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"Some say the world will end in fire,
Some say in ice.
From what I've tasted of desire
I hold with those who favor fire.
But if it had to perish twice,
I think I know enough of hate
To say that for destruction ice
Is also great
And would suffice."

Robert Frost,*"Fire and Ice"* (1920)

Entropy- A Tale of Ice and Fire

(Review of some exceptional Tsallis indexes)

Iulia-Elena Hirica, Cristina-Liliana Pripoae, Gabriel-Teodor Pripoae, and Vasile Preda

Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary

Abstract In this review paper, we recall, in a unifying manner, our recent results concerning the Lie symmetries of nonlinear Fokker-Plank equations, associated to the (weighted) Tsallis and Kaniadakis entropies. The special values of the Tsallis parameters, highlighted by the classification of these symmetries, clearly indicate algebraic and geometric invariants which differentiate the Lie algebras involved.
We compare these values with the ones previously obtained by several authors, and we try to establish connections between our theoretical families of entropies and specific entropies arising in several applications found in the literature.
We focus on the discovered correlations, but we do not neglect dissimilarities, which might provide -in the future- deeper details for an improved extended panorama of the Tsallis entropies.

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Keywords Tsallis entropy; Fokker-Plank equations; special values of Tsallis parameters

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1 Introduction

1.1 History

Born around the middle of the 19-th Century, in the childhood era of Thermodynamics, entropy soon became a cornerstone of the modern world. Its applications range from Statistical mechanics to Information theory, covering (sometimes unexpected) topics from Physics, Chemistry, Biology, Social sciences. Used especially by scientists, in order to model/measure the subtle duality order/disorder, entropy attained new meanings, as it quickly transgressed the thin frontier with Philosophy and even with the common mundane discourse: cold/warm, information/unknown, pattern/chaos, all seems to spring from entropy-related principles ([89,93,95,105]). For example, we find quite inspiring and genuine the literary exegesis [70] of John Milton's *Paradise lost*, which points to entropy as a symbolic vector to ice and fire, in a Dantesque allegory of Inferno. Moreover, the motto of our paper remembers the ideational roots of a well-known modern saga, a Robert Frosts' poem written -eventually- after a discussion in 1919 with astronomer Harlow Shapley.

Entropy is among the notions with the fastest increase in notoriety but, at the same time, among the poorest understood and even controversial ones ([89]). Its apologists and its critics fight on an unsafe ground, due to the great number of entropy related notions. The reader can find in [93] a detailed panorama of the "Universe of entropies". In what follows, we recall but a few of them, for the limited purpose of our study.

The Boltzmann–Gibbs–Shannon (BGS) entropy is the seed notion most the subsequent entropies were derived from. One of its generalization is due to Sharma and Mittal ([99]), later completed by Taneja ([110]); this family of entropies, abbreviated STM, depends on a pair of real parameters. Two particular cases were independently defined and extensively studied: the Tsallis entropy ([113,117,119,122,125]), depending on parameters $q \in \mathbb{R}$, $q \neq 1$; the Kaniadakis entropy ([56]), depending on parameters $k \in [-1, 1]$. Theoretically, all these infinite families of entropies are interesting and of equal interest. Instead, with regards to applications, it is natural to suspect that only a few of them will be useful. The following question becomes haunting:

Among so many "entropies", how to detect the "important" ones ?

The situation is similar with the emergence of a plethora of relativistic models for the large scale Universe, in the first two thirds of the 20-th Century. After they were drastically sorted out by the discovery of the Universe background radiation, many of these models (corresponding to static universes) entered a shadowed zone, but the "winners" (modeling the expanding universes) became standard. In their turn, these "winners" compete today to predict if the visible expanding Universe will eventually grow forever, running toward a thermal (frozen) death, or if it will attain a maximum size and, after that, will begin to collapse, ending in a hot Big Crunch.

Will this approach work for entropy models, as well ?

A promising method may be provided by some algebraic and geometric invariants of specific stochastic PDEs, canonically associated to the entropy models. An example is the nonlinear Fokker-Planck equation (NFPE); this valuable tool is useful to model stochastic phenomena, based on a given entropy. The Lie symmetries of a NFPE can be obtained,

by a standard procedure, and provide (local) groups of transformations which invariate the NFPE's solutions. For the special case of STM entropies, for Tsallis q-entropies and for Kaniadakis k-entropies, these Lie symmetries were studied in [96, 102, 132].

In our papers [50, 91], we generalized these results, for the case of weighted STM, Tsallis and Kaniadakis entropies, respectively. In the final classification of symmetries, the Lie algebras differentiate by means of six "exceptional" values of the Tsallis parameter q and by two "exceptional" values of the Kaniadakis parameter k.

We made a closer look to the Tsallis "exceptional cases", which seem to generate a "spectrum-like" for the set of all Tsallis constants. The fact that these "exceptional" constants are able to control the symmetries of a stochastic phenomenon (with the largest symmetry got for the BGS entropy) suggests that they can act, within the specified model, as "universal constants".

To what extent do these (theoretical) "universal constants" correspond to (practical) "universal constants" behind the fundamental phenomena of Nature ?

In the present paper we show, by case studies, how our selection of the respective six "exceptional" Tsallis parameters corresponds, in a relevant way, to remarkable selected recorded experimental data.

1.2 Our contribution

In Section 2, we recall some of our recent results ([50, 91]) concerning the Lie symmetries associated to the nonlinear Fokker-Plank equations derived from weighted Kaniadakis or Tsallis entropies. Six special values of the Tsallis parameters, highlighted by the classification of these symmetries, clearly indicate algebraic and geometric invariants which differentiate the Lie algebras involved.

In Section 3, we compare these values with the ones previously obtained by several authors, and we try to establish connections between our theoretical families of Tsallis entropies and specific entropies arising in several applications found in the literature.

We focus on the discovered correlations, but we do not neglect dissimilarities, which might provide -in the future- deeper details for an improved extended panorama of the Tsallis entropies.

1.3 Conventions

In what follows, by "differentiable" we understand "smooth"; this generic (and implicit) condition is not -sometimes- necessary, as many results need weaker conditions. In order to not complicate the formalism, we did not point out the existence conditions for integrals and functions.

2 Preliminary notions and results

In this section, we recall some notions and results from [49, 50, 91, 102], concerning the nonlinear Fokker-Planck equation (NFPE), including important examples of entropies, needed for our next discussion in Section 3.

2.1 The NFPE in One Dimension

Denote by $p = p(x,t)$ a parameterized probability density function (PDF) on $\mathbb{R}^2$, where t has the meaning of time. Denote by U an open subset of $\mathbb{R}^2$. We fix a (non-negative) diffusion function $\mathfrak{D}$ and a drift function $\mathfrak{d}$, both depending on $(x,t,p) \in U \times \mathbb{R}$. The associated NFPE is

$$\frac{\partial}{\partial t}p = -\frac{\partial}{\partial x}\Big[\mathfrak{d}\cdot p\Big] + \frac{\partial^2}{\partial x^2}\Big[\mathfrak{D}\cdot p\Big]. \tag{2.1}$$

We can write it $\Delta p(x,t) = 0$, where

$$\begin{aligned}\Delta :=& \frac{\partial}{\partial t} + \Big(\mathfrak{d} + \mathfrak{d}_p I - 2\mathfrak{D}_x - 2\mathfrak{D}_{xp} I\Big)\cdot\frac{\partial}{\partial x} - \\ &- \Big(\mathfrak{D} + \mathfrak{D}_p I\Big)\cdot\frac{\partial^2}{\partial x^2} - \Big(2\mathfrak{D}_p + \mathfrak{D}_{pp} I\Big)\cdot\Big(\frac{\partial}{\partial x}\Big)^2 + \Big(\mathfrak{d}_x - \mathfrak{D}_{xx}\Big)\cdot I\end{aligned} \tag{2.2}$$

and I denotes the identity. In the particular case when $\mathfrak{D}$ and $\mathfrak{d}$ do not depend on p, we recover the linear Fokker–Planck equation, namely

$$p_t = (-\mathfrak{d} + 2\cdot\mathfrak{D}_x)\cdot p_x + \mathfrak{D}\cdot p_{xx} + (\mathfrak{D}_{xx} - \mathfrak{d}_x)\cdot p. \tag{2.3}$$

In the previous setting, let $L = \xi\cdot\partial_x + \eta\cdot\partial_t + \phi\cdot\partial_p$ be a linear differential operator on $U\times\mathbb{R}$, with the property that a function R exists, satisfying the equality $[L,\Delta] = R\cdot\Delta$. Such an operator transforms solutions of (2.1) into solutions of (2.1) and is called Lie symmetry operator. Taking together all these operators, we obtain the symmetry Lie algebra of the NFPE (2.1); its local Lie group contains information about the solutions symmetries.

2.2 Examples of entropies

We fix an arbitrary PDF $\rho = \rho(x)$ and a differentiable function $\varphi = \varphi(x)$. Their associated entropy is

$$H[\rho] = -\int_{\mathbb{R}} \rho(x)\cdot\varphi(\rho(x))dx. \tag{2.4}$$

In addition, we fix a differentiable "weighting" function $w = w(x)$. We suppose w everywhere positive. The w-weighted entropy associated to (2.4) is

$$H^w[\rho] = -\int_{\mathbb{R}} w(x)\cdot\rho(x)\cdot\varphi(\rho(x))dx. \tag{2.5}$$

In applications, the functions φ and w must satisfy additional constraints, imposed by the models' hypothesis. More details in an extended framework may be found in our paper [49].

Example 1. ([96, 132]) (i) In particular, if $\varphi(x) := log(x)$, then, from (2.4), we recover the Boltzmann–Gibbs–Shannon (BGS) entropy.

(ii) For each value of the parameter $q \in \mathbb{R}$, $q \neq 1$, we can define the Tsallis q-logarithm

$$\varphi^{T}_{\{q\}}(x) := \frac{x^{1-q}-1}{1-q}, \tag{2.6}$$

which produces the Tsallis q-entropy in (2.4). The BGS entropy may be obtained as the limit of the Tsallis q-entropy, for $q \to 1$.

(iii) For each value of the parameter $k \in [-1,1]$, $k \neq 0$, we can define the Kaniadakis k-logarithm

$$\varphi^{K}_{\{k\}}(x) := \frac{x^{k}-x^{-k}}{2k}, \tag{2.7}$$

which produces the Kaniadakis k-entropy in (2.4) (also known as the k-deformed entropy). The BGS entropy may be obtained as the limit of the Kaniadakis k-entropy, for $k \to 0$.

(iv) For each value of the parameters $k \in [-1,1]\backslash\{0\}$ and $r \in \mathbb{R}$, we can define the Sharma–Taneja–Mittal (STM) (k,r)-logarithm

$$\varphi^{STM}_{\{(k,r)\}}(x) := x^{r} \cdot \frac{x^{k}-x^{-k}}{2k}, \tag{2.8}$$

which produces the STM (k,r)-entropy in (2.4) (also known as the (k,r)-deformed entropy). If $r = \pm \mid k \mid$, we obtain the particular case of the Tsallis q-logarithm (we denoted $q = 1 \mp 2 \mid k \mid$). If $r = 0$, we get the particular case of the Kaniadakis k- logarithm. If $(k,r) \to (0,0)$, then we derive the BGS entropy.

In order to assure that some n-momentum be finite (for a non-negative integer n), one requires ([96]) as additional hypothesis that $(\mid k \mid, r) \in R(n)$, where the two-parameter region $R(n)$ is defined by

$$-\mid k \mid \leq r \leq \mid k \mid \quad \text{if} \quad 0 \leq \mid k \mid < \frac{1}{2(n+1)},$$

$$\mid k \mid -\frac{1}{n+1} \leq r \leq -\mid k \mid +\frac{1}{n+1} \quad \text{if} \quad \frac{1}{2(n+1)} \leq \mid k \mid < \frac{1}{n+1}.$$

2.3 The Sinkala's classification and the six "exceptional" values of the Tsallis parameter

Consider $k \in [-1,1]$, $k \neq 0$, $r \in \mathbb{R}$ and the NFPEs associated to (k,r)-STM entropies. The corresponding Lie symmetries were determined in [102] (and were recovered in [50, 91], from our similar studies, based on the weighted Tsallis and weighted Kaniadakis entropies). The only such entropies which produce Lie symmetries, apart the generic ones

$$\mathcal{X}_1 = \partial_t, \quad \mathcal{X}_2 = e^{-t}\partial_x, \tag{2.9}$$

fit into one of the two cases: (i) $r = \pm k$ (ii) $r = -1 \pm k$.

The Case without Momentum Restrictions. The conditions imposed upon the parameters k and r provide specific families of entropies, which were classified in the following cases A-C, (i)–(iv).

The ("exceptional") case A. (i) $r=k$, $k=-\frac{2}{3}$, $(q=\frac{7}{3})$; (ii) $r=-k$, $k=\frac{2}{3}$, $(q=\frac{7}{3})$; (iii) $r=-1+k$, $k=-\frac{1}{6}$, $(q=\frac{4}{3})$; (iv) $r=-1-k$, $k=\frac{1}{6}$, $(q=\frac{4}{3})$.

One obtains a Lie algebra spanned by (2.9) and by

$$\mathcal{X}_3=e^{-2t/3}\Big(x\partial_x-\partial_t-p\partial_p\Big),\ \mathcal{X}_4=xe^t\Big(x\partial_x-3p\partial_p\Big),\ \mathcal{X}_5=x\partial_x-\partial_t-\frac{3}{2}p\partial_p. \tag{2.10}$$

The ("exceptional") case B. (i) $r=k$, $k=-1$, $(q=3)$; (ii) $r=-k$, $k=1$, $(q=3)$; (iii) $r=-1+k$, $k=-\frac{1}{2}$, $(q=2)$; (iv) $r=-1-k$, $k=\frac{1}{2}$, $(q=2)$.

One obtains a Lie algebra spanned by (2.9) and by

$$\mathcal{X}_3=x\partial_x-\partial_t-p\partial_p,\quad \mathcal{X}_4=tx\partial_x-t\partial_t-p\Big(t+\frac{1}{2}\Big)\partial_p. \tag{2.11}$$

The ("generic") case C. (i) $r=k$, $k\notin\Big\{-\frac{1}{2},-\frac{2}{3},-1\Big\}$, $(q=1-2k,\ q\notin\Big\{2,\frac{7}{3},3\Big\})$; (ii) $r=-k$, $k\notin\Big\{\frac{1}{2},\frac{2}{3},1\Big\}$, $(q=1+2k,\ q\notin\Big\{2,\frac{7}{3},3\Big\})$; (iii) $r=-1+k$, $k\notin\Big\{\pm\frac{1}{2},-\frac{1}{6}\Big\}$, $(q=1-2k,\ q\notin\Big\{2,\frac{4}{3}\Big\})$; (iv) $r=-1-k$, $k\notin\Big\{\pm\frac{1}{2},\frac{1}{6}\Big\}$, $(q=1+2k,\ q\notin\Big\{2,\frac{4}{3}\Big\})$.

One obtains a Lie algebra spanned by (2.9) and by

$$\mathcal{X}_3=e^{-2(1+\delta)t}\cdot\Big(x\partial_x-\partial_t-p\partial_p\Big),\quad \mathcal{X}_4=x\partial_x-\partial_t+\frac{p}{\delta}\cdot\partial_p, \tag{2.12}$$

where $\delta=k$, for (i); $\delta=-k$, for (ii); $\delta=k-\frac{1}{2}$, for (iii); $\delta=-k-\frac{1}{2}$, for (iv).

In [91], we remarked that the previous Lie symmetries point out "exceptional" cases which reveal that the respective Tsallis entropies differ fundamentally from the resting cases. We conjectured that this entropies correspond to some optimal situation, the "maximal symmetry" being associated to the "maximal stability" (in agreement with [26, 117]).

The Case with Momentum Restrictions. The next result significantly restricts the number of the previous STM entropies.

Theorem 1. ([91]) *Consider the family of (k,r)-STM entropies, with momentum convergence order n. If such an entropy admits additional Lie symmetries, beyond (2.9), then it is a Tsallis q-entropy, for $q\in\Big(\frac{n-1}{n+1},\frac{n+2}{n+1}\Big)$, where $q\neq 1$, for $n>0$ and $q\neq 1,\frac{1}{2},\frac{3}{2}$, for $n=0$.*

Remark. (i) For $n=0$, Theorem 1 highlights two additional "exceptional" values: $q=\frac{1}{2}$ and $q=\frac{3}{2}$. In the next section, we shall show how all the previous values correspond to interesting applications and are already presented in the literature, through theoretical and/or experimental approaches.

(ii) We point out that the results of this section are strongly dependent on the context. It is quite plausible that these "exceptional" values of the parameter q are related - somehow- to the NFP operator spectrum. Starting with different PDEs, we should expect to find other remarkable values of the Tsallis parameter q.

3 Exceptional values of the Tsallis parameter

3.1 The value $q = \frac{1}{2}$

The paper [54] considers two well-known entropies: Renyi's and Tsallis-Havrda-Charvat's. Although MaxEnt distributions are comparable for both entropies, the applications in physics are quite distinct. The maximizers for the hybrid entropy $\mathcal{D}_q$ are described using the Lambert W function. For $q = \frac{1}{2}$, the hybrid entropy is concave.

In [60], one considers the dynamic Tsallis entropy and its applications for model systems. For the model of the ideal gas, the relaxation of particle density fluctuations is investigated in the case $q = \frac{1}{2}$.

In [92], the value $q = \frac{1}{2}$ arises in special cases, corresponding to linear equations. This Tsallis distribution is used when the number of expectation values is big. It has physical significance, in connection with the notion of enstrophy.

The papers [18,74] reproduce experiments on turbulence in electron plasma, using the value $q = \frac{1}{2}$.

Tsallis entropy with adequate values for q is very useful in information measurement and in the statistical range of entropy. In [31], the author studied the relation between the entropy with $q = \frac{1}{2}$ and probability distribution under two-level system.

In [76], the authors study a transportation network equilibrium model; the PDF of the generalized extreme value distribution is considered, with $q = \frac{1}{2}$.

Additional examples may be found in [75, 119].

3.2 The value $q = \frac{4}{3}$

In [4], the authors suggest that the Tsallis q-exponential distribution may be useful to model the (1945- 2008) citations of the Institute of Scientific Information. They show that a suitable choice is to use the Tsallis q-exponential distribution, with $q = \frac{4}{3}$.

Verlinde conjectured ([130]) that gravity is connected to an entropic force. Three realizations of this conjecture are presented in [86] and two more in [27]. In all five cases, a Tsallis entropy is considered, with $q = \frac{4}{3}$.

In [87], the entropic force is modeled by means of a Tsallis entropy, for $q = \frac{4}{3}$; this is not true in the BGS case. It is pointed out that Tsallis entropy is appropriate to long-range interactions and has applications in gravitation modeling.

Additional examples may be found in [16, 19, 24, 47, 48, 51, 52, 57–59, 79, 94, 97]. For the approximate value $q = 1.3$, see [3, 14, 17, 20, 23, 30, 32, 33, 53, 55, 61, 73, 126, 129, 133]. For the approximate value $q = 1.4$, see [7, 13, 17, 21, 22, 25, 78, 83, 115, 128].

3.3 The value $q = \frac{3}{2}$

In [40], the authors investigate the possibility to generate random variables, by using the q-product. If $q < \frac{3}{2}$, the Lyapunov's Central Limit Theorem can be applied. When $q > \frac{3}{2}$, it is shown that the attracting distribution is a Levy's one.

In [77], the constraining technique of Niven is used for the Tsallis entropy function, to determine the equivalent constrained forms. An example of a system with equispaced energy levels is given, together with a visual illustration of some characteristics of Tsallisian statistical mechanics. A special behavior is discovered for the value $q = \frac{3}{2}$.

In [60], dynamic Tsallis entropy essentially extends possibilities of the stochastic description of model physical systems. The time dependent entropy S_q^n and its frequency spectrum are considered for $n = 2$ and for $q = \frac{3}{2}$. This model is important in real complex systems of wildlife, where dynamic states of physiological and pathological systems may appear.

Tsallis thermostatistics is affected by divergence problems. In [84], this fact is validated by computing the non-extensive q-partition function of harmonic oscillators in dimension greater or equal to 3. In this setting, the q- bound $q = \frac{3}{2}$ is considered.

The paper [120] reviews several applications of the q- exponentials and the q- Gaussians. For $q \sim \frac{3}{2}$, the following examples suggest a q-Gaussian distribution: the velocity distribution of Hydra viridissima cells [127]; the velocity distribution of defect turbulence [34]. Conversely, simulations of silo drainage velocity distributions hint for the value $q \sim \frac{3}{2}$ [9, 123].

In [18], the Hamiltonian Mean Field model is considered. One proves that the hypothesis $q \sim 1.5$ is sufficient, in order for the distributions of angles to converge asymptotically towards a q-Gaussian form.

The paper [72] studies to what extent the entropic index q influences critical temperatures and condensate fractions. Existence is proved in some relativistic and non-relativistic systems, only for some values of the parameter q. The critical value $q_c = \frac{3}{2}$ appears in the study of the condensate fraction.

In [76], the variance of the generalized extreme value distribution was defined for $q < \frac{3}{2}$. When $q \geq \frac{3}{2}$, the distribution has no variance. The approach is useful in the field of transportation modeling.

Additional examples may be found in [61, 73, 79, 94, 134].

3.4 The value $q = 2$

In [37], the Tsallis distribution is used for modeling one of the most promising walking mechanisms. A numerical example for the generalized simulated annealing (GSA) studies to what extent the convergence to the optimal solution is influenced by the parameter q. The fast simulated annealing (FSA) is associated to the value $q = 2$.

In [90], the Tsallis entropy with index $q = 2$ is considered and new families of copula are obtained, using several distributions with bounded support.

In [100], the Lambert-Tsallis function was defined. As an application, the disentropy was introduced. The Tsallis 2- entanglement is studied and it is compared with disentanglement for bipartite of qubit mixed states.

In [106], the author analyses several mathematical notions and structures behind Tsallis statistics. The value $q = 2$ is used in the q- Stirlings formula, with applications in statistic models based on Tsallis entropy.

Experimental results in [38] show that PDFs of a ion cooled by buffer gases are q-Gaussian. An increase of the mass of buffer molecules, from that of He to about 200, leads to an increase of parameter q, from about 1 to about 1.9.

In [113], a generalized entropy expression is postulated, using some notion from multifractals formalism. An example illustrates the peculiar characteristics that appeared. It is shown that the value $q = 2$ reduced the internal "energy", when a non-degenerate two-level system was considered.

Additional examples may be found in [15, 33, 35, 63, 94, 107].

3.5 The value $q = \frac{7}{3}$

In [91], we remarked that, in addition to $q = \frac{7}{3}$, one must look for $q = 2.3$ or $q = 2.4$ as well; these approximated values may also correspond to important Tsallis entropies ([77, 80]).

The paper [5] studies a new unstable, nonlinear econometric model, based on a Tsallis cross-entropy. Many economic and financial models belong to this category. For the CESP production model, it is shown that minimum LS errors are attained when $q \sim \frac{7}{3}$. The fact that the q belong to $(\frac{5}{3}, 3)$ characterizes the Lévy distributions.

In [43], the value $q = 2.3$ was shown to be suitable for modeling Soderblom law, for late-type stars.

In [88], the Shiner-Davison-Landsberg Complexity Measure is considered. The Tsallis curve has a maximum at $q \sim 2.3$, corresponding to a maximal complexity.

3.6 The value $q = 3$

In [43], the authors show that the value $q = 3$ is appropriate for modeling the Skumanich law, for early-type G stars.

In [64], numerical experiments are considered, in order to prove, with generalized statistics methods, the outlier-resistance for the data-inversion method. The authors consider a problem originating from geophysics, namely a process to obtain estimates of subsurface properties. For a problem of seismic inversion, satisfactory results are obtained when $q \to 3$.

In [26], a new Tsallis unstable, nonlinear, cross-entropy econometric model is constructed. Several functions from this category are studied, each one with a different stochastic form. The value $q = 3$ is the maximum one in order to have stable laws.

In [36], an appropriate two-steps algorithm is given, provided by the inspiration of Mantegna. The values of q vary within $(1.0, 3.0)$. The investigation uses the Kolmogorov's statistic.

The paper [101] investigates the Tsallis relative entropy, using the keyword extraction methodology for a number of words from a certain Book, in the particular case when $q = 3$.

In [100], it is shown that the equation which gives the Lambert W function has no real-valued solutions when $q = 3$, apart from the trivial ones.

In [46], one proves that the maximal value of the Tsallis entropy (w.r.t. q-expectation values) is reached for q-canonical PDFs. Similarly, a Tsallis entropy maximal value (w.r.t. q-variance) is reached for q-Gaussian distributions. The supremum value is $q = 3$.

In [124], it is proved that, for $1 \leq q < 3$, some central limit theorem q-variants exist, which allow strongly correlated summed random variables.

Additional examples may be found in [37, 69, 81, 114, 119].

3.7 Other mappings

The previous "exceptional" values $\frac{1}{2}$, $\frac{4}{3}$, $\frac{3}{2}$, 2, $\frac{7}{3}$, 3 of parameter q can be translated, by means of several mappings:

(i) Consider the mapping $q \to 2-q$. The new Tsallis q-logarithm is:

$$x \rightsquigarrow \frac{x^{q-1}-1}{q-1}.$$

The new "exceptional" values are $-\frac{3}{2}$, $-\frac{2}{3}$, $-\frac{1}{2}$, 0, $\frac{1}{3}$, 1, respectively.

(ii) Consider the mapping $q \to \frac{1}{q}$. The new Tsallis q-logarithm is:

$$x \rightsquigarrow \frac{q}{q-1}(x^{\frac{q-1}{q}}-1).$$

The respective "exceptional" values become 2, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{7}$, $\frac{1}{3}$, respectively.

(iii) In [121], the author generalizes the two previous mappings, by considering the parameterized correspondence $q \to q_a(q) := \frac{(a+2)-aq}{a-(a-2)q}$, for $a \in \mathbb{R}$. In particular, one obtains $q_a(1) = 1$, $q_2(q) = 2-q$ and $q_0(q) = \frac{1}{q}$.

For $a > 2$: when q belongs to $(-\infty, 1]$, then q_a strictly decreases from $\frac{a}{a-2}$ to 1; when q belongs to $[1, \frac{a}{a-2}]$, then q_a strictly decreases from 1 to $-\infty$; when q belongs to $(\frac{a}{a-2}, \infty)$, the q_a strictly decreases from ∞ to $\frac{a}{a-2}$.

For $a < 2$: when q belongs to $(-\infty, \frac{a}{a-2})$, then q_a strictly decreases from $\frac{a}{a-2}$ to $-\infty$; when q belongs to $(\frac{a}{a-2}, 1]$, then q_a strictly decreases from ∞ to 1; when q belongs to $[1, \infty)$, then q_a strictly decreases from 1 to $\frac{a}{a-2}$.

An eventual new search into the literature may complete the list of the examples from subsections 3.1-3.6 with new ones, via the previous translations.

3.8 Other special values not arising from the previous procedure

In [116], it is shown that, for a given class of complex systems, only several independent values of q are considered (the other values depending on the previous ones). An example is provided by the data of Voyager 1, concerning the solar wind, which point out the q-triplet: $(q_{se}, q_{st}, q_{re}) = (-\frac{1}{2}, \frac{7}{4}, 4)$, where $q_{se} := q_{sensitivity}$, $q_{st} := q_{stationary\ state}$, $q_{re} := q_{relaxation}$.

The paper [80] uses Tsallis q-triplets for the experiment test of q-statistics and in the context of fractal dynamics. Tsallis theory is considered in different systems: solar plasmas, atmospheric dynamics, seismogenesis and brain dynamics. The q-values are: $q_{st} = 1.98 \pm 0.06$, in the case of the plasma velocity; $q_{st} = 2.05 \pm 0.04$, for magnetospheric plasma magnetic fields. In the situation of solar wind magnetic cloud, the value is $q_{st} = 2.02 \pm 0.04$.

In [116], the author studies the logistic map near its edge of chaos. Numerical simulations lead to the q-triplet: $(q_{se}, q_{st}, q_{re}) = (0.2444..., 1.65 \pm 0.05, 2.2497...)$. The same particular q-triplet is mentioned in [120]. The special Tsallis index $q = 0.24...$ appears in many similar applications (see [118] and references therein).

In [80], the authors consider the estimated q_{st} indexes for some human body organs activity, magnetospheric dynamics, solar activity, cosmic phenomena. Namely, the values are: Atmosphere (Temperature) $q_{st} = 1.89 \pm 0.08$; Cardiac (hrv) $q_{st} = 1.26 \pm 0.10$; Brain (seizure) $q_{st} = 1.63 \pm 0.14$; Seismic (Magnitude) $q_{st} = 1.77 \pm 0.09$; Seismic (Inter-event) $q_{st} = 2.28 \pm 0.12$; Atmosphere (Rainfall) $q_{st} = 2.21 \pm 0.06$; Magnetosphere (Ey storm) $q_{st} =$

2.49 ± 0.07; Magnetosphere (Electrons storm) $q_{st} = 2.15 \pm 0.07$; Magnetosphere (Protons storm) $q_{st} = 2.49 \pm 0.05$; Solar Wind (Bz cloud) $q_{st} = 2.02 \pm 0.04$; Solar (Sunspot Index) $q_{st} = 1.53 \pm 0.04,$; Solar (Flares Index) $q_{st} = 1.8700$; Solar (Protons) $q_{st} = 2.31 \pm 0.13$; Solar (Electrons) $q_{st} = 2.13 \pm 0.06$; Cosmic Ray (C) $q_{st} = 1.44 \pm 0.05$. Cosmic Stars (Brightness) $q_{st} = 1.64 \pm 0.03$;

Additional examples may be found in [1, 2, 6, 8, 10–12, 19, 22, 28, 29, 35, 38, 39, 41, 42, 44, 45, 62, 65–68, 71, 81, 82, 84, 85, 88, 98, 103, 104, 108, 109, 111, 112, 117, 121, 131, 135, 136].

4 Conclusions

Our paper reviews the emergence of the "exceptional" values $\frac{1}{2}, \frac{4}{3}, \frac{3}{2}, 2, \frac{7}{3}, 3$ of the Tsallis entropy parameter q, from a double perspective. Firstly, these numbers appear, theoretically, in the process of determining the Lie symmetry properties corresponding to the NFPEs, in the case when the basic entropy is the Tsallis one; each such number corresponds to a specific degree of symmetry behavior at the Lie algebras level. Secondly, we gave examples of papers devoted to applications of specific Tsallis entropies, where the respective "exceptional" values of the parameter q appear in a natural, empirically way.

This coincidence leads us to conjecture the existence of a kind of "spectrum" of the family of Tsallis q-entropies, containing the six values pointed out by us and maybe additional ones. The versatility of the Tsallis parameter q seems to be censored by a rigidity principle, limiting the number of its values which may be useful in real-life applications.

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