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Estimate of third Hankel determinant for a subfamily of analytic functions

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Abstract In this article, our aim is to study analytic functions related with Salagean operator and associated with the right half of the lemniscate of Bernoulli. We find the estimates of the third Hankel determinant for new family of analytic functions. It is important to mention that our results generalize a number of existence results in the literature.

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1 Introduction

Let A denote the class of all analytic (holomorphic) functions λ defined in the open unit disk $\mathbf{U} = \{\omega : \omega \in \mathbb{C} \text{ and } |\omega| < 1\}$, which are normalized by the following conditions $\lambda(0) = 0$ and $\lambda'(0) = 1$. Thus each $\lambda \in A$ has the following Maclaurin series expansion of the form:

$$\lambda(\omega) = \omega + \sum_{t=2}^{\infty} c_t \omega^t \quad \omega \in \mathbf{U}. \tag{1.1}$$

Further, let S denote a subfamily of A, containing all functions univalent in $\mathbf{U}$. Here we list some important subfamilies that will be used in our subsequent work. A function $\lambda \in S$ is starlike with respect to origin if it satisfies the following condition:

$$\Re\left(\frac{\omega\lambda'(\omega)}{\lambda(\omega)}\right) > 0, \qquad (\omega \in \mathbf{U}).$$

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The class of all such functions is denoted by S^*. For $\lambda \in S^*$, one can find that:

$$|c_n| \leq n \quad \text{for } n = 2, 3, \ldots \quad \text{(see [8])}. \tag{1.2}$$

A function $\lambda \in S$ is said to be close-to-convex function if there exist $\delta \in S^*$ and

$$\Re\left(\frac{\omega\lambda'(\omega)}{\delta(\omega)}\right) > 0.$$

The class of close-to-convex functions is denoted by K. By SL^* we mean the class of functions $\lambda \in A$ that satisfy the following inequality:

$$\left|\left(\frac{\omega\lambda'(\omega)}{\lambda(\omega)}\right)^2 - 1\right| < 1, \quad (\omega \in \mathbf{U}).$$

Thus geometrically a function $\lambda \in SL^*$ if $\frac{\omega\lambda'(\omega)}{\lambda(\omega)}$ lies in the region bounded by the right half of the lemniscate of Bernoulli given by the following relation:

$$\left|k^2 - 1\right| < 1, \quad (\omega \in \mathbf{U}),$$

where $k = \frac{\omega\lambda'(\omega)}{\lambda(\omega)}$. The above defined class was introduced by Sokol et al., see [31]. Let $\delta \in A$ and having series representation of the form

$$\delta(\omega) = \omega + \sum_{t=2}^{\infty} d_t \omega^t, \tag{1.3}$$

then the convolution of λ and δ denoted by $\lambda * \delta$ is defined as:

$$(\lambda * \delta)(\omega) = \omega + \sum_{t=2}^{\infty} c_t d_t \omega^t = (\delta * \lambda)(\omega), \tag{1.4}$$

where $\lambda(\omega)$ is given by (1.1). Another important tool which we will use in our current research is subordination. A function λ is subordinate to δ in $\mathbf{U}$ and written as: $\lambda \prec \delta$, if there exists a Schwarz function $k(\omega)$ which is holomorphic in $\mathbf{U}$ with $|k(\omega)| < 1$, such that $\lambda = \delta(k(\omega))$. Furthermore, if the function δ is univalent in $\mathbf{U}$, then we have the following equivalence (see [18]):

$$\lambda(\omega) \prec \delta(\omega) \text{ and } \lambda(U) \subset \delta(U).$$

Let P be the class of analytic functions p with $Re(p(\omega)) > 0$ in $\mathbf{U}$ and

$$p(\omega) = 1 + \sum_{t=1}^{\infty} p_t \omega^t . \tag{1.5}$$

In recent years, several interesting subfamilies of analytic functions related with class p have been introduced and investigated (see [6, 20, 21, 23, 28, 30, 34]).

Definition 1.1. *[26]. Let $\lambda \in A$. Then Salagean differential operator $D^n : A \to A$ is defined by the following equation*

$$(D^n\lambda(\omega)) = \omega + \sum_{t=2}^{\infty} t^n c_t \omega^t, \quad \omega \in \mathbf{U}. \tag{1.6}$$

Note that $D^0\lambda(\omega) = \lambda(\omega)$, $D^1\lambda(\omega) = \omega\lambda'(\omega)$ and $D^2\lambda(\omega) = \omega\lambda'(\omega) + \omega^2\lambda''(\omega)$, where D^1 is the well known Alexander operator (see [1]).

Definition 1.2. *A function $\lambda \in A$ is in the class KL_n^* if it satisfies the following inequality:*

$$\left| \left(\frac{\omega (D^n\lambda(\omega))'}{D^n\delta(\omega)} \right)^2 - 1 \right| < 1. \tag{1.7}$$

for some $\delta \in S^*$. Equivalently, we have:

$$\frac{\omega (D^n\lambda(\omega))'}{D^n\delta(\omega)} \prec \sqrt{1+\omega}. \tag{1.8}$$

Thus clearly, a function $\lambda \in KL_n^*$ is such that $\frac{\omega(D^n\lambda(\omega))'}{D^n\delta(\omega)}$ lies in the region bounded by the right half of the lemniscate of Bernoulli. It is worth mentioning that many geometric properties of analytic functions were studied by several authors using conditions on coefficients of Maclaurin's series expansion. In 1916, Bieberbach conjectured that the *nth* coefficient of each univalent function having series (1.1) is less than or equal to n. This formulation conjectured was proved by Bieberbach in 1985.

Closely related to the Bieberbach conjecture is the problem of finding sharp estimates for the coefficients of odd univalent functions, which has the most general form of the square-root transformation of a function $\lambda \in S$:

$$I(\omega) = \sqrt{\lambda(\omega^2)} = \omega + c_3\omega^3 + c_5\omega^5 + \ldots$$

Let $n \geq 0$ and $q \geq 1$. Then the *qth* Hankel determinant is defined as follows:

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & \cdot & \cdot & \cdot & c_{n+q-1} \\ c_{n+1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n+q-1} & \cdot & \cdot & \cdot & \cdot & c_{n+2(q-1)} \end{vmatrix}. \tag{1.9}$$

The above Hankel determinant has been studied by many authors. In particular, sharp upper bounds on $H_2(2)$ were obtained by several authors (see [9, 19, 29, 33]) for various classes of normalized analytic functions. $H_2(1) = |c_3 - c_2^2|$ is the well-known Fekete-Szego functional. This functional is further generalized as $|c_3 - \mu c_2^2|$ for some real or complex μ. In fact, Fekete and Szego gave sharp estimates of $|c_3 - \mu c_2^2|$ for real μ and $\lambda \in S$, the class of normalized univalent functions in $\mathbf{U}$. It is also known that the functional $|c_2c_4 - c_3^2|$ is equivalent to $H_2(2)$. Babalola [4] studied the Hankel determinant $H_3(1)$ for some

subclasses of analytic functions. For more literature see [13, 15, 16, 22, 35]. The Hankel determinant of order three given below in (1.10) and the estimation of this determinant $|H_{3,1}(\lambda)|$ is very hard as compared to derive the bound of $|H_{2,2}(\lambda)|$,

$$H_{3,1}(\lambda) = \begin{vmatrix} 1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{vmatrix} = -c_5c_2^2 + 2c_2c_3c_4 - c_3^3 + c_5c_3 - c_4^2. \tag{1.10}$$

Babalola [4] is the first one who tried to find $H_{3,1}(\lambda)$ for a class of analytic functions. Some recent work on univalent function can be found in [2, 3, 5, 7, 10, 17, 25, 27].

Later on many authors found sharp estimates for the various classes of analytic functions related with the lemniscate of Bernoulli and the Salagean operator. We prove our results by utilizing some techniques similar to those that were used earlier by Sokol and Stankiewicz (see [31]).

2 Set of Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([14]). *Let*

$$p(\omega) = 1 + c_1\omega + c_2\omega^2 + \dots \quad ,$$

be in the class P. Then for any complex number μ

$$\left|c_2 - \mu c_1^2\right| = \begin{cases} -4\mu + 2 & \text{if } \mu \le 0, \\ 2 & \text{if } 0 \le \mu \le 1, \\ 4\mu - 2 & \text{if } \mu \ge 1. \end{cases} \tag{2.1}$$

When $\mu < 0$ or $\mu > 1$, equality holds true in (2.1) if and only if

$$p(\omega) = \frac{1+\omega}{1-\omega},$$

or one of its rotations. If $0 < \mu < 1$, then equality holds true in (2.1) if and only if

$$p(\omega) = \frac{1+\omega^2}{1-\omega^2},$$

or one of its rotations. If $\mu = 0$, equality holds true in (2.1) if and only if

$$p(\omega) = \left(\frac{1+\nu}{2}\right)\frac{1+\omega}{1-\omega} + \left(\frac{1-\nu}{2}\right)\frac{1-\omega}{1+\omega} \quad 0 \le \nu \le 1,$$

or one of its rotations. If $\mu = 1$, then the equality in (2.1) holds true if $p(\omega)$ is a reciprocal of one of the functions such that the equality holds true in the case when $\mu = 0$.

Lemma 2.2 ([11,12]). *Let*

$$p(\omega) = 1 + c_1\omega + c_2\omega^2 + \dots \quad ,$$

be in the class P. Then

$$2c_2 = c_1^2 \ + x\left(4 - c_1^2\right),$$

for some x, $(|x| \leq 1)$ and

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - \left(4 - c_1^2\right)c_1 x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\omega,$$

for some ω, $(|\omega| \leq 1)$.

Lemma 2.3 ([24]). *Let*

$$p(\omega) = 1 + c_1\omega + c_2\omega^2 + \dots \quad ,$$

be in the class P of functions with positive real part in $\mathbf{U}$. Then

$$|c_k| \leq 2, \quad k \in \mathbb{N},$$

and the inequality is sharp.

3 Main Results

Theorem 3.1. *Let $\lambda \in KL_n^*$ be of the form of (1.1). Then*

$$\left|c_3 - \mu c_2^2\right| \leq \begin{cases} \frac{1}{2}\left(1 - \frac{25\mu}{2^{n+3}}\right) + \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right), & \text{if} \quad \mu < \frac{2^n \cdot 16}{3^{n+1} \cdot 25}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right) \\ \frac{1}{2}, & \text{if} \ \frac{2^n \cdot 16}{3^{n+1} \cdot 25}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right) \leq \mu \leq \frac{2^n \cdot 16}{25}\left(1 + \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right) \\ \frac{1}{2}\left(\frac{25\mu}{2^{n+3}} - 1\right) - \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right), & \text{if} \quad \mu > \frac{2^n \cdot 16}{25}\left(1 + \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right). \end{cases} \tag{3.1}$$

It is also asserted that:

(i) For $\mu < \frac{2^n \cdot 16}{3^{n+1} \cdot 25}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)$

$$\left|c_3 - \mu c_2^2\right| + \frac{1}{3}\left[3\mu - \frac{2^n \cdot 16}{3^n \cdot 25}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right]|c_2|^2 \leq \frac{1}{2},$$

if $\frac{2^n \cdot 16}{3^{n+1} \cdot 25}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right) < \mu \leq \frac{1}{3^{n+1}}\left(\frac{240}{2^{2n}} - 126\right) + 24$, and

(ii) For $\mu > \frac{2^n \cdot 16}{25}\left(1 + \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right)$

$$\left|c_3 - \mu c_2^2\right| + \frac{1}{3}\left[\frac{2^n \cdot 3.16}{25}\left(1 + \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right) - 3\mu\right]|c_2|^2 \leq \frac{1}{2},$$

if $\frac{1}{3^{n+1}}\left(\frac{240}{2^{2n}} - 126\right) + 24 < \mu \leq \frac{2^n \cdot 16}{25}\left(1 + \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right)$.

Proof. If $\lambda \in KL_n^*$ then it follows from (1.8) that:

$$\frac{\omega \left(D^n \lambda \left(\omega\right)\right)'}{D^n \delta \left(\omega\right)} \prec \varphi \left(\omega\right), \tag{3.2}$$

where

$$\varphi \left(\omega\right) = \left(1+\omega\right)^{\frac{1}{2}}.$$

We define a function $p\left(\omega\right)$ by:

$$p\left(\omega\right) = \frac{1+k\left(\omega\right)}{1-k\left(\omega\right)} = 1 + p_1\omega + p_2\omega^2 + p_3\omega^3 + \ldots \quad . \tag{3.3}$$

It is clear that $p\left(\omega\right) \in P$. From (3.2) and (3.3), we have

$$\frac{\omega \left(D^n \lambda \left(\omega\right)\right)'}{D^n \delta \left(\omega\right)} = \varphi \left(k\left(\omega\right)\right). \tag{3.4}$$

Consider

$$\begin{aligned} \varphi \left(k\left(\omega\right)\right) &= \left(1+k\left(\omega\right)\right)^{\frac{1}{2}} = \left(\frac{2p\left(\omega\right)}{p\left(\omega\right)+1}\right)^{\frac{1}{2}} \\ &= 1 + \frac{1}{4}p_1\omega + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right)\omega^2 + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3\right)\omega^3 \\ &\quad + \left(\frac{1}{4}p_4 - \frac{5}{16}p_1p_3 + \frac{39}{128}p_2p_1^2 - \frac{5}{32}p_2^2 - \frac{141}{2048}p_1^4\right)\omega^4 + \ldots. \quad . \end{aligned} \tag{3.5}$$

Using (1.1) and (1.6) we have

$$\omega \left(D^n \lambda \left(\omega\right)\right)' = \omega + \sum_{t=2}^{\infty} t^{n+1} c_k \omega^k.$$

Let $D^n \delta \left(z\right) = \omega + \sum\limits_{t=2}^{\infty} t^n d_k \omega^k$.

Now

$$\begin{aligned} \frac{\omega \left(D^n \lambda \left(\omega\right)\right)'}{D^n \delta \left(\omega\right)} &= \frac{\omega + 2^{n+1}c_2\omega^2 + 3^{n+1}c_3\omega^3 + 4^{n+1}c_4\omega^4 + 5^{n+1}c_5\omega^5 + \ldots}{\omega + 2^n d_2\omega^2 + 3^n d_3\omega^3 + 4^n d_4\omega^4 + 5^n d_5\omega^5 + \ldots} \\ &= \left(1 + 2^{n+1}c_2\omega + 3^{n+1}c_3\omega^2 + 4^{n+1}c_4\omega^3 + 5^{n+1}c_5\omega^4 + \ldots\right) \\ &\quad \left(\begin{array}{c} 1 - 2^n d_2\omega - 3^n d_3\omega^2 - 4^n d_4\omega^3 - 5^n d_5\omega^4 + 2^{2n}d_2^2\omega^2 + \\ 3^{2n}d_3^2\omega^4 + 2^{n+1}3^n d_2 d_3\omega^3 - 2^{3n}d_2^3\omega^3 - 3^{n+1}.2^n d_2 d_3\omega^3 + 2^{4n}d_2^4\omega^4 + \ldots \end{array}\right). \end{aligned}$$

This implies that

$$\begin{aligned}\frac{\omega\left(D^n\lambda\left(\omega\right)\right)'}{D^n\delta\left(\omega\right)} &= 1-2^nd_2\omega-3^nd_3\omega^2-4^nd_4\omega^3-5^nd_5\omega^4+2^{2n}d_2^2\omega^2+3^{2n}d_3^2\omega^4\\ &\quad+2^{n+1}3^nd_2d_3\omega^3-2^{3n}d_2^3\omega^3+2^{4n}d_2^4\omega^4+2^{n+1}c_2\omega+3^{n+1}c_3\omega^2\\ &\quad+4^{n+1}c_4\omega^3+5^{n+1}c_5\omega^4-2^{2n+1}c_2d_2\omega^2-2^{n+1}3^nc_2d_3\omega^3+2^{n+1}4^nc_2d_4\omega^4\\ &\quad+2^{3n+1}c_2d_2^2\omega^3+2^{2n+2}3^nc_2d_2d_3\omega^4-2^{n+1}2^{3n}c_2d_2^3\omega^4-3^{n+1}2^nc_3d_2\omega^3\\ &\quad-3^{n+1}3^nc_3d_3\omega^4-+3^{n+1}2^{2n}c_3d_2^2\omega^4-4^{n+1}2^nc_4d_2\omega^4\\ &= 1+2^n\left(2c_2-d_2\right)\omega+\left[3^n\left(3c_3-d_3\right)+d_2\left(d_2-2c_2\right)\right]\omega^2\\ &\quad+\left[4^n\left(4c_4-d_4\right)+2^n3^nd_2\left(2d_3-3c_3\right)-2^n\left(2.3^nc_2d_3-2^{2n}d_2^3\right)+2^{n+1}c_2d_2^2\right]\omega^3\\ &\quad+\left[\begin{array}{c}5^n\left(5c_5-d_5\right)+3^{2n}d_3\left(d_3-3c_3\right)-2^{3n}d_2\left(d_2^3-2c_4\right)+\\ 2^{2n}\left(3^{n+1}c_3d_2^2-2^{n+1}c_3d_4\right)+2^n.3^n\left(2^{n+2}c_2d_2d_3-3d_2d_3-2.3^{2n}c_2d_2^3\right)\end{array}\right]\omega^4\\ &\quad+\ldots\ . \end{aligned}\tag{3.6}$$

Similarly,

$$\begin{aligned}\frac{\omega\left(D^n\delta\left(\omega\right)\right)'}{D^n\delta\left(\omega\right)} &= 1+\left(2^nd_2\right)\omega+\left[2\cdot3^nd_3+2^{2n}d_2^2\right]\omega^2+\left[3\cdot4^nd_4+2^{n+2}3^nd_2d_3+3\cdot2^{3n}d_2^3\right]\omega^3\\ &\quad+\left[\begin{array}{c}5^n\cdot4d_5-3^{2n}\cdot2d_3^2+3^n\cdot2^nd_2^2d_3\left(3^n+3^n\cdot2^n+3\cdot2^n\right)\\ +4^n\cdot2^{n+1}d_2d_4+2^nd_2\left(2^{3n}-2\cdot3^{3n}d_2^3\right)\end{array}\right]\omega^4+\ldots\ .\end{aligned}\tag{3.7}$$

Comparing the coefficients of (3.3) and (3.7), we have

$$d_2 = \frac{1}{2^n}p_1,\tag{3.8}$$

$$d_3 = \frac{1}{2\cdot3^n}\left(p_2+p_1^2\right),\tag{3.9}$$

$$d_4 = \frac{1}{3\cdot4^n}\left(p_3+2p_1p_2-p_1^3\right),\tag{3.10}$$

$$d_5 = \frac{1}{5^n\cdot4}\left(\begin{array}{c}p_4+p_1^4\left(\frac{2^n}{3}-\frac{1}{2^{n+1}}\left(3^n+3^n\cdot2^n+3\cdot2^n\right)+1\right)+\frac{1}{2}p_2^2-\\ p_2p_1^2\left(\frac{1}{3}+\frac{1}{2^{n+1}}\left(3^n+3^n\cdot2^n+3\cdot2^n\right)\right)-\frac{2}{3}p_1p_3-2^{3n}p_1\end{array}\right).\tag{3.11}$$

Using (3.8)-(3.11) in (3.6), we obtain

$$\begin{aligned}\frac{z\left(D^n\lambda\left(z\right)\right)'}{D^n\delta\left(z\right)} &= 1+2^n\left(2c_2-\frac{1}{2^n}p_1\right)\omega\\ &\quad+\left[3^n\left(3c_3-\frac{1}{2\cdot3^n}\left(p_2+p_1^2\right)\right)+\frac{1}{2^n}p_1\left(\frac{1}{2^n}p_1-2c_2\right)\right]\omega^2\\ &\quad+\left[\begin{array}{c}4^n\left(4c_4-\frac{1}{3\cdot4^n}\left(p_3+2p_1p_2-p_1^3\right)\right)+2^n3^n\frac{1}{2^n}p_1\left(\frac{1}{3^n}\left(p_2+p_1^2\right)-3c_3\right)-\\ 2^n\left(2\cdot3^n\frac{1}{2\cdot3^n}\left(p_2+p_1^2\right)c_2-2^{2n}\left(\frac{1}{2^n}p_1\right)^3\right)+2^{n+1}c_2\left(\frac{1}{2^n}p_1\right)^2\end{array}\right]\omega^3\\ &\quad+\left[\begin{array}{c}5^n\left(5c_5-\frac{1}{5^n\cdot4}\left(\begin{array}{c}p_4+p_1^4\left(\frac{2^n}{3}+1\right)-\frac{1}{2^{n+1}}\left(3^n+3^n\cdot2^n+3\cdot2^n\right)\\ \left(p_2p_1^2+p_1^4\right)+\frac{1}{2}p_2^2-\frac{p_2p_1^2}{3}-\frac{2}{3}p_1p_3-2^{3n}p_1\end{array}\right)\right)\\ +3^{2n}\cdot\frac{1}{2\cdot3^n}\left(p_2+p_1^2\right)\left(\frac{1}{2\cdot3^n}\left(p_2+p_1^2\right)-3c_3\right)-2^{2n}p_1\left(\frac{1}{2^{3n}}p_1^3-2c_4\right)+\\ 2^{2n}\left(\begin{array}{c}3^{n+1}c_3\frac{1}{2^{2n}}p_1^2-2^{n+1}\frac{1}{3\cdot4^n}\\ \left(p_3+2p_1p_2-p_1^3\right)c_3\end{array}\right)+2^n\cdot3^n\left(\begin{array}{c}2^{n+2}c_2\frac{1}{2^n}p_1\frac{p_2+p_1^2}{2\cdot3^n}-\\ \frac{3p_1\left(p_2+p_1^2\right)}{2^{n+1}\cdot3^n}-2\cdot3^{2n}\frac{c_2p_1^3}{2^{3n}}\end{array}\right)\end{array}\right]\omega^4\\ &\quad+\ldots\ .\end{aligned}\tag{3.12}$$

Therefore, upon comparing the corresponding coefficients of (3.5) and (3.12), we have

$$c_2 = \frac{1}{2^{n+1}}\frac{5}{4}p_1, \tag{3.13}$$

$$c_3 = \frac{1}{3^{n+1}}\left[\left(\frac{5}{2^{2n+2}}-\frac{21}{32}\right)p_1^2+\frac{3}{4}p_2\right], \tag{3.14}$$

$$c_4 = \frac{1}{4^{n+1}}\left[\frac{7}{12}p_3+\frac{35}{48}p_1p_2-\frac{869}{384}p_1^3\right], \tag{3.15}$$

$$c_5 = \frac{1}{5^{n+1}}\begin{bmatrix} \frac{1}{4}p_4-\frac{5}{16}p_1p_3+\frac{39}{128}p_2p_1^2-\frac{5}{32}p_2^2- \\ \frac{141}{2048}p_1^4+5^nd_5-3^{2n}d_3\left(d_3-3c_3\right)+ \\ 2^{3n}d_2\left(d_2^3-2c_4\right)-2^{2n}\left(3^{n+1}c_3d_2^2-2^{n+1}c_3d_4\right) \\ -2^n\cdot 3^n\left(2^{n+2}c_2d_2d_3-3d_2d_3-2\cdot 3^{2n}c_2d_2^3\right) \end{bmatrix}. \tag{3.16}$$

We obtain

$$\left|c_3-\mu c_2^2\right| = \frac{1}{4}\left|\frac{3}{3^{n+1}}p_2-\left[\frac{25\mu}{2^n\cdot 16}-\frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}}-\frac{21}{8}\right)\right]p_1^2\right|. \tag{3.17}$$

Finally, by applying Lemma 2.1 and (2.1) in conjunction with (3.17), we obtain the result asserted by Theorem 3.1. □

Corollary 3.2. *For $n=0$ the above class KL_n^* reduces to KL^*, in (3.1), we have*

$$\left|c_3-\mu c_2^2\right| \le \begin{cases} \frac{1}{48}\left(62-75\mu\right), & \text{if } \mu < \frac{38}{75}, \\ \frac{1}{2}, & \text{if } \frac{38}{75}\le\mu\le\frac{86}{75}, \\ \frac{1}{48}\left(75\mu-62\right), & \text{if } \mu > \frac{86}{75}. \end{cases}$$

It is also asserted that:

(i) For $\mu < \frac{38}{75}$

$$\left|c_3-\mu c_2^2\right|+\frac{1}{3}\left[3\mu-\frac{38}{25}\right]|c_2|^2 \le \frac{1}{2}, \quad \text{if} \quad \frac{38}{75}<\mu\le\frac{62}{75},$$

and

(ii) For $\mu > \frac{86}{75}$

$$\left|c_3-\mu c_2^2\right|+\frac{1}{3}\left[\frac{86}{25}-3\mu\right]|c_2|^2 \le \frac{1}{2}, \quad \text{if} \quad \frac{62}{75}<\mu\le\frac{86}{25},$$

where by (3.13)-(3.16), we obtain

$$\begin{aligned} c_2 &= \frac{5}{8}p_1, \\ c_3 &= \frac{1}{4}p_2+\frac{19}{96}p_1^2, \\ c_4 &= \frac{7}{48}p_3+\frac{35}{192}p_1p_2-\frac{869}{1536}p_1^3, \\ c_5 &= \frac{1}{5}\begin{bmatrix} \frac{1}{4}p_4-\frac{5}{16}p_1p_3+\frac{39}{128}p_2p_1^2-\frac{5}{32}p_2^2-\frac{141}{2048}p_1^4+ \\ \frac{1}{4}\left(p_4-\frac{7}{6}p_1^4+\frac{1}{2}p_2^2-\frac{17}{6}p_2p_1^2-\frac{2}{3}p_1p_3-p_1\right) \\ -\frac{1}{2}\left(p_2+p_1^2\right)\left(\frac{1}{2}\left(p_2+p_1^2\right)-3\left(\frac{1}{4}p_2+\frac{19}{96}p_1^2\right)\right) \\ +p_1\left(p_1^3-\frac{7}{24}p_3-\frac{35}{96}p_1p_2+\frac{869}{768}p_1^3\right) \\ -\left(\frac{3}{4}p_2+\frac{19}{32}p_1^4-\frac{2}{3}\left(\frac{1}{4}p_2+\frac{19}{96}p_1^2\right)\left(p_3+2p_1p_2-p_1^3\right)\right) \\ -\left(\frac{5}{4}p_1^2\left(p_2+p_1^2\right)-\frac{3}{2}p_1\left(p_2+p_1^2\right)-\frac{5}{4}p_1^4\right) \end{bmatrix}, \end{aligned}$$

which is the main result in [32]. Similarly, for $n = 1, 2$, the Salagean differential operator D^n reduces to $D^1\lambda(\omega) = \omega\lambda'(\omega)$ and $D^2\lambda(\omega) = \omega\lambda'(\omega) + \omega^2\lambda''(\omega)$.

Theorem 3.3. *Let $\lambda \in KL_n^*$ be of the form (1.1). Then*

$$\left|c_2c_4 - c_3^2\right| \leq \frac{9}{4}. \tag{3.18}$$

Proof. Using (3.13)-(3.15), and after some simplification, we have

$$\begin{aligned}
c_2c_4 - c_3^2 &= \frac{1}{2^{n+1}}\frac{5}{4}p_1\left[\frac{1}{4^{n+1}}\left(\frac{7}{12}p_3 + \frac{35}{48}p_1p_2 - \frac{869}{384}p_1^3\right)\right] - \\
&\quad \left[\frac{1}{3^{n+1}}\left(\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)p_1^2 + \frac{3}{4}p_2\right)\right]^2 \\
&= \frac{1}{2^{n+1}\cdot 4^{n+1}}\frac{5}{4}p_1\left[\frac{7}{12}p_3 + \frac{35}{48}p_1p_2 - \frac{869}{384}p_1^3\right] \\
&\quad - \left[\frac{1}{3^{2n+2}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)^2 p_1^4 + \frac{9}{16}p_2^2 + 2\frac{1}{3^{n+1}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)\left(\frac{3}{4}p_1^2p_2\right)\right] \\
&= \frac{1}{2^{n+1}\cdot 4^{n+1}}\left[\frac{35}{48}p_1p_3 + \frac{175}{192}p_1^2p_2 - \frac{4345}{1536}p_1^4\right] \\
&\quad - \left[\frac{1}{3^{2n+2}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)^2 p_1^4 + \frac{9}{16}p_2^2 + \frac{1}{3^n}p_1^2p_2\cdot\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)\right].
\end{aligned}$$

Using the values of p_2 and p_3 from Lemma 2.2 along with the triangular inequality and replacing $|x| \leq 1$ by υ and p_1 by p,we have

$$\begin{aligned}
\left|c_2c_4 - c_3^2\right| &\leq \frac{1}{2^{n+1}\cdot 4^{n+1}}\left[\frac{35}{192}p_1\cdot 4p_3 + \frac{175}{384}p_1^2\cdot 2p_2 - \frac{4345}{1536}p_1^4\right] \\
&\quad - \left[\frac{1}{3^{2n+2}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)^2 p_1^4 + \frac{9}{64}(2p_2)^2 + \left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)\left(\frac{1}{2\cdot 3^n}p_1^2\cdot 2p_2\right)\right],
\end{aligned}$$

which implies

$$\begin{aligned}
\left|c_2c_4 - c_3^2\right| &\leq \frac{1}{2^{n+1}.4^{n+1}}\left[\begin{array}{c}\frac{35}{192}\left(p^4 + 8p^2\upsilon - 2p^4\upsilon - 4p^2\upsilon^2 + p^4\upsilon^2 + 8p - 8p\upsilon^2 - 2p^3 + 2p^3\upsilon^2\right) + \\ \frac{175}{384}\left(p^4 + 4p^2\upsilon - p^4\upsilon\right) - \frac{4345}{1536}p^4\end{array}\right] \\
&\quad - \frac{1}{3^{2n+2}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)^2 p^4 - \frac{9}{64}\left(p^4 + 16\upsilon^2 + \upsilon^2p^4 - 8\upsilon^2p^2 + 8p^2\upsilon - 2p^4\upsilon\right) \\
&\quad - \frac{1}{2\cdot 3^n}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)\left(p^4 + 4p^2\upsilon - p^4\upsilon\right) \\
&= F(p, \upsilon).
\end{aligned} \tag{3.19}$$

Differentiating $F(p, \upsilon)$ with respect to υ, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial \upsilon} &= \frac{1}{2^{n+1}\cdot 4^{n+1}}\left[\frac{35}{192}\left(\begin{array}{c}8p^2 - 2p^4 - 8p^2\upsilon + 2p^4\upsilon - \\ 16p\upsilon + 4p^3\upsilon\end{array}\right) + \frac{175}{384}\left(4p^2 - p^4\right)\right] \\
&\quad - \frac{9}{64}\left(32\upsilon + 2\upsilon p^4 - 16\upsilon p^2 + 8p^2 - 2p^4\right) - \frac{1}{2\cdot 3^n}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)\left(4p^2 - p^4\right).
\end{aligned}$$

It is obvious that

$$\frac{\partial F(p,\upsilon)}{\partial \upsilon} > 0,$$

which shows that $F(p,\upsilon)$ is an increasing function on the closed interval $[0,1]$ and the maximum value occurs at $\upsilon = 1$, that is

$$\max\{F(p,\upsilon)\} = F(p,1) = G(p).$$

We have

$$\begin{aligned} G(p) &= \frac{1}{2^{n+1}\cdot 4^{n+1}}\left[\frac{35}{192}4p^2 + \frac{175}{384}4p^2 - \frac{4345}{1536}p^4\right] - \frac{1}{3^{2n+2}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)^2 p^4 \\ &\quad -\frac{9}{4} - \frac{2p^2}{3^n}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right). \end{aligned} \tag{3.20}$$

Differentiating the above relation with respect to p, we have

$$\begin{aligned} G'(p) &= \frac{1}{2^{n+1}\cdot 4^{n+1}}\left[\frac{35}{192}\cdot 8p + \frac{175}{384}\cdot 8p - \frac{4345}{1536}\cdot 4p^3\right] \\ &\quad -\frac{1}{3^{2n+2}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)^2 \cdot 4p^3 - \frac{4p}{3^n}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right). \end{aligned}$$

Differentiating the above relation once again with respect to p, we obtain:

$$G''(p) = \frac{1}{2^{n+1}\cdot 4^{n+1}}\left(\frac{35}{28} + \frac{175}{48} - \frac{4345p^2}{128}\right) - \frac{4p^2}{3^{2n+1}}\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)^2 - \frac{4}{3^n}\left(\frac{10}{2^{2n}} - \frac{21}{4}\right).$$

For $p = 0$, this shows that the maximum value of $G(p)$ occurs at $p = 0$. Hence we obtain (3.18), which completes the proof. □

Theorem 3.4. *Let $\lambda \in KL_n^*$ be of the form (1.1). Then*

$$|c_2c_3 - c_4| \le \frac{7}{4^{n+1}\cdot 6}.$$

Proof. We make use of Equations (3.13)-(3.15), along with Lemma 2.2. Since $p_1 \le 2$, by Lemma 2.3, let $p_1 = p$ and assume without restriction that $p \in [0,2]$. Then, taking the absolute value and applying the triangle inequality with $\upsilon = |x|$, we obtain:

$$\begin{aligned} c_2c_3 - c_4 &= \frac{5}{3^{n+1}\cdot 2^{n+1}\cdot 4}p_1\frac{1}{3^{n+1}}\left[\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)p_1^2 + \frac{3}{4}p_2\right] \\ &\quad -\frac{1}{4^{n+1}}\left[\frac{7}{12}p_3 + \frac{35}{48}p_1p_2 - \frac{869}{384}p_1^3\right] \\ &= \frac{1}{2^{n+1}.3^{n+1}}\left[\left(\frac{5}{2^{2n+2}} - \frac{21}{32}\right)\frac{5}{4}p_1^3 + \frac{15}{16}p_1p_2\right] \\ &\quad -\frac{1}{4^{n+1}}\left[\frac{7}{12}p_3 + \frac{35}{48}p_1p_2 - \frac{869}{384}p_1^3\right]. \end{aligned}$$

Therefore

$$\begin{aligned}|c_2c_3-c_4| &\leq \frac{1}{2^{n+1}\cdot 3^{n+1}}\left[\left(\frac{5}{2^{2n+2}}-\frac{21}{32}\right)\frac{5}{4}p^3+\frac{15}{32}p\left(p^2+4\upsilon-\upsilon p^2\right)\right]\\ &\quad -\frac{1}{4^{n+1}}\left[\begin{array}{c}\frac{7}{48}\left(p^3+8p\upsilon-2p^3\upsilon-4p\upsilon^2+p^3\upsilon^2+8-8\upsilon^2-2p^3+2p^2\upsilon^2\right)\\ +\frac{35}{96}p\left(p^2+4\upsilon-\upsilon p^2\right)-\frac{869}{384}p^3\end{array}\right].\end{aligned}$$

Using the similar procedure in the above theorem, we have

$$\begin{aligned}|c_2c_3-c_4| &\leq \frac{1}{2^{n+1}\cdot 3^{n+1}}\left[\left(\frac{5}{2^{2n+2}}-\frac{21}{32}\right)\frac{5}{4}p^3+\frac{15}{32}\left(p^3+4p\upsilon-\upsilon p^3\right)\right]\\ &\quad -\frac{1}{4^{n+1}}\left[\begin{array}{c}\frac{7}{48}\left(p^3+8p\upsilon-2p^3\upsilon-4p\upsilon^2+p^3\upsilon^2+8-8\upsilon^2-2p^2+2p^2\upsilon^2\right)+\\ \frac{35}{96}\left(p^3+4p\upsilon-\upsilon p^3\right)-\frac{869}{384}p^3\end{array}\right]\\ &= F\left(p,\upsilon\right).\end{aligned}$$

Now differentiating above relation with respect to υ we have

$$\begin{aligned}F^{'}\left(\upsilon\right) &= \frac{1}{2^{n+1}.3^{n+1}}\left(\frac{15}{32}\left(4p-p^3\right)\right)\\ &\quad -\frac{1}{4^{n+1}}\left[\frac{7}{48}\left(\begin{array}{c}8p-2p^3-8p\upsilon+\\ 2p^3\upsilon-16\upsilon+4p^2\upsilon\end{array}\right)+\frac{35}{96}\left(4p-p^3\right)-\frac{869}{384}p^3\right].\end{aligned}$$

For $0\leq\upsilon\leq 1$ and fixed $p\in(0,2)$ it can easily be seen that

$$F^{'}\left(\upsilon\right)\leq 0.$$

This shows that $F\left(p,\upsilon\right)$ is a decreasing function, therefore

$$max\left(p,\upsilon\right)=F\left(p,0\right)=G\left(p\right),$$

where

$$\begin{aligned}G\left(p\right) &= \frac{1}{2^{n+1}\cdot 3^{n+1}}\left[\left(\frac{5}{2^{2n+2}}-\frac{21}{32}\right)\frac{5}{4}p^3+\frac{15}{32}p^3\right]\\ &\quad -\frac{1}{4^{n+1}}\left[\frac{7}{48}\left(p^3+8-2p^2\right)+\frac{35}{96}p^3-\frac{869}{384}p^3\right].\end{aligned}\tag{3.21}$$

Differentiating the above relation with respect to p, we have:

$$\begin{aligned}G^{'}\left(p\right) &= \frac{1}{2^{n+1}\cdot 3^{n+1}}\left[\left(\frac{5}{2^{2n+2}}-\frac{21}{32}\right)\frac{5}{4}\left(3p^2\right)+\frac{15}{32}\left(3p^2\right)\right]\\ &\quad -\frac{1}{4^{n+1}}\left[\frac{7}{48}\left(3p^2-4p\right)+\frac{35}{96}3p^2-\frac{869}{384}\left(3p^2\right)\right].\end{aligned}$$

and also

$$\begin{aligned}G^{''}\left(p\right) &= \frac{1}{2^{n+1}\cdot 3^{n+1}}\left[\left(\frac{5}{2^{2n+2}}-\frac{21}{32}\right)\frac{5}{4}\left(6p\right)+\frac{15}{32}\left(6p\right)\right]\\ &\quad -\frac{1}{4^{n+1}}\left[\frac{7}{48}\left(6p-4\right)-\frac{35}{16}p-\frac{869}{64}p\right]\\ &= G^{''}\left(p\right)\leq 0\end{aligned}$$

for $p=0$. Thus clearly, $p=0$ is the point of maximum. Hence we get the required result asserted by the above theorem. □

Theorem 3.5. *If a function λ of the form of (1.1) and (3.13)-(3.16) is in the class KL_n^*, then using (1.2) we have*

$$|c_2| \leq \frac{5}{2^{n+2}}, \tag{3.22}$$

$$|c_3| \leq \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}}-\frac{9}{8}\right), \tag{3.23}$$

$$|c_4| \leq \frac{673}{4^{n+1}\cdot 48}, \tag{3.24}$$

$$\begin{aligned} |c_5| &\leq \frac{1}{5^{n+1}}\left|\begin{array}{c} \frac{1}{2}-\frac{5}{4}+\frac{39}{16}-\frac{5}{8}-\frac{141}{128}+1-9+ \\ 4\left(\frac{2^n}{3}-\frac{1}{2^{n+1}}\left(3^n+3^n\cdot 2^n+3.2^n\right)+1\right)-\frac{1}{2^n}\left(3^n+3^n\cdot 2^n+3.2^n\right) \\ -\frac{2^{3n}}{2}+3\left(\frac{5}{2^{2n}}-\frac{9}{8}\right)+2^{2n+1}\left(\frac{8}{2^{3n}}+\frac{673}{4^{n+1}.24}\right) \\ -\frac{20}{2^{2n}}+\frac{34}{8}-\frac{2^{n+2}}{3^{n+2}.4^n}\left(\frac{5}{2^{2n}}-\frac{9}{8}\right)-30+18+\frac{3^{3n}.20}{2^{3n}} \end{array}\right| \\ &= \frac{1}{5^{n+1}}\left|\begin{array}{c} -\frac{1941}{128}+\frac{4.2^n}{3}-\frac{3}{2^n}\left(3^n+3^n\cdot 2^n+3.2^n\right)-\frac{2^{3n}}{2}+\frac{15}{2^{2n}}+ \\ 2^{2n+1}\left(\frac{8}{2^{3n}}+\frac{673}{4^{n+1}.24}\right)-\frac{20}{2^{2n}}-\frac{2^{n+2}}{3^{n+2}.4^n}\left(\frac{5}{2^{2n}}-\frac{9}{8}\right)+\frac{3^{3n}.20}{2^{3n}} \end{array}\right|. \end{aligned} \tag{3.25}$$

These estimates are sharp.

Proof. The proof of Theorem 3.5 is similar to that of a known result which was proved by Sokol (see [30]). Therefore, we here choose to omit the details involved in the proof of Theorem 3.5. □

Corollary 3.6. *(see [32]) For $n=0$ the above class KL_n^* reduces to KL^*, and (3.22)-(3.25) reduce to*

$$\begin{aligned} |c_2| &\leq \frac{5}{4}, \\ |c_3| &\leq \frac{31}{24}, \\ |c_4| &\leq \frac{673}{192}, \\ |c_5| &\leq \frac{1}{5}\left[-\frac{1941}{128}+\frac{4}{3}-\frac{1}{2}+2\left(8+\frac{673}{24}\right)-\frac{4}{9}\left(5-\frac{9}{8}\right)\right]=\frac{899}{64}. \end{aligned}$$

Theorem 3.7. *Let $\lambda \in KL_n^*$ be of the form of (1.1). Then*

$$\begin{aligned} H_3(1) \leq\ & \frac{3}{4\cdot 3^n}\left(\frac{5}{2^{2n}}-\frac{9}{8}\right)+\frac{4711}{4^{2n+2}\cdot 288} \\ & +\frac{1}{5^{n+1}}\left|\begin{array}{c} -\frac{1941}{128}+\frac{4\cdot 2^n}{3}-\frac{3}{2^n}\left(3^n+3^n\cdot 2^n+3\cdot 2^n\right)-\frac{2^{3n}}{2}+\frac{15}{2^{2n}}+ \\ 2^{2n+1}\left(\frac{8}{2^{3n}}+\frac{673}{4^{n+1}\cdot 24}\right)-\frac{20}{2^{2n}}-\frac{2^{n+2}}{3^{n+2}\cdot 4^n}\left(\frac{5}{2^{2n}}-\frac{9}{8}\right)+\frac{3^{3n}\cdot 20}{2^{3n}} \end{array}\right| \\ & \cdot\left|\frac{2}{2\cdot 3^n}-\frac{25}{2^{n+4}}+\frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}}-\frac{21}{8}\right)\right|. \end{aligned} \tag{3.26}$$

Proof. Since

$$|H_3(1)| \leq |c_3|\left|\left(c_2c_4-c_3^2\right)\right|+|c_4|\left|\left(c_2c_3-c_4\right)\right|+|c_5|\left|\left(c_1c_3-c_2^2\right)\right|, \tag{3.27}$$

by Theorem 3.3, we have

$$\left|c_2c_4 - c_3^2\right| \leq \frac{9}{4}. \tag{3.28}$$

Also by Theorem 3.4, we obtain

$$|c_2c_3 - c_4| \leq \frac{7}{4^{n+1}\cdot 6}. \tag{3.29}$$

In addition using the fact that $c_1 = 1$ and Theorem 3.1 with $\mu = 1$, we have

$$\left|c_3 - c_2^2\right| = \left|\frac{2}{2\cdot 3^n} - \left[\frac{25}{2^{n+4}} - \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right]\right|. \tag{3.30}$$

Combining (3.23)-(3.25) and (3.28)-(3.30), in conjunction with equation (3.27), we obtain

$$\begin{aligned}|H_3(1)| \leq\ & \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{9}{8}\right)\frac{9}{4} + \frac{673}{4^{n+1}\cdot 48}\frac{7}{4^{n+1}\cdot 6} \\ & + \frac{1}{5^{n+1}}\left[\begin{array}{l} -\frac{1941}{128} + \frac{4\cdot 2^n}{3} - \frac{3}{2^n}\left(3^n + 3^n\cdot 2^n + 3\cdot 2^n\right) - \frac{2^{3n}}{2} + \frac{15}{2^{2n}} + \\ 2^{2n+1}\left(\frac{8}{2^{3n}} + \frac{673}{4^{n+1}\cdot 24}\right) - \frac{20}{2^{2n}} - \frac{2^{n+2}}{3^{n+2}\cdot 4^n}\left(\frac{5}{2^{2n}} - \frac{9}{8}\right) + \frac{3^{3n}\cdot 20}{2^{3n}} \end{array}\right] \\ & \cdot\frac{2}{2\cdot 3^n} - \left[\frac{25}{2^{n+4}} - \frac{1}{3^{n+1}}\left(\frac{5}{2^{2n}} - \frac{21}{8}\right)\right].\end{aligned}$$

Thus we get the desired result of Theorem 3.7. □

4 Conclusions

In this article, using the concept of subordination, we studied Hankel determinant $H_{3,1}(\lambda)$ for a new class of analytic functions and modified the existing results by employing the Salagean operator and lemniscate of Bernoulli. We also presented many special cases.

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