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Stability properties of a Cournot-type oligopoly model with time delay

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Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary

Abstract
The paper studies a Cournot game involving n private firms, where the firms' interactions are analyzed in a discrete-time setting with time delay. The past production levels of other private firms have an impact on the production levels of each private firm. We performed a local stability analysis on the nonlinear system associated with the model and found that it has only one equilibrium point, which is asymptotically stable if and only if some conditions involving the number of firms, the degree of product differentiation and the time delay are fulfilled. In addition, numerical examples are presented in this paper, which illustrate the theoretical results obtained.

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Keywords equilibrium point; local stability; oligopoly game; bifurcations

1 Introduction

In the field of mathematical economics, the study of oligopoly theory has been a subject of great interest. Within this field, the branch of chaotic dynamics, which draws on concepts from chaos theory and bifurcation theory, has gained significant attention and has been widely applied in various contexts.

Dynamic duopoly games have been extensively studied in both discrete and continuous time settings, particularly focusing on firms on the market which compete with homogeneous products [15, 18]. Papers [4, 11] pertain to dynamic Cournot oligopoly games, considering n firms in the competition. Paper [4] investigates a scenario where three firms are considered, characterized by bounded rationality, utilizing the utility CES function. Moreover, [20] presented an oligopoly model that assumed a linear demand function

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and constant marginal costs. The paper [13] investigated Cournot duopoly models with adaptive expectations and in recent studies, the same strategies have been incorporated into the Cournot-Theocharis model [8]. Additionally, [21] presented a Cournot game for n-person asymmetric information. Other examples of Cournot games involved the consideration of heterogeneous firms [1, 2, 10].

The literature on mixed oligopoly extensively investigates the calculation of the most advantageous financial assistance, assuming that private firms aim to maximize their profits while the public firm seeks to optimize welfare in a framework of simultaneous Cournot-Nash competition [5, 16, 19].

The paper [3] investigates the stability of the Nash equilibrium in a dynamic model involving n competing firms. The study analyzes the conditions under which the Nash equilibrium in this dynamic model remains stable. The recent paper [9] focuses on the interactions between a single public firm and n private firms in a market setting. Specifically, the paper examines the dynamics of a discrete-time Cournot game involving these firms, taking into account two time delays. The analysis explores the effects of these time delays on the strategic behavior and equilibrium outcomes in the market.

In the theory of delay difference equations, a classical result which provides necessary and sufficient conditions that ensure asymptotic stability of the equilibrium has been provided by Kuruklis [14]. Moreover, an alternative approach, that appears to be simpler compared to the method presented by Kuruklis, has been explored in [17]. Additionally, in the papers [7,12], the framework of delayed linear difference systems has been analyzed, extending the existing stability conditions.

In the present analysis, we examine a dynamic oligopoly game involving n private firms that produce differentiated products. The production levels of these private firms are determined based on reaction functions. The main contribution of our study is the determination of the stability properties of the Nash equilibrium, involving the number of private firms, the degree of product differentiation, the adjustment parameter, and the presence of time delays.

The structure of this paper is outlined as follows: in Section 2, the mathematical model is presented, and an equilibrium point is identified. The local stability analysis for the equilibrium point is discussed in Section 3. The theoretical results are illustrated through numerical simulations in Section 4. Finally, the paper concludes with a summary, conclusions, and a discussion of future research directions.

2 The mathematical model and the equilibrium point

In our mathematical model, the state variable q_i, $i = \overline{1,n}$, represents the output produced by the private firm i with the retail price p_i, in a market with n competitors. We assume that all private firms have naive expectation.

We consider that the production levels of private firms are adjusted with respect to the past productions of other private firms, occurring at time $t-\tau$, where $\tau \in \mathbb{Z}^+$. Consequently, we focus on analyzing the following discrete-time mathematical model with

time delay:

$$q_j(t+1) = \max\left\{0, (1-\delta)q_j(t) + \alpha\frac{\delta}{2} - \frac{\delta}{2}\sum_{i=1, i\neq j}^{n} q_i(t-\tau)\right\}, \quad j = \overline{1,n} \tag{2.1}$$

where the parameter $\delta \in (0,1)$ represents the degree of product differentiation and α is a positive parameter.

The unique equilibrium point for the discrete dynamical system given in (2.1) is:

$$E_n^\star = (q^\star, q^\star, ..., q^\star), \text{ where } q^\star = \frac{\alpha}{n+1}.$$

A similar model has been recently explored in [6], mainly considering specific values for the time delay, $\tau \in \{1,2,3,4\}$. The aim of the present work is to show that in fact, it is possible to obtain necessary and sufficient conditions for the asymptotic stability of the equilibrium in the general case, without any restriction on the time delay. For simplicity, we assume equal marginal costs for the n private firms, however it is important to emphasize that the generality of the theoretical results is preserved.

3 Local stability analysis

The linearized system at the equilibrium point $E_n^\star$ takes the following form:

$$y(t+1) = Ay(t) + By(t-\tau) \tag{3.1}$$

where

$$y(t) = \begin{bmatrix} q_1(t) - q^e & q_2(t) - q^e & \dots & q_n(t) - q^e \end{bmatrix}^T$$

and the matrices A and B are:

$$A = (1-\delta)I, \quad B = \begin{bmatrix} 0 & \dots & -\delta/2 & -\delta/2 \\ -\delta/2 & \dots & -\delta/2 & -\delta/2 \\ \vdots & \dots & \vdots & \vdots \\ -\delta/2 & \dots & -\delta/2 & 0 \end{bmatrix}$$

The characteristic equation of system (3.1) can be obtained using the $\mathcal{Z}$-transform method, and is given below:

$$\det\left(\lambda I - A - B\lambda^{-\tau}\right) = 0. \tag{3.2}$$

If we replace the matrix A by $(1-\delta)I$, the previous equation can be written:

$$\det\left[(\lambda + \delta - 1)I - B\lambda^{-\tau}\right] = 0$$

and if we multiply with $\lambda^{n\tau}$, the characteristic equation has the form

$$\det\left[\lambda^\tau(\lambda + \delta - 1)I - B\right] = 0.$$

Hence, the roots of the characteristic equation satisfy one of the equations

$$\lambda^{\tau+1} - (1-\delta)\lambda^{\tau} - \beta_i = 0 \quad \text{where} \quad \beta_i \in \sigma(B). \tag{3.3}$$

The eigenvalues of the circulant matrix B are the simple eigenvalue $\beta_1 = (1-n)\dfrac{\delta}{2}$ and $\beta_2 = \dfrac{\delta}{2}$ with the order of multiplicity $(n-1)$.

Therefore, all the roots of the characteristic equation (3.2) are inside the unit disk if and only if the roots of both equations (3.3) (with $i \in \{1,2\}$) are inside the unit disk.

Remark 1. *If $\tau = 0$, the equilibrium point $E_n^\star$ is asymptotically stable if and only if the following inequality is satisfied:*

$$(n+1)\delta < 4. \tag{3.4}$$

Characteristic equations of the form (3.3) have been extensively investigated previously in [7, 12, 14, 17]. To obtain our main stability result, we refer to Corollary 3.4 from [12]. With this mind, we first introduce the necessary notations.

We define the function

$$s_{\delta,\tau}(\theta) = \sin(\tau+1)\theta - (1-\delta)\sin\tau\theta$$

and we denote by $\theta_{\delta,\tau}$ the first positive root of the function $s_{\delta,\tau}$, which belongs to the interval $\left(\dfrac{\pi}{2\tau+1}, \dfrac{\pi}{\tau+1}\right)$ (see Lemma 2.1 from [12]).

We can formulate the following necessary and sufficient condition for the asymptotic stability of the equilibrium point:

Proposition 1. *For any $\tau > 0$, the equilibrium point $E_n^\star$ is asymptotically stable if and only if*

$$(n-1)\frac{\delta}{2} < \sqrt{1 + (1-\delta)^2 - 2(1-\delta)\cos\theta_{\delta,\tau}}. \tag{3.5}$$

Proof. For the asymptotic stability of the equilibrium point $E_n^\star$, Corollary 3.4 from [12] provides the necessary and sufficient condition expressed as the following inequality:

$$-\sqrt{1 + (1-\delta)^2 - 2(1-\delta)\cos\theta^1_{\delta,\tau}} < \beta_i < \delta, \quad i \in \{1,2\}. \tag{3.6}$$

As the previous inequality is clearly satisfied for the eigenvalue β_2, we must only ensure that it is also satisfied for β_1, which provides the equivalent inequality (3.5). □

Remark 2. *If δ and τ are fixed parameters, the maximum number of firms for which the equilibrium point $E_n^\star$ is asymptotically stable is given by:*

$$n_{max} = 1 + \left\lceil \frac{2}{\delta}\sqrt{1 + (1-\delta)^2 - 2(1-\delta)\cos\theta_{\delta,\tau}} - 1 \right\rceil.$$

Consequently, the stability region of $E_n^\star$ corresponding to a larger number of firms is included in the stability the stability region corresponding to a smaller number of firms, if the (τ,δ) parameter plane is considered (see Fig 1). In addition, Fig. 1 also indicates that as the number of private firms increases, the stability of the positive equilibrium requires lower values of the degree of product differentiation.

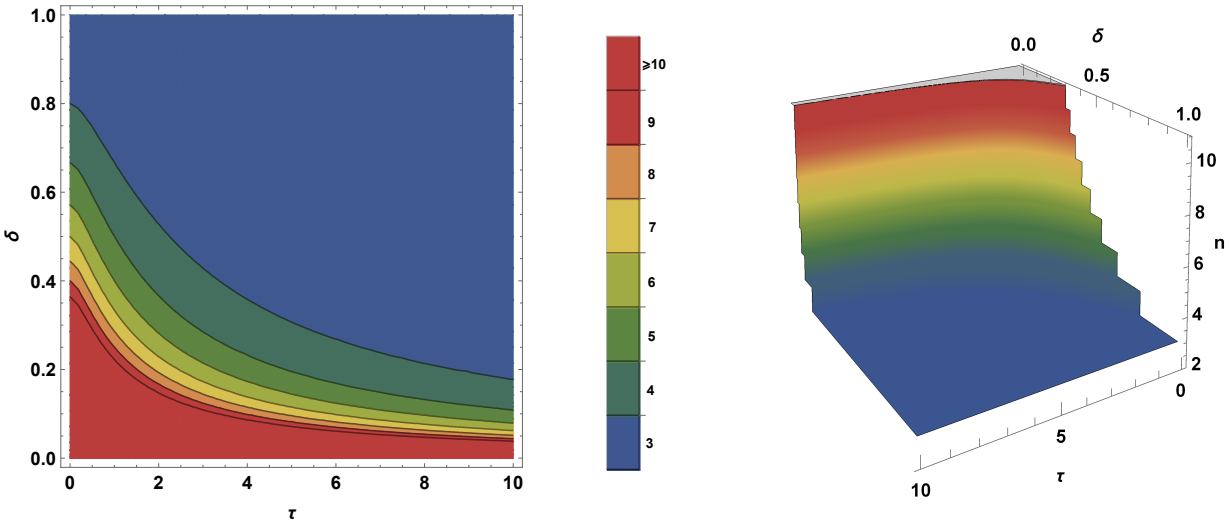


Figure 1: Stability regions of the positive equilibrium point $E_n^\star$ of system (2.1) in the (τ, δ) parameter plane, for different values of $n \in \{3, 4, 5, \dots\}$ (colored blue to red). Here, $\alpha = 0.7$.

Example 1. *If* $\tau = 1$ *then the function* $s_{\delta,1} = \sin 2\theta - (1-\delta)\sin\theta$ *is considered, and hence* $\cos\theta_{\delta,1} = \dfrac{1-\delta}{2}$. *Therefore, the inequality* (3.5) *is reduced to:*

$$(n-1)\frac{\delta}{2} < 1.$$

Example 2. *If* $\tau = 2$ *as the function* $s_{\delta,2} = \sin 3\theta - (1-\delta)\sin 2\theta$ *is taken into account, we obtain:*

$$4\cos^2\theta_{\delta,2} - 2(1-\delta)\cos\theta_{\delta,2} - 1 = 0$$

and the inequality (3.5) *can be written as:*

$$(n-1)\frac{\delta}{2} < \sqrt{\frac{3 - 2\delta + \delta^2 + (\delta-1)\sqrt{\delta^2 - 2\delta + 5}}{2}}.$$

4 Numerical simulations

To illustrate our theoretical findings, we consider two scenarios: one with 4 private firms and another with 6 private firms. In both cases, we use a fixed parameter value of $\alpha = 0.7$. With this parameter value, the positive equilibrium points for the two cases are computed as follows:

$$E_4^\star = (0.14, 0.14, 0.14, 0.14)$$
$$E_6^\star = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1).$$

For the case of $n = 4$ private firms, considering the time delay $\tau = 2$, numerical computations show that inequality (3.5) is satisfied if and only if $\delta \in (0, 0.52752)$, which

gives the interval for which the local asymptotic stability of the equilibrium $E_4^\star$ is guaranteed. At the critical value $\delta^\star = 0.52752$ a Neimark-Sacker bifurcation takes place in a neighborhood of the equilibrium, which causes the appearance of an attractive limit cycle, as shown in Figure 3.

On the other hand, for $n = 6$ private firms, considering the time delay $\tau = 8$, the stability of the equilibrium $E_6^\star$ is lost for $\delta^\star = 0.09639$, and a Neimark-Sacker bifurcation takes place.

In both cases, the phase portraits presented in Figures 3 and 5 demonstrate coherence with the bifurcation diagrams, providing visual representation of diverse dynamic regimes.

It is interesting to notice that in both cases, high values of δ are associated with the existence of an asymptotically stable periodic cycle, of period 6 (for $n = 4$) and of period 18 (for $n = 6$), respectively.

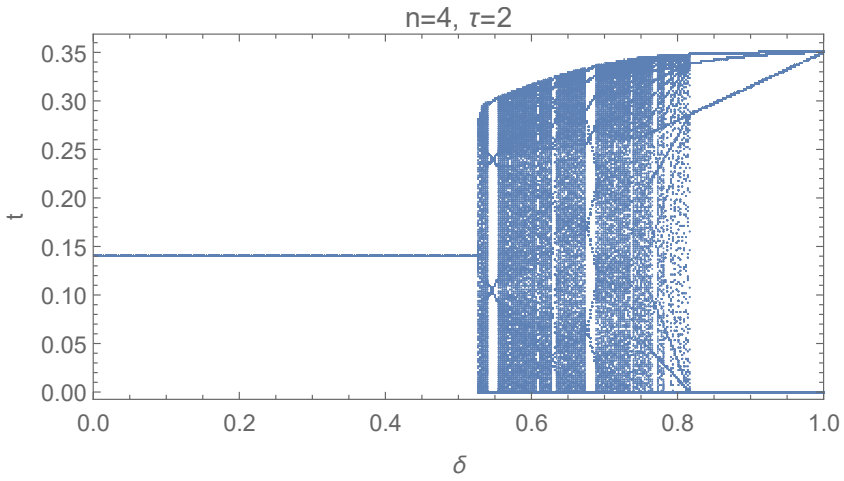


Figure 2: Bifurcation diagram for system (2.1) with $n = 4$ private firms, with respect to δ. Time delay: $\tau = 2$.

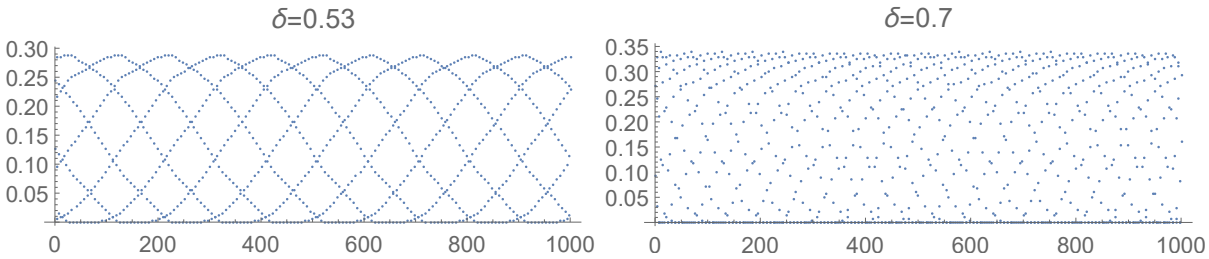


Figure 3: Phase portraits for system (2.1) with $n = 4$ private firms, for various values of δ. Time delay: $\tau = 2$.

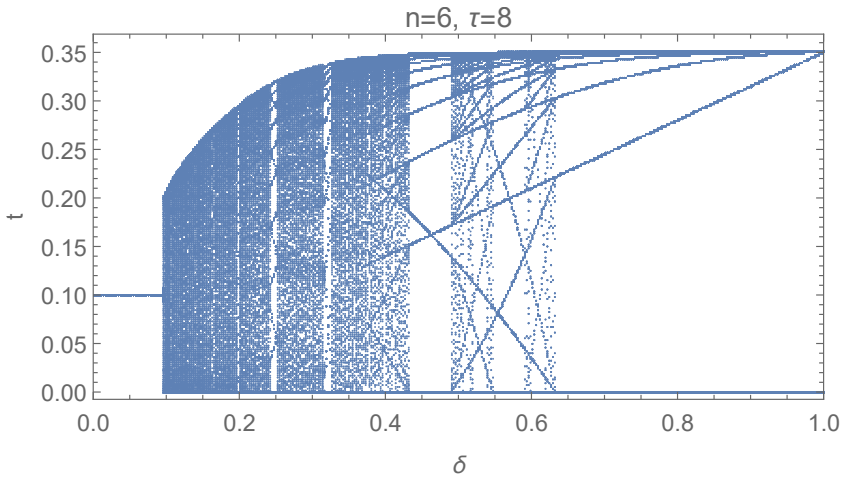


Figure 4: Bifurcation diagram for system (2.1) with $n = 6$ private firms , with respect to δ. Time delay: $\tau = 8$.

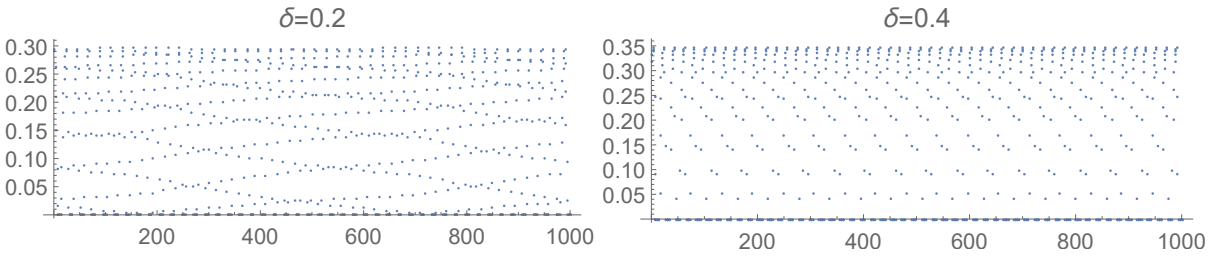


Figure 5: Phase portraits for system (2.1) with $n = 6$ private firms, for various values of δ. Time delay: $\tau = 6$.

5 Conclusions

The current study focuses on examining the dynamics of an oligopoly game involving the coexistence of n private firms. An equilibrium point has been established for the associated discrete-time mathematical model incorporating a single time delay. Necessary and sufficient conditions are found for the local asymptotic stability of the unique equilibrium point $E^\star$, formulated as a simple inequality involving the number of firms n, the discrete time delay τ and the degree of product differentiation δ. We deduce that the stability region in the (τ, δ) parameter plane decreases as n increases, which can also be visualised numerically. Numerical simulations are also presented, by exploring bifurcation diagrams for specific parameter values.

As a direction for future research, we plan to explore diverse bifurcation scenarios of this model.

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