

On uniform dichotomy in mean of stochastic skew-evolution semiflows in Banach spaces

Tímea Melinda Személy Fülöp

Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary

Abstract In this paper we consider three concepts of uniform dichotomy in mean for stochastic skew-evolution semiflows: uniform exponential dichotomy in mean, uniform polynomial dichotomy in mean and uniform h -dichotomy in mean. Some characterizations of these notions and connections between these concepts are given. The obtained results can be considered generalizations to the dichotomy case of the results from [29].

AMS Subject Classification (2020) 34D05; 93D20; 37L55

Keywords stochastic skew-evolution semiflows; growth rate; uniform h -dichotomy in mean

1 Introduction

The concept of dichotomy was introduced by O. Perron (see [20]), while studying the connection between the conditional stability of an equation $\dot{x}(t) = A(t)x$ and the existence of bounded solutions of the equation $\dot{x}(t) = A(t)x + f(t, x)$. The property of exponential dichotomy for linear differential equations has gained importance since the appearance of two fundamental monographs due to J. L. Massera & J. J. Schäffer in 1966 (see [14]) and J. L. Daleckii & M.G. Krein in 1974 (see [9]).

The exponential dichotomy in a stochastic setting was discussed by many authors, such as A. M. Ateiwi in [2] or T. Caraballo et al. in [7].

The notion of skew-evolution semiflow introduced by M. Megan and C. Stoica in [18], is a generalization of the notions of: evolution operators, semigroups of operators and skew-product semiflows (see [1, 8, 10, 16, 17, 21]).

The asymptotic behaviours of evolution cocycles have been treated from various perspectives by many authors and we emphasize the papers [8, 10, 15–17, 19, 24, 25].

Another line of research regarding the topics on dichotomies is represented by the relationship between dichotomy and admissibility. Some of the most recent results in this direction were obtained by D. Dragičević, A. L. Sasu and B. Sasu (see [11]) who give new admissibility criteria for polynomial dichotomies of discrete nonautonomous systems.

For the study of the asymptotic behaviours in mean of stochastic skew-evolution semiflows we mention the works of D. Stoica (see [26–28]) for uniform exponential stability in mean, uniform exponential instability in mean and P. V. Hai (see [13]) for uniform polynomial stability in mean and uniform polynomial instability in mean.

A general framework for the property of splitting that includes, as a particular case, the dichotomy is studied by C. Stoica in [23] and R. Boruga and M. Megan (see [6]) give in their work some characterizations for the uniform exponential dichotomy respectively uniform polynomial dichotomy using new techniques to prove the theorems in polynomial case.

The main aim of this paper is to give characterization theorems for the concept of uniform dichotomy in mean of stochastic skew-evolution semiflows in Banach spaces.

The paper is organized as follows. In Section 2, we recall some basic notions and the concept of a uniform dichotomy in mean for stochastic skew-evolution semiflows. In Section 3 we prove some characterizations of uniform exponential dichotomy in mean. In Section 4 we consider the concept of uniform h -dichotomy in mean, where $h : \mathbb{R}_+ \rightarrow [1, \infty)$ is a growth rate (more precisely h is a nondecreasing and bijective function with $\lim_{t \rightarrow \infty} h(t) = \infty$) for stochastic skew-evolution semiflows in Banach spaces and we prove some characterizations of uniform dichotomy with growth rates and also we present the connection between uniform exponential dichotomy in mean and uniform h -dichotomy in mean. In Section 5 we use the obtained results from Section 2 to prove the characterization theorems for uniform polynomial dichotomy in mean.

2 Preliminaries

Let $(\Omega, \mathcal{B}, \mathcal{P})$ be a probability space. We denote by $\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s \geq 0\}$ and $T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$. We also denote by X a real or complex Banach space and by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X .

Definition 2.1. A measurable random field $\varphi : \Delta \times \Omega \rightarrow \Omega$ is called a stochastic evolution semiflow on Ω if

$$(s1) \quad \varphi(t, t, \omega) = \omega \text{ for all } (t, \omega) \in \mathbb{R}_+ \times \Omega,$$

$$(s2) \quad \varphi(t, s, \varphi(s, t_0, \omega)) = \varphi(t, t_0, \omega) \text{ for all } t \geq s \geq t_0 \geq 0 \text{ and all } \omega \in \Omega.$$

Definition 2.2. A map $\Phi : \Delta \times \Omega \rightarrow \mathcal{B}(X)$ is called a stochastic evolution cocycle associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \rightarrow \Omega$ if

$$(c1) \quad \Phi(t, t, \omega) = I \text{ (the identity operator on } X) \text{ for all } (t, \omega) \in \mathbb{R}_+ \times \Omega,$$

$$(c2) \quad \Phi(t, s, \varphi(s, t_0, \omega))\Phi(s, t_0, \omega) = \Phi(t, t_0, \omega) \text{ for all } t \geq s \geq t_0 \geq 0 \text{ and all } \omega \in \Omega.$$

If Φ is a stochastic evolution cocycle over a stochastic evolution semiflow φ , then the pair $C = (\Phi, \varphi)$ is called a *stochastic skew-evolution semiflow*.

Example 2.1. If $\Phi : \Delta \times \Omega \rightarrow \mathcal{B}(X)$ is a stochastic evolution cocycle associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \rightarrow \Omega$ and $h : \mathbb{R}_+ \rightarrow [1, \infty)$ is a growth rate (i.e. h is bijective and nondecreasing), then

$$\varphi_h : \Delta \times \Omega \rightarrow \Omega, \quad \varphi_h(t, s, \omega) = \varphi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

is a stochastic evolution semiflow on Ω , and

$$\Phi_h : \Delta \times \Omega \rightarrow \mathcal{B}(X), \quad \Phi_h(t, s, \omega) = \Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

is a stochastic evolution cocycle associated to the stochastic evolution semiflow φ_h .

Example 2.2. If $\Phi : \Delta \times \Omega \rightarrow \mathcal{B}(X)$ is a stochastic evolution cocycle associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \rightarrow \Omega$ and $h : \mathbb{R}_+ \rightarrow [1, \infty)$ is a growth rate (see Example 2.1), then

$$\psi_h : \Delta \times \Omega \rightarrow \Omega, \quad \psi_h(t, s, \omega) = \varphi(h^{-1}(t+1), h^{-1}(s+1), \omega)$$

is a stochastic evolution semiflow on Ω , and

$$\Psi_h : \Delta \times \Omega \rightarrow \mathcal{B}(X), \quad \Psi_h(t, s, \omega) = \Phi(h^{-1}(t+1), h^{-1}(s+1), \omega)$$

is a stochastic evolution cocycle associated to the stochastic evolution semiflow ψ_h .

Remark 2.1. Interesting examples of stochastic skew-evolution semiflows are given in [13], [28] and [29].

Definition 2.3. The mapping $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ is said to be a family of projectors if

$$P^2(s, \omega) = P(s, \omega),$$

for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$.

Remark 2.2. If $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ is a family of projectors for $C = (\Phi, \varphi)$, then $Q : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ defined by

$$Q(s, \omega) = I - P(s, \omega)$$

is also a family of projectors, called the *complementary family of projectors* P .

Definition 2.4. The family of projectors $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ is invariant to the stochastic skew-evolution semiflow $C = (\Phi, \varphi)$ if

$$\Phi(t, s, \omega)P(s, \omega) = P(t, \varphi(t, s, \omega))\Phi(t, s, \omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$.

Example 2.3. If $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ is a family of projectors invariant to $C = (\Phi, \varphi)$, then $P_h : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$, where $P_h(s, \omega) = P(h^{-1}(e^s), \omega)$ is a family of projectors invariant to $C = (\Phi_h, \varphi_h)$, and

$$\begin{aligned} P_h(s, \omega) \cdot P_h(s, \omega) &= P(h^{-1}(e^s), \omega) \cdot P(h^{-1}(e^s), \omega) \\ &= P(h^{-1}(e^s), \omega) \\ &= P_h(s, \omega). \end{aligned}$$

Example 2.4. If $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ is a family of projectors invariant to $C = (\Psi, \psi)$, then $P_h : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$, $P_h(s, \omega) = P(h^{-1}(t+1), \omega)$ is a family of projectors invariant to $C = (\Psi_h, \psi_h)$, and

$$\begin{aligned} P_h(s, \omega) \cdot P_h(s, \omega) &= P(h^{-1}(s+1), \omega) \cdot P(h^{-1}(s+1), \omega) \\ &= P(h^{-1}(s+1), \omega) \\ &= P_h(s, \omega). \end{aligned}$$

We denote by $\mathcal{L}(\Omega, X, P)$ the Banach space of all Bochner measurable functions $f : \Omega \rightarrow X$ with

$$E(f) = \int_{\Omega} \|f(\omega)\| dP(\omega) < \infty.$$

where $E(f)$ represents the mean of stochastic process f and $\mathcal{P}$ is the probability measure.

3 Characterizations for uniform exponential dichotomy in mean

Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow on $\Delta \times \Omega$ and $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ be a family of projectors invariant to $C = (\Phi, \varphi)$. Then, the pair (C, P) is called a *dichotomic pair*.

Definition 3.1. The pair (C, P) is called *uniformly exponentially dichotomic in mean (u.e.d.m.)* if there exist $N > 1$ and $\nu > 0$ with

$$(ued_1 m) \quad E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq Ne^{-\nu(t-s)}E\|P(s, \omega)x\|,$$

$$(ued_2 m) \quad E\|Q(s, \omega)x\| \leq Ne^{\nu(t-s)}E\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Definition 3.2. The pair (C, P) has *uniform exponential growth in mean (u.e.g.m.)* if there exist $M \geq 1$ and $\alpha > 0$ with

$$(ueg_1 m) \quad E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq Me^{\alpha(t-s)}E\|P(s, \omega)x\|,$$

$$(ueg_2 m) \quad ME\|\Phi(t, s, \omega)Q(s, \omega)x\| \leq e^{-\alpha(t-s)}E\|Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Theorem 3.1. *If the pair (C, P) has uniform exponential growth in mean (u.e.g.m.) then the pair (C, P) is uniformly exponentially dichotomic in mean (u.e.d.m.) if and only if there exist $r > 1$ and $c \in (0, 1)$ with*

$$(ueH_1m) \quad E\|\Phi(r+s, s, \omega)P(s, \omega)x\| \leq cE\|P(s, \omega)x\|,$$

$$(ueH_2m) \quad E\|Q(s, \omega)x\| \leq cE\|\Phi(r+s, s, \omega)Q(s, \omega)x\|,$$

for all $s \geq 0$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. Necessity. If (C, P) is u.e.d.m. then there are constants $N > 1$ and $\nu > 0$ with $r \stackrel{def}{=} 1 + \left\lceil \frac{\ln N}{\nu} \right\rceil$ where $\lceil \cdot \rceil$ denotes the integer part and $c \stackrel{def}{=} Ne^{-\nu r} \in (0, 1)$ because $e^{\nu r} > e^{\ln N} = N$.

From (ued_1m) we obtain

$$E\|\Phi(r+s, s, \omega)P(s, \omega)x\| \leq Ne^{-\nu r}E\|P(s, \omega)x\| = c \cdot E\|P(s, \omega)x\|.$$

From (ued_2m) , we have

$$c \cdot E\|\Phi(r+s, s, \omega)Q(s, \omega)x\| \geq Ne^{-\nu r} \frac{e^{(r+s)\nu}}{Ne^{\nu s}} E\|Q(s, \omega)x\| = E\|Q(s, \omega)x\|.$$

Sufficiency. We suppose that there exist $r > 1$ and $c \in (0, 1)$ such that (ueH_1m) and (ueH_2m) are satisfied for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ and $x \in L(\Omega, X, \mathcal{P})$. If $(t, s) \in \Delta$ then there exist $n \in \mathbb{N}$ and $\delta \in [0, r)$ such that $t - s = nr + \delta$.

We have

$$\begin{aligned} E\|\Phi(t, s, \omega)P(s, \omega)x\| &= E\|\Phi(t, s + nr, \varphi(s + nr, s, \omega))\Phi(s + nr, s, \omega)P(s, \omega)x\| \\ &= E\|\Phi(t, s + nr, \varphi(s + nr, s, \omega))P(s + nr, \varphi(s + nr, s, \omega))\Phi(s + nr, s, \omega)P(s, \omega)x\| \\ &\leq Me^{\alpha r} E\|P(s + nr, \varphi(s + nr, s, \omega))\Phi(s + nr, s, \omega)P(s, \omega)x\| \\ &= Me^{\alpha r} E\|\Phi(s + nr, s, \omega)P(s, \omega)x\| \\ &= Me^{\alpha r} E\|\Phi(s + nr, s + (n-1)r, \varphi(s + (n-1)r, s, \omega))\Phi(s + (n-1)r, s, \omega)P(s, \omega)x\| \\ &\leq Me^{\alpha r} cE\|\Phi(s + (n-1)r, s, \omega)P(s, \omega)x\| \\ &\leq \dots \leq Me^{\alpha r} c^n E\|P(s, \omega)x\| \\ &= Me^{\alpha r} e^{n \ln c} E\|P(s, \omega)x\| \\ &= Me^{\alpha r} e^{\frac{t-s+\delta}{r} \ln c} E\|P(s, \omega)x\| \\ &= M \frac{e^{\alpha r}}{c^{\frac{\delta}{r}}} e^{-\nu(t-s)} E\|P(s, \omega)x\| \\ &= Ne^{-\nu(t-s)} E\|P(s, \omega)x\|. \end{aligned}$$

Making a similar computation we have

$$\begin{aligned} E\|\Phi(t, s, \omega)Q(s, \omega)x\| &= \\ &= E\|\Phi(t, s + nr, \varphi(s + nr, s, \omega))Q(s + nr, \varphi(s + nr, s, \omega))\Phi(s + nr, s, \omega)Q(s, \omega)x\| \\ &\geq \frac{1}{M} e^{-\alpha r} E\|Q(s, \omega)x\| \\ &\geq \frac{1}{Mce^{\alpha r}} E\|\Phi(s + (n-1)r, s, \omega)Q(s, \omega)x\| \end{aligned}$$

$$\begin{aligned}
&\geq \cdots \geq \frac{1}{M} \cdot \frac{1}{e^{\alpha r} e^{n \ln c}} E \|Q(s, \omega)x\| \\
&= \frac{1}{M} \cdot \frac{1}{e^{\alpha r} e^{\frac{t-s-\delta}{r} \ln c}} E \|Q(s, \omega)x\| \\
&= \frac{1}{M} \cdot \frac{1}{e^{\alpha r - \frac{\delta \ln c}{r}}} \cdot \frac{1}{e^{(t-s) \frac{\ln c}{r}}} E \|Q(s, \omega)x\| \\
&= \frac{1}{N} \cdot \frac{1}{\frac{e^{\alpha r}}{c^{\frac{\delta}{r}}}} e^{-(t-s) \frac{\ln c}{r}} E \|Q(s, \omega)x\| \\
&= \frac{1}{N} e^{\nu(t-s)} E \|Q(s, \omega)x\|,
\end{aligned}$$

where $N = \frac{M e^{\alpha r}}{c^{\frac{\delta}{r}}} > 1$ and $\nu = -\frac{\ln c}{r}$, for all $s \geq 0$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

It follows that the pair (C, P) is uniformly exponentially dichotomic in mean, so the proof is complete. $\square$

Theorem 3.2. *If the pair (C, P) has uniform exponential growth in mean (u.e.g.m.) then the pair (C, P) is uniformly exponentially dichotomic in mean (u.e.d.m.) if and only if there exist $L > 1$ with*

$$(uel_1m) \quad (t-s)E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq LE\|P(s, \omega)x\|,$$

$$(uel_2m) \quad (t-s)E\|Q(s, \omega)x\| \leq LE\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. Necessity. Using the inequality $\frac{u}{e^{\nu u}} \leq \frac{1}{\nu e}$ for all $u \geq 1$ and (ued_1m) we have that

$$\begin{aligned}
(t-s)E\|\Phi(t, s, \omega)P(s, \omega)x\| &\leq (t-s)Ne^{-\nu(t-s)}E\|P(s, \omega)x\| \\
&= \frac{N(t-s)}{e^{\nu(t-s)}}E\|P(s, \omega)x\| \\
&\leq \frac{N}{\nu e}E\|P(s, \omega)x\| \\
&\leq LE\|P(s, \omega)x\|
\end{aligned}$$

where $L = 1 + \frac{N}{\nu e}$.

Similarly, from (ued_2m) we obtain

$$\begin{aligned}
(t-s)E\|Q(s, \omega)x\| &\leq (t-s)Ne^{-\nu(t-s)}E\|\Phi(t, s, \omega)Q(s, \omega)x\| \\
&= \frac{N(t-s)}{e^{\nu(t-s)}}E\|\Phi(t, s, \omega)Q(s, \omega)x\| \\
&\leq \frac{N}{\nu e}E\|\Phi(t, s, \omega)Q(s, \omega)x\| \\
&\leq LE\|\Phi(t, s, \omega)Q(s, \omega)x\|
\end{aligned}$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Sufficiency. Let $c = \frac{L}{r}$ and $r = L + 1$.

By (uel_1m) we have

$$E\|\Phi(r+s, s, \omega)P(s, \omega)x\| \leq \frac{L}{r}E\|P(s, \omega)x\| = cE\|P(s, \omega)x\|$$

which implies (ueH_1m) .

Respectively, by (uel_2m) we have

$$c \cdot E\|\Phi(r+s, s, \omega)Q(s, \omega)x\| = \frac{L}{r}E\|\Phi(r+s, s, \omega)Q(s, \omega)x\| \geq c \cdot E\|Q(s, \omega)x\|$$

which implies (ueH_2m) . □

Theorem 3.3. *If the pair (C, P) has uniform exponential growth in mean (u.e.g.m.) then the pair (C, P) is uniformly exponentially dichotomic in mean (u.e.d.m.) if and only if there exists $g : [1, \infty) \rightarrow \mathbb{R}_+$ with $\lim_{t \rightarrow \infty} g(t) = \infty$ and*

$$(ueM_1m) \quad g(t-s)E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq E\|P(s, \omega)x\|,$$

$$(ueM_2m) \quad g(t-s)E\|Q(s, \omega)x\| \leq E\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. Necessity. It follows from Theorem 3.2 for $g(t) = \frac{s}{L}$.

Sufficiency. By (ueM_1m) we have that

$$E\|\Phi(r+s, s, \omega)P(s, \omega)x\| \leq \frac{1}{g(r)}E\|P(s, \omega)x\| \leq c \cdot E\|P(s, \omega)x\|$$

where $c = \frac{1}{g(r)}$. Thus (ueH_1m) is satisfied.

Similarly, by (ueM_2m) we obtain

$$c \cdot E\|\Phi(r+s, s, \omega)Q(s, \omega)x\| \geq c \cdot g(r)E\|Q(s, \omega)x\| = c \cdot E\|Q(s, \omega)x\|$$

which implies (ueH_2m) , for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$. □

Remark 3.1. Theorem 3.1, Theorem 3.2 and Theorem 3.3 are generalizations of some results presented by C. Stoica in [22] but also A. Găină, M. Megan and C. F. Popa in [12] present the variant of these theorems for discrete-time case when Φ is an evolution cocycle on a Banach space.

4 Characterizations for uniform h -dichotomy in mean

Let $\Phi : \Delta \times \Omega \rightarrow \mathcal{B}(X)$ be a stochastic evolution cocycle associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \rightarrow \Omega$ and let $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ be a family of projectors invariant to $C = (\Phi, \varphi)$. Let $h : \mathbb{R}_+ \rightarrow [1, \infty)$ be a growth rate (i.e. h is bijective and nondecreasing).

Definition 4.1. The pair (C, P) is called uniformly h -dichotomic in mean (u.h.d.m.) if there exist $N > 1$ and $\nu > 0$ with

$$(uhd_1m) \quad h(t)^\nu E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq Nh(s)^\nu E\|P(s, \omega)x\|,$$

$$(uhd_2m) \quad h(t)^\nu E\|Q(s, \omega)x\| \leq Nh(s)^\nu E\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathbb{P})$.

Definition 4.2. The pair (C, P) has uniform h -growth in mean (u.h.g.m.) if there exist $M \geq 1$ and $\alpha > 0$ with

$$(uhg_1m) \quad h(s)^\alpha E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq Mh(t)^\alpha E\|P(s, \omega)x\|,$$

$$(uhg_2m) \quad Mh(t)^\alpha E\|\Phi(t, s, \omega)Q(s, \omega)x\| \geq h(s)^\alpha E\|Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathbb{P})$.

The next theorem presents the connection between uniform h -dichotomy in mean and uniform exponential dichotomy in mean and extends the results obtained by T. M. Személyi Fülöp, M. Megan and D. I. Borlea (Pătraşcu) in [29] for uniform stability in mean to uniform dichotomy in mean.

Theorem 4.1. The dichotomic pair (C, P) is uniformly h -dichotomic in mean if and only if the pair (C_h, P_h) where

$$\Phi_h : \Delta \times \Omega \rightarrow \mathcal{B}(X), \quad \Phi_h(t, s, \omega) \stackrel{\text{def}}{=} \Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

$$\varphi_h : \Delta \times \Omega \rightarrow \mathcal{B}(X), \quad \varphi_h(t, s, \omega) \stackrel{\text{def}}{=} \varphi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

$$P_h : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X), \quad P_h(s, \omega) \stackrel{\text{def}}{=} P(h^{-1}(e^s), \omega)$$

is uniformly exponentially dichotomic in mean.

Proof. Necessity. If (C, P) is uniformly h -dichotomic in mean and $(t, s, \omega) \in \Delta \times \Omega$ then there exist the constants $N > 1$ și $\nu > 0$ such that

$$\begin{aligned} e^{\nu t} E\|\Phi_h(t, s, \omega)P_h(s, \omega)x\| &= e^{\nu t} E\|\Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)P(h^{-1}(e^s), \omega)x\| \\ &\leq e^{\nu t} N \left(\frac{h(h^{-1}(e^s))}{h(h^{-1}(e^t))} \right)^\nu E\|P(h^{-1}(e^s), \omega)x\| \\ &= e^{\nu t} N \cdot \frac{e^{\nu s}}{e^{\nu t}} E\|P(h^{-1}(e^s), \omega)x\| \\ &= N \cdot e^{\nu s} E\|P(h^{-1}(e^s), \omega)x\| \\ &= N \cdot e^{\nu s} E\|P_h(s, \omega)x\|, \end{aligned}$$

$$\begin{aligned} e^{\nu t} E\|Q_h(s, \omega)x\| &= e^{\nu t} E\|Q(h^{-1}(e^s), \omega)x\| \\ &\leq e^{\nu t} N \left(\frac{h(h^{-1}(e^s))}{h(h^{-1}(e^t))} \right)^\nu E\|\Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)Q(h^{-1}(e^s), \omega)x\| \\ &= e^{\nu t} N \cdot \frac{e^{\nu s}}{e^{\nu t}} E\|\Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)Q(h^{-1}(e^s), \omega)x\| \\ &= N \cdot e^{\nu s} E\|\Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)Q(h^{-1}(e^s), \omega)x\| \\ &= N \cdot e^{\nu s} E\|\Phi_h(t, s, \omega)Q_h(s, \omega)x\|, \end{aligned}$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

It follows that (C_h, P_h) is uniformly exponentially dichotomic in mean, so the necessity is proved.

Sufficiency. We suppose that (C_h, P_h) is uniformly exponentially dichotomic in mean and $(t, s, \omega) \in \Delta \times \Omega$. Let $N > 1$ and $\nu > 0$. Then for (uhd_1m) we have

$$\begin{aligned} h(t)^\nu E\|\Phi(t, s, \omega)P(s, \omega)x\| &= h(t)^\nu E\|\Phi(h^{-1}(h(t)), h^{-1}(h(s)), \omega)P(h^{-1}(h(t)))x\| \\ &= h(t)^\nu E\|\Phi(h^{-1}(e^{\ln h(t)}), h^{-1}(e^{\ln h(s)}), \omega)P(h^{-1}(e^{\ln h(t)}))x\| \\ &\leq Nh(t)^\nu e^{-\nu(\ln h(t) - \ln h(s))} E\|P(s, \omega)x\| \\ &= Nh(s)^\nu E\|P(s, \omega)x\|. \end{aligned}$$

Also, for (uhd_2m) via similar computation we obtain

$$\begin{aligned} h(t)^\nu E\|Q(s, \omega)x\| &= h(t)^\nu E\|Q(h^{-1}(h(t)))x\| \\ &= h(t)^\nu E\|Q(h^{-1}(e^{\ln h(t)}))x\| \\ &\leq Nh(t)^\nu e^{\nu(\ln h(t) - \ln h(s))} E\|\Phi(t, s, \omega)Q(s, \omega)x\| \\ &= Nh(s)^\nu E\|\Phi(t, s, \omega)Q(s, \omega)x\|, \end{aligned}$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$. It follows that (C, P) is uniformly h -dichotomic in mean, so the proof is complete. $\square$

Theorem 4.2. *If the pair (C, P) has uniform h -growth in mean (u.h.g.m.) then the pair (C, P) is uniformly h dichotomic in mean (u.h.d.m.) if and only if there exist $r > e$ and $c \in (0, 1)$ with*

$$(uhH_1m) \quad E\|\Phi(h^{-1}(rs), h^{-1}(s), \omega)P(h^{-1}(s), \omega)x\| \leq cE\|P(h^{-1}(s), \omega)x\|,$$

$$(uhH_2m) \quad E\|Q(h^{-1}(s), \omega)x\| \leq cE\|\Phi(h^{-1}(rs), h^{-1}(s), \omega)Q(h^{-1}(s), \omega)x\|,$$

for all $s \geq 0$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. It follows from Theorem 3.1 and Theorem 4.1. $\square$

Theorem 4.3. *If the pair (C, P) has uniform h -growth in mean (u.h.g.m.) then the pair (C, P) is uniformly h dichotomic in mean (u.h.d.m.) if and only if there exists $L > 1$ with*

$$(uhl_1m) \quad E\|\Phi(t, s, \omega)P(s, \omega)x\| \ln \frac{h(t)}{h(s)} \leq LE\|P(s, \omega)x\|,$$

$$(uhl_2m) \quad E\|Q(s, \omega)x\| \ln \frac{h(t)}{h(s)} \leq LE\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. It follows from Theorem 3.2 and Theorem 4.1. $\square$

Theorem 4.4. *If the pair (C, P) has uniform h -growth in mean (u.h.g.m.) then the pair (C, P) is uniformly h dichotomic in mean (u.h.d.m.) if and only if there exists $g : [1, \infty) \rightarrow \mathbb{R}_+$ with $\lim_{t \rightarrow \infty} g(t) = \infty$, $g(e) = 1$ and there exists $L > 1$ such that:*

$$(uhM_1m) \quad g\left(\frac{h(t)}{h(s)}\right) E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq LE\|P(s, \omega)x\|,$$

$$(uhM_2m) \quad g\left(\frac{h(t)}{h(s)}\right) E\|Q(s, \omega)x\| \leq LE\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. It follows from Theorem 3.3 and Theorem 4.1. $\square$

Remark 4.1. Theorem 4.2, Theorem 4.3 and Theorem 4.4 are generalizations of some results presented by R. Boruga (Toma), D. I. Borlea (Pătraşcu) and D. M. M. Toth in [5] and extend the results obtained by T. M. Személyi Fülöp, M. Megan and D. I. Borlea (Pătraşcu) in [29] for uniform stability in mean to uniform dichotomy in mean.

5 Characterizations for uniform polynomial dichotomy in mean

Let $\Phi : \Delta \times \Omega \rightarrow \mathcal{B}(X)$ be a stochastic evolution cocycle associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \rightarrow \Omega$ and let $P : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X)$ be a family of projectors invariant to $C = (\Phi, \varphi)$.

Definition 5.1. The pair (C, P) is called *uniformly polynomially dichotomic in mean (u.p.d.m.)* if there exist $N > 1$ and $\nu > 0$ with

$$(upd_1m) \quad (t+1)^\nu E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq N(s+1)^\nu E\|P(s, \omega)x\|,$$

$$(upd_2m) \quad (t+1)^\nu E\|Q(s, \omega)x\| \leq N(s+1)^\nu E\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Definition 5.2. The pair (C, P) has *uniform h -growth in mean (u.p.g.m.)* if there exist $M \geq 1$ and $\alpha > 0$ with

$$(upg_1m) \quad (s+1)^\alpha E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq M(t+1)^\alpha E\|P(s, \omega)x\|,$$

$$(upg_2m) \quad M(t+1)^\alpha E\|\Phi(t, s, \omega)Q(s, \omega)x\| \geq (s+1)^\alpha E\|Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

The next theorem presents the connection between uniform polynomial dichotomy in mean and uniform exponential dichotomy in mean.

Theorem 5.1. The dichotomic pair (C, P) is *uniformly polynomially dichotomic in mean* if and only if the dichotomic pair (C_1, P_1) where $C_1 = (\Phi_1, \varphi_1)$ with

$$\Phi_1 : \Delta \times \Omega \rightarrow \mathcal{B}(X), \quad \Phi_1(t, s, \omega) \stackrel{def}{=} \Phi(e^t - 1, e^s - 1, \omega)$$

$$\varphi_1 : \Delta \times \Omega \rightarrow \mathcal{B}(X), \quad \varphi_1(t, s, \omega) \stackrel{def}{=} \varphi(e^t - 1, e^s - 1, \omega)$$

and

$$P_1 : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{B}(X), \quad P_1(s, \omega) \stackrel{def}{=} P(e^s - 1, \omega)$$

is *uniformly exponentially dichotomic in mean*.

Proof. The proof follows immediately from Theorem 4.1 if we take $h(t) = t + 1$. $\square$

Theorem 5.2. *If the pair (C, P) has uniform polynomial growth in mean (u.p.g.m.) then the pair (C, P) is uniformly polynomially dichotomic in mean (u.p.d.m.) if and only if there exist $r \in \mathbb{R}^*$ and $c \in (0, 1)$ with*

$$(upH_1m) \quad E\|\Phi(rs, s, \omega)P(s, \omega)x\| \leq cE\|P(s, \omega)x\|,$$

$$(upH_2m) \quad E\|Q(s, \omega)x\| \leq cE\|\Phi(rs, s, \omega)Q(s, \omega)x\|,$$

for all $s \geq 1$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. It follows from Theorem 4.2 by taking $h(t) = t + 1$. $\square$

Theorem 5.3. *If the pair (C, P) has uniform polynomial growth in mean (u.p.g.m.) then the pair (C, P) is uniformly polynomially dichotomic in mean (u.p.d.m.) if and only if there exists $L > 1$ with*

$$(upl_1m) \quad E\|\Phi(t, s, \omega)P(s, \omega)x\| \ln \frac{t+1}{s+1} \leq LE\|P(s, \omega)x\|,$$

$$(upl_2m) \quad E\|Q(s, \omega)x\| \ln \frac{t+1}{s+1} \leq LE\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. It follows immediately from Theorem 4.3 if we take $h(t) = t + 1$. $\square$

Theorem 5.4. *If the pair (C, P) has uniform polynomial growth in mean (u.p.g.m.) then the pair (C, P) is uniformly polynomially dichotomic in mean (u.p.d.m.) if and only if there exists a nondecreasing function $g : [1, \infty) \rightarrow \mathbb{R}_+$ with $\lim_{t \rightarrow \infty} g(t) = \infty$. Then there exists $L > 1$ with:*

$$(upM_1m) \quad g\left(\frac{t+1}{s+1}\right) E\|\Phi(t, s, \omega)P(s, \omega)x\| \leq LE\|P(s, \omega)x\|,$$

$$(upM_2m) \quad g\left(\frac{t+1}{s+1}\right) E\|Q(s, \omega)x\| \leq E\|\Phi(t, s, \omega)Q(s, \omega)x\|,$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in \mathcal{L}(\Omega, X, \mathcal{P})$.

Proof. It follows from Theorem 4.4 by taking $h(t) = t + 1$. $\square$

Remark 5.1. Theorem 5.2, Theorem 5.3 and Theorem 5.4 are generalizations to the uniform polynomial dichotomy in mean of the results from the theory of uniform polynomial dichotomy obtained by R. Boruga and M. Megan in [3], and R. Boruga in [4] for evolution operators in Banach spaces. These theorems extend the results obtained by A. Găină, M. Megan and C. F. Popa in [12] for discrete-time skew-evolution cocycles in Banach spaces.

Acknowledgement

The author would like to gratefully acknowledge helpful support, suggestions and comments from her research supervisor *Professor Emeritus Mihail Megan*, the Head of the Research Seminar on Mathematical Analysis and Applications in Control Theory at the West University of Timișoara, Romania.

References

- [1] L. Arnold, *Random Dynamical Systems*, Springer Monographs in Mathematics, Berlin, 1998.
- [2] A. M. Atewi, *About bounded solutions of linear stochastic Ito systems*, Miskolc Math. Notes **3** (2002), 3-12.
- [3] R. Boruga, M. Megan, *On uniform polynomial dichotomy in Banach spaces*, Bul. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz. **63** (2018), 32-40.
- [4] R. Boruga, *Majorization criteria for polynomial stability and instability of evolution operators*, Bul. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz **64** (2019), 55-63.
- [5] R. Boruga (Toma), D. I. Borlea (Pătraşcu), D. M. M. Toth, *On uniform stability with growth rates in Banach spaces*, 2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics (SACI) (2021), 000393-000396.
- [6] R. Boruga, M. Megan, *On some characterizations for uniform dichotomy of evolution operators in Banach spaces*, Mathematics **10** (19) (2022), 3704, 1-21.
- [7] T. Caraballo, J. Duan, K. Lu, B. Schmalfuss, *Invariant manifolds for random and stochastic partial differential equations*, Adv. Nonlinear Stud. **10** (2010), 23-52.
- [8] C. Chicone, Yu. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surveys and Monographs **70**, American Mathematical Society, 1999.
- [9] J. L. Daleckii, M. G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces*, Trans. Math. Monographs **43**, American Mathematical Society, 1974.
- [10] D. Dragičević, A. L. Sasu, B. Sasu, *On polynomial dichotomies of discrete nonautonomous systems on the half-line*, Carpathian J. Math. **38** (2022), 663-680.
- [11] D. Dragičević, A. L. Sasu, B. Sasu, *Admissibility and polynomial dichotomy of discrete nonautonomous systems*, Carpathian J. Math. **38** (2022), 737-762.
- [12] A. Găină , M. Megan , C. F. Popa, *Uniform dichotomy concepts for discrete-time skew evolution cocycles in Banach Spaces*, Mathematics **9** (17) (2021), 2177, 1-11.
- [13] P. V. Hai, *Polynomial behavior in mean of stochastic skew-evolution semiflows* (2019), arXiv:1902.04214.
- [14] J. L. Massera, J. J. Schäffer, *Linear Differential Equations and Function Spaces*, Pure Appl. Math. **21**, Academic Press, New York-London, 1966.
- [15] M. Megan, A. L. Sasu, B. Sasu, *Stabilizability and controllability of systems associated to linear skewproduct semiflows*, Rev. Mat. Complut. **15** (2002), 599-618.
- [16] M. Megan, A. L. Sasu, B. Sasu, *On uniform exponential dichotomy of linear skew product semiflows*, Bull. Belg. Math. Soc. Simon Stevin **10** (2003), 1-21.
- [17] M. Megan, C. Stoica, L. Buliga, *On asymptotic behaviors for linear skew-evolution semiflows in Banach spaces*, Carpathian J. Math. **23** (2007), 117-125.

- [18] M. Megan, C. Stoica, *Exponential instability of skew-evolution semiflows in Banach spaces*, Studia Univ. Babeş-Bolyai, Seria Math **LIII** (1) (2008), 17-24.
- [19] M. Megan, C. Stoica, *Concepts of dichotomy for skew-evolution semiflows in Banach spaces*, Ann. Acad. Rom. Sci. Ser. Math. Appl. **2** (2010), 125-140.
- [20] O. Perron, *Die stabilitätsfrage bei differential gleichungen*, Math. Z. **32** (1930), 703-728.
- [21] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.
- [22] C. Stoica, *On exponential stability for skew-evolution semiflows on Banach spaces* (2008), arXiv:0804.1479v1.
- [23] C. Stoica, *An approach to evolution cocycles from a stochastic point of view*, AIP Conf. Proc. **2425** (1) (2022), 420029.
- [24] C. Stoica, D. Borlea, *On h -dichotomy for skew-evolution semiflow in Banach spaces*, Theory Appl. Math. Comput. Sci. **2** (1) (2012), 29-36.
- [25] C. Stoica, M. Megan, *Exponential dichotomy and trichotomy for skew-evolution semiflows on Banach spaces*, Preprint Univ. Bordeaux (2008), 1-6, arXiv 0804.3558.
- [26] D. Stoica, *Uniform exponential dichotomy of stochastic cocycles*, Stochastic Process. Appl. **120** (10) (2010), 1920-1928.
- [27] D. Stoica, *Exponential stability for stochastic cocycle*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity A **7** (2009), 111-118.
- [28] D. Stoica, *Asymptotic behaviors of stochastic evolution cocycle*, Journal of Engineering, Ann. Fac. Eng. Hunedoara **VII** (3) (2010), 98-101.
- [29] T. M. Személy Fülöp, M. Megan, D. I. Borlea (Pătraşcu), *On uniform stability with growth rates of stochastic skew-evolution semiflows in Banach spaces*, Axioms **10** (2021) 182.

Timea Melinda Személy Fülöp

Department of Mathematics

Faculty of Mathematics and Computer Science

West University of Timișoara

V. Pârvan Blvd., No. 4

300223 Timișoara, Romania

E-mail: timeea.21@gmail.com, timea.moraru@e-uvt.ro