

# On uniform $h$ -dichotomy of skew-evolution cocycles in Banach spaces

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*Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary*

**Abstract** The paper presents some characterizations for the study of dichotomy of dynamical systems. Integral characterizations of Datko type are considered for the notion of uniform dichotomy with growth rates, also called uniform  $h$ -dichotomy, for skew-evolution cocycles in Banach spaces.

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## 1 Introduction

In the last years, the dichotomy concepts for evolution equations were intensively studied and also an important progress was made in the study of the asymptotic behavior of skew-evolution cocycles in Banach spaces.

The notion of dichotomy in the case of differential equations was introduced by Perron in [31]. An impressive development was made starting with the works of Daleckii and Krein [9], Massera and Schäffer [19], Coppel [8]. For recent contributions to the study of exponential dichotomy, we refer to Boruga and Megan [7], Dragičević, Sasu and Sasu [12], Găină, Megan and Popa [15], Lupa and Megan [18], Megan, Sasu and Sasu [23], Megan and Stoica [27], Sasu and Sasu [37], Stoica and Megan [39], Zhou, Lu and Zhang [40, 41].

The concept of polynomial dichotomy was introduced by Barreira and Valls in [1] and it was treated in a large number of papers. In the following years, the polynomial behaviors were studied by Boruga and Megan [3, 5, 7], Dragičević, Sasu and Sasu [13, 14], Găină, Megan and Popa [15], Hai [16], Mihiţ and Lăpădat [28].

In this paper, we focus on uniform dichotomy with growth rates or uniform  $h$ -dichotomy, where  $h : \mathbb{R}_+ \rightarrow [1, \infty)$  is a growth rate, more precisely  $h$  is a nondecreasing and bijective function. This concept is a generalization of uniform exponential dichotomy

and uniform polynomial dichotomy. We mention that, for the first time, the case of growth rates for the concept of stability under some perturbations was studied by Pinto [33] in 1984. The next steps were made by Megan [20, 21], followed by Bento, Lupa, Megan and Silva [2], Boruga (Toma), Megan and Toth [4, 6].

Megan, Stoica and Buliga introduced the notion of skew-evolution cocycle in [25] in the continuous case and Megan and Stoica in [26] in the discrete case. Their work has been continued in the studies developed by Hai [16], Megan and Stoica [27], Mihit and Lăpădat [28], Stoica and Megan [39], Găină, Megan and Popa [15], thus generalizing the notions of evolution operator,  $C_0$ -semigroup and skew-product (semi)flow studied by Sacker and Sell [35], Megan, Sasu and Sasu [22–24], Petre and Megan [32] (see also the references therein).

The characterizations presented in this paper have their origins in the works of Datko [10, 11]. Contributions in this direction have been brought by Pazy [30], Rolewicz [34], Littman [17], Neerven [29], Megan, Sasu and Sasu [22, 24], Megan and Stoica [26], Sasu [36] and Sasu, Megan and Sasu [38], Lupa and Megan [18], Bento, Lupa, Megan and Silva [2], Hai [16], Boruga (Toma), Megan and Toth [4], Boruga and Megan [5], Boruga (Toma), Megan and Toth [6], Boruga and Megan [7] for both stability and dichotomy. We mention that Megan, Sasu and Sasu proved in 2002 for the first time in the literature a stability theorem of Datko type for skew-product semiflows in [22].

The purpose of this paper is to give necessary and sufficient conditions in order to deduce theorems of Datko type [10] for the notion of uniform h-dichotomy in the case of skew-evolution cocycles in Banach spaces. The particular cases of these integral characterizations are obtained for the concepts of uniform exponential dichotomy and uniform polynomial dichotomy.

## 2 Preliminaries

Let  $X$  be a metric space,  $V$  a Banach space,  $\mathcal{B}(V)$  the Banach space of all bounded linear operators acting on  $V$  and the sets

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}$$

$$T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

**Definition 2.1.** A mapping  $\varphi : \Delta \times X \rightarrow X$  is said to be an evolution semiflow on  $X$  if the following properties are satisfied:

$$(es_1) \quad \varphi(s, s, x_0) = x, \text{ for all } (s, x_0) \in \mathbb{R}_+ \times X;$$

$$(es_2) \quad \varphi(t, s, \varphi(s, t_0, x_0)) = \varphi(t, t_0, x_0), \text{ for all } (t, s, t_0, x_0) \in T \times X.$$

**Definition 2.2.** Let  $\varphi : \Delta \times X \rightarrow X$  be an evolution semiflow on  $X$ . A mapping  $\Phi : \Delta \times X \rightarrow \mathcal{B}(V)$  is said to be a skew-evolution semiflow on  $X \times V$  over the evolution semiflow  $\varphi$  if :

$$(ses_1) \quad \Phi(s, s, x_0) = I \text{ (the identity operator on } X), \text{ for all } (s, x_0) \in \mathbb{R}_+ \times X;$$

$(ses_2) \Phi(t, s, \varphi(s, t_0, x_0))\Phi(s, t_0, x_0) = \Phi(t, t_0, x_0)$ , for all  $(t, s, t_0, x_0) \in T \times X$ .

If  $\Phi$  is a skew-evolution semiflow over the evolution semiflow  $\varphi$ , then the pair  $C = (\Phi, \varphi)$  is called a skew-evolution cocycle.

**Definition 2.3.** A mapping  $E : \Delta \rightarrow \mathcal{B}(V)$  is called an evolution operator on  $V$  if the following conditions hold:

$(e_1)$   $E(t, t) = I$  (the identity operator), for all  $t \in \mathbb{R}_+$ ;

$(e_2)$   $E(t, s)E(s, t_0) = E(t, t_0)$ , for all  $(t, s, t_0) \in T$ .

**Example 2.1.** For each evolution operator we can associate a mapping

$$\Phi_E : \Delta \times X \rightarrow \mathcal{B}(V), \Phi_E(t, s, x_0) = E(t, s),$$

which is in fact a skew-evolution cocycle over  $\varphi$ .

**Example 2.2.** Let  $X = \mathbb{R}_+$  and the evolution semiflow  $\varphi : \Delta \times X \rightarrow X$  defined by  $\varphi(t, s, x_0) = x_0$ . Then  $\Phi_E : \Delta \times \mathbb{R}_+ \rightarrow \mathcal{B}(V)$  defined by

$$\Phi_E(t, s, x_0) = E(t - s + x_0, x_0),$$

is a skew-evolution cocycle over  $\varphi$ .

**Definition 2.4.** The mapping  $P : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$  is said to be a family of projectors if the following relation holds:

$$P^2(s, x_0) = P(s, x_0), \text{ for all } (s, x_0) \in \mathbb{R}_+ \times X.$$

**Remark 2.1.** If  $P : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$  is a family of projectors, then also  $Q : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$  is a family of projectors defined by  $Q(s, x_0) = I - P(s, x_0)$  called the complementary family of projectors  $P$ .

**Definition 2.5.** The skew-evolution cocycle  $C = (\Phi, \varphi)$  is called strongly measurable if, for all  $(s, x_0, v_0) \in \mathbb{R}_+ \times X \times V$ , the mapping  $t \mapsto \|\Phi(t, s, x_0)v_0\|$  is measurable on  $[s, \infty)$ .

**Definition 2.6.** The family of projectors  $P : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$  is said to be invariant to the skew-evolution cocycle  $C = (\Phi, \varphi)$  if the following relation holds:

$$\Phi(t, s, x_0)P(s, x_0) = P(t, \varphi(t, s, x_0))\Phi(t, s, x_0),$$

for all  $(t, s, x_0) \in \Delta \times X$ .

**Definition 2.7.** The pair  $(C, P)$  is said to be uniformly h-dichotomic if there exist two constants  $N \geq 1$  and  $\nu > 0$  with:

$$(uhd_1) \ h(t)^\nu \|\Phi(t, s, x_0)P(s, x_0)v_0\| \leq Nh(s)^\nu \|P(s, x_0)v_0\|;$$

$$(uhd_2) \ h(t)^\nu \|Q(s, x_0)v_0\| \leq Nh(s)^\nu \|\Phi(t, s, x_0)Q(s, x_0)v_0\|,$$

for all  $(t, s, x_0, v_0) \in \Delta \times X \times V$ .

**Remark 2.2.** The pair  $(C, P)$  is said to be uniformly h-dichotomic if and only if there exist  $N \geq 1$  and  $\nu > 0$  with:

$$(uhd'_1) \quad h(t)^\nu \|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\| \leq Nh(s)^\nu \|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|;$$

$$(uhd'_2) \quad h(t)^\nu \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| \leq Nh(s)^\nu \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|,$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

Particular cases:

1. For  $h(t) = e^t$  we obtain the property of *uniform exponential dichotomy*.
2. For  $h(t) = t + 1$  the concept of *uniform polynomial dichotomy* is obtained.

**Definition 2.8.** The pair  $(C, P)$  has uniform h-growth if there exist  $M \geq 1$  and  $\omega > 0$  with:

$$(uhg_1) \quad h(s)^\omega \|\Phi(t, s, x_0)P(s, x_0)v_0\| \leq Mh(t)^\omega \|P(s, x_0)v_0\|;$$

$$(uhg_2) \quad h(s)^\omega \|Q(s, x_0)v_0\| \leq Mh(t)^\omega \|\Phi(t, s, x_0)Q(s, x_0)v_0\|,$$

for all  $(t, s, x_0, v_0) \in \Delta \times X \times V$ .

**Remark 2.3.** The pair  $(C, P)$  has uniform h-growth if and only if there exist  $M \geq 1$  and  $\omega > 0$  with:

$$(uhg'_1) \quad h(s)^\omega \|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\| \leq Mh(t)^\omega \|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|;$$

$$(uhg'_2) \quad h(s)^\omega \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| \leq Mh(t)^\omega \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|,$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

Particular cases:

1. For  $h(t) = e^t$  the property of *uniform exponential growth* is obtained.
2. If  $h(t) = t + 1$ , the skew-evolution cocycle has *uniform polynomial growth*.

**Remark 2.4.** If the pair  $(C, P)$  is uniformly h-dichotomic, than it also has uniform h-growth. The converse implication is not true, as we can see in the following:

**Example 2.3.** Let be  $V = \mathbb{R}^2$  and  $\varphi : \Delta \times X \rightarrow X$  a evolution semiflow on  $X$ . The mapping  $\Phi : \Delta \times X \rightarrow \mathcal{B}(V)$  defined by

$$\Phi(t, s, x_0)(v_1, v_2) = \left( \frac{h(t)}{h(s)}v_1, \frac{h(s)}{h(t)}v_2 \right)$$

is a skew-evolution semiflow on  $X \times V$  over  $\varphi$ .

Let  $P : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$  be the family of projectors defined by

$$P(s, x_0)(v_1, v_2) = (v_1, 0).$$

Then the complementary family of projectors  $Q : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$  is defined by

$$Q(s, x_0)(v_1, v_2) = (0, v_2),$$

for all  $(s, x_0, v_1, v_2) \in \mathbb{R}_+ \times X \times V$ .

We have that  $(C, P)$  has uniform h-growth, but is not uniformly h-dichotomic.

### 3 Integral characterizations for uniform h-dichotomy

Let  $C = (\Phi, \varphi)$  be a strongly measurable skew-evolution cocycle and the invariant family of projectors  $P : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$ .

We consider some classes of functions which have been also used in [4] and [6]:

- (i)  $\mathcal{H}_0$  the set of functions  $h : \mathbb{R}_+ \rightarrow [1, \infty)$  with the property that  $h(t) \geq t + 1$ , for all  $t \geq 0$ .
- (ii)  $\mathcal{H}$  the set of functions  $h : \mathbb{R}_+ \rightarrow [1, \infty)$  with the property that exists  $H > 1$  such that  $h(t + 1) \leq Hh(t)$ , for all  $t \geq 0$ .
- (iii)  $\mathcal{H}_1$  the set of functions  $h : \mathbb{R}_+ \rightarrow [1, \infty)$  with the property that for all  $\alpha \in (0, 1)$ , exists  $H_1 > 1$  such that  $\int_s^\infty \frac{1}{h(t)^\alpha} dt \leq \frac{H_1}{h(s)^\alpha}$ , for all  $s \geq 0$  and  $\int_0^t h(s)^\alpha ds \leq H_1 h(t)^\alpha$ , for all  $t \geq 0$ .
- (iv)  $\mathcal{H}_2$  the set of functions  $h : \mathbb{R}_+ \rightarrow [1, \infty)$  with the property that for all  $\alpha \in (0, 1)$ , exists  $H_2 > 1$  such that  $\int_s^\infty \frac{1}{h(t)^{\alpha+1}} dt \leq \frac{H_2}{h(s)^\alpha}$ , for all  $s \geq 0$  and  $\int_0^t h(s)^\alpha ds \leq H_2 h(t)^\alpha$ , for all  $t \geq 0$ .

**Theorem 3.1.** *We suppose that  $(\Phi, \varphi)$  is a strongly measurable skew-evolution cocycle,  $(C, P)$  has uniform h-growth and  $h \in \mathcal{H}_0 \cap \mathcal{H} \cap \mathcal{H}_1$ . The pair  $(C, P)$  is uniformly h-dichotomic if and only if there exist  $D > 1$  and  $d \in [0, 1)$  with*

$$(uhD_1) \quad \int_s^\infty h(t)^d \|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\| dt \leq Dh(s)^d \|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|,$$

for all  $(s, t_0, x_0, v_0) \in \Delta \times X \times V$ ;

$$(uhD_2) \quad \int_{t_0}^t h(s)^{-d} \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| ds \leq Dh(t)^{-d} \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|,$$

for all  $(t, t_0, x_0, v_0) \in \Delta \times X \times V$ .

*Proof.* The relation  $(uhd_1) \iff (uhD_1)$  results from the proof of Theorem 4 from [4] where  $\Phi(t, s)x$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$ .

The equivalence  $(uhd_2) \iff (uhD_2)$  follows from Theorem 3.3 in [6] having the evolution operator  $\Phi(t, s)x$  replaced by the skew-evolution cocycle and the family of projectors  $\Phi(t, s, x_0)Q(s, x_0)v_0$ .  $\square$

**Corollary 3.2.** *We suppose that  $(\Phi, \varphi)$  is a strongly measurable skew-evolution cocycle and  $(C, P)$  has uniform exponential growth. The pair  $(C, P)$  is uniformly exponentially dichotomic if and only if there exist  $D > 1$  and  $d \in [0, 1)$  with*

$$(ueD'_1) \quad \int_s^\infty e^{dt} \|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\| dt \leq De^{ds} \|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|,$$

for all  $(s, t_0, x_0, v_0) \in \Delta \times X \times V$ ;

$$(ueD'_2) \int_{t_0}^t e^{-ds} \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| ds \leq De^{-dt} \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|,$$

for all  $(t, t_0, x_0, v_0) \in \Delta \times X \times V$ .

*Proof.* It is similar with Theorem 2 from [7], but  $\Phi(t, s)P(s)x$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$  and  $\Phi(t, s)Q(s)x$  by  $\Phi(t, s, x_0)Q(s, x_0)v_0$ .  $\square$

The connection between uniform polynomial dichotomy and uniform exponential dichotomy for skew-evolution cocycles is given in the following theorem:

**Theorem 3.3.** Let  $\varphi_1 : \Delta \times X \rightarrow X$  be an evolution semiflow on  $X$  defined by  $\varphi_1(t, s, x_0) = \varphi(e^t - 1, e^s - 1, x_0)$ ,  $\Phi_1 : \Delta \times X \rightarrow \mathcal{B}(V)$  a skew-evolution semiflow over  $\varphi_1$  defined by  $\Phi_1(t, s, x_0) = \Phi(e^t - 1, e^s - 1, x_0)$  and the families of projectors  $P_1, Q_1 : \mathbb{R}_+ \times X \rightarrow \mathcal{B}(V)$  defined by  $P_1(s, x_0) = P(e^s - 1, x_0)$  and  $Q_1(s, x_0) = Q(e^s - 1, x_0)$ .

The pair  $(C, P)$  is uniformly polynomially dichotomic if and only if the pair  $(C_1, P_1)$  is uniformly exponentially dichotomic.

*Proof.* It follows from Proposition 1 by R. Boruga and M. Megan [7] where  $\Phi(t, s)P(s)x$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$  and  $\Phi(t, s)Q(s)x$  by  $\Phi(t, s, x_0)Q(s, x_0)v_0$ .  $\square$

**Corollary 3.4.** We suppose that  $(\Phi, \varphi)$  is a strongly measurable skew-evolution cocycle and  $(C, P)$  has uniform polynomial growth. The pair  $(C, P)$  is uniformly polynomially dichotomic if and only if there exist  $D > 1$  and  $d \in (0, 1)$  with

$$(upD_1) \int_s^\infty (t+1)^{d-1} \|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\| dt \leq D(s+1)^d \|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|;$$

$$(upD_2) \int_{t_0}^t (s+1)^{-d-1} \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| ds \leq D(t+1)^{-d} \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|,$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

*Proof.* It is analogous to Theorem 9 from [7] where  $\Phi(t, s)P(s)x$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$  and  $\Phi(t, s)Q(s)x$  by  $\Phi(t, s, x_0)Q(s, x_0)v_0$ .  $\square$

**Theorem 3.5.** Let be  $(\Phi, \varphi)$  a strongly measurable skew-evolution cocycle,  $h \in \mathcal{H} \cap \mathcal{H}_1$  and  $(C, P)$  has uniform  $h$ -growth. The pair  $(C, P)$  is uniformly  $h$ -dichotomic if and only if there exist the constants  $D > 1$  and  $d \in [0, 1)$  with

$$(uhD'_1) \int_{t_0}^t \frac{h(s)^{-d}}{\|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|} ds \leq \frac{Dh(t)^{-d}}{\|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\|} \text{ with } \Phi(t, t_0, x_0)P(t_0, x_0)v_0 \neq 0;$$

$$(uhD'_2) \int_s^\infty \frac{h(t)^d}{\|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|} dt \leq \frac{Dh(s)^d}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|} \text{ with } Q(t_0, x_0)v_0 \neq 0,$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

*Proof. Necessity.*

The relation  $(uhd_1) \implies (uhD'_1)$  results from the proof of Theorem 5 in [4], where  $\Phi(t, s)x_0$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$ .

For  $(uhd_2) \implies (uhD'_2)$  we consider  $d \in (0, \nu)$  and we obtain

$$\begin{aligned} \int_s^\infty \frac{h(t)^d}{\|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|} dt &\leq N \int_s^\infty h(t)^d \left( \frac{h(t)}{h(s)} \right)^{-\nu} \frac{1}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|} dt \\ &= \frac{Nh(s)^\nu}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|} \int_s^\infty (h(t))^{d-\nu} dt \\ &\leq \frac{Nh(s)^\nu}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|} H_1 h(s)^{d-\nu} \\ &\leq D \frac{h(s)^d}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|}, \end{aligned}$$

where  $D = 1 + NH_1$ .

*Sufficiency.* The equivalence  $(uhD'_1) \implies (uhd_1)$  follows from Theorem 5 obtained in [4] where  $\Phi(t, s)x_0$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$ .

For relation  $(uhD'_2) \implies (uhd_2)$  we consider two cases:

*Case 1.1:*  $t \geq s + 1$  and  $Q(t_0, x_0)v_0 \neq 0$ .

$$\begin{aligned} \frac{h(t)^d}{\|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|} &= \int_{t-1}^t \frac{h(t)^d}{\|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|} d\tau \\ &\leq M \int_{t-1}^t \left( \frac{h(t)}{h(\tau)} \right)^\omega \frac{h(t)^d}{\|\Phi(\tau, t_0, x_0)Q(t_0, x_0)v_0\|} d\tau \\ &= M \int_{t-1}^t \left( \frac{h(t)}{h(\tau)} \right)^{\omega+d} \frac{h(\tau)^d}{\|\Phi(\tau, t_0, x_0)Q(t_0, x_0)v_0\|} d\tau \\ &\leq M \int_{t-1}^t \left( \frac{h(t)}{h(t-1)} \right)^{\omega+d} \frac{h(\tau)^d}{\|\Phi(\tau, t_0, x_0)Q(t_0, x_0)v_0\|} d\tau \\ &\leq MH^{\omega+d} \int_s^\infty \frac{h(\tau)^d}{\|\Phi(\tau, t_0, x_0)Q(t_0, x_0)v_0\|} d\tau \\ &\leq MH^{\omega+d} D \frac{h(s)^d}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|}. \end{aligned}$$

So we have

$$h(t)^d \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| \leq MDH^{\omega+d} h(s)^d \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|, \quad (3.1)$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

Case 1.2:  $t \in [s, s+1)$  with  $Q(t_0, x_0)v_0 \neq 0$ .

$$\begin{aligned} h(t)^d \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| &\leq Mh(t)^d \left(\frac{h(t)}{h(s)}\right)^\omega \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\| \\ &= M \left(\frac{h(t)}{h(s)}\right)^{\omega+d} h(s)^d \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\| \\ &\leq MH^{\omega+d} h(s)^d \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|. \end{aligned}$$

So

$$h(t)^d \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| \leq MH^{\omega+d} h(s)^d \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|, \quad (3.2)$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

From (3.1) and (3.2) it follows that there exist  $N = MDH^{\omega+d} + 1$  such that

$$h(t)^d \|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\| \leq Nh(s)^d \|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|,$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ . □

**Corollary 3.6.** *Suppose that  $(\Phi, \varphi)$  is a strongly measurable skew-evolution cocycle and  $(C, P)$  has uniform exponential growth. The pair  $(C, P)$  is uniformly exponentially dichotomic if and only if there exist  $D > 1$  and  $d \in [0, 1)$  with*

$$(ueD_1'') \int_{t_0}^t \frac{e^{-ds}}{\|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|} ds \leq \frac{De^{-dt}}{\|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\|} \text{ with } \Phi(t, t_0, x_0)P(t_0, x_0)v_0 \neq 0;$$

$$(ueD_2'') \int_s^\infty \frac{e^{dt}}{\|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|} dt \leq \frac{De^{ds}}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|} \text{ with } Q(t_0, x_0)v_0 \neq 0,$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

*Proof.* It is analogous to Theorem 4 from [7] where  $\Phi(t, s)P(s)x$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$  and  $\Phi(t, s)Q(s)x$  by  $\Phi(t, s, x_0)Q(s, x_0)v_0$ . □

**Corollary 3.7.** *Suppose that  $(\Phi, \varphi)$  is a strongly measurable skew-evolution cocycle and  $(C, P)$  has uniform polynomial growth. The pair  $(C, P)$  is uniformly polynomially dichotomic if and only if there exist  $D > 1$  and  $d \in (0, 1)$  with*

$$(upD_1') \int_{t_0}^t \frac{(s+1)^{-d-1}}{\|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|} ds \leq \frac{D(t+1)^{-d}}{\|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\|} \text{ with } \Phi(t, t_0, x_0)P(t_0, x_0)v_0 \neq 0;$$

$$(upD_2') \int_s^\infty \frac{(t+1)^{d-1}}{\|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|} dt \leq \frac{D(s+1)^d}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|} \text{ with } Q(t_0, x_0)v_0 \neq 0,$$

for all  $(t, s, t_0, x_0, v_0) \in T \times X \times V$ .

*Proof.* It is similar to Theorem 11 from [7], where the evolution operator is replaced by the skew-evolution cocycle. □

**Theorem 3.8.** *Suppose that  $(\Phi, \varphi)$  is a strongly measurable skew-evolution cocycle,  $(C, P)$  has uniform h-growth and  $h \in \mathcal{H}_0 \cap \mathcal{H} \cap \mathcal{H}_2$ . Then, the pair  $(C, P)$  is uniformly h-dichotomic if and only if there exist two constants  $D > 1$  and  $d \in (0, 1)$  with*

$$(uhD_1'') \int_{t_0}^t \frac{h(s)^{-d}}{\|\Phi(s, t_0, x_0)P(t_0, x_0)v_0\|} ds \leq \frac{Dh(t)^{-d}}{\|\Phi(t, t_0, x_0)P(t_0, x_0)v_0\|},$$

for all  $(t, t_0, x_0, v_0) \in \Delta \times X \times V$  with  $\Phi(t, t_0, x_0)P(t_0, x_0)v_0 \neq 0$ ;

$$(uhD_2'') \int_s^\infty \frac{h(t)^{d-1}}{\|\Phi(t, t_0, x_0)Q(t_0, x_0)v_0\|} dt \leq \frac{Dh(s)^d}{\|\Phi(s, t_0, x_0)Q(t_0, x_0)v_0\|},$$

for all  $(s, t_0, x_0, v_0) \in \Delta \times X \times V$  with  $Q(t_0, x_0)v_0 \neq 0$ .

*Proof.* The equivalence  $(uhd_1) \iff (uhD_1'')$  is a consequence of the proof of Theorem 5 in [4], where  $\Phi(t, s)x_0$  is replaced by  $\Phi(t, s, x_0)P(s, x_0)v_0$ .

The relation  $(uhd_2) \iff (uhD_2'')$  follows from Corollary 3.6 in [6], where  $\Phi(t, s)x_0$  is replaced by  $\Phi(t, s, x_0)Q(s, x_0)v_0$ .  $\square$

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