v) \\
&< d(\mathfrak{J}u, \mathfrak{M}v) \\
&\leq \max\left\{d(\mathfrak{J}u, \mathfrak{M}v), \frac{1}{2}d(\mathfrak{B}u, \mathfrak{J}u), \frac{1}{2}d(\mathfrak{F}v, \mathfrak{M}v), d(\mathfrak{J}u, \mathfrak{F}v), d(\mathfrak{B}u, \mathfrak{M}v)\right\}.
\end{aligned}$$

Thus all the conditions of Theorem 5.1 are satisfied. Therefore, there is a unique point $u' \in \mathfrak{X}$ such that $\mathfrak{B}u' = \mathfrak{F}u' = \mathfrak{J}u' = \mathfrak{M}u' = u'$, consequently, u' is a unique solution of (6.1). □

7 Conclusion

In this paper, we proved four unique common fixed point theorems for two occasionally weakly biased pairs maps. We investigated in two sides; in the first side, we exploited in a metric space, while in the second side, we took benefit from reducing the first axiom of the metric space to fall in the dislocated metric space. This later one is unique because in the first condition, the distance between a point and itself can be positive. Our results extend and improve Theorems 3.1 and 3.3 of [6] and some results therein, for instance, Theorem 1 of [1].

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Hakima Bouhadjera

Laboratory of Applied Mathematics, Badji Mokhtar-Annaba University, P.O. Box 12, 23000 Annaba, Algeria

E-mail: b_hakima2000@yahoo.fr