sciendo

DOI: 10.2478/awutm-2022-0017

Annals of West University of Timisoara
Mathematics and Computer Science
58, 2, (2022), 56– 66

On some growth concepts for dichotomic behaviors of evolution operators

Rovana Boruga (Toma)

Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary

Abstract The aim of the present paper is to emphasize some growth concepts for the dichotomic behavior of evolution operators in Banach spaces. In fact, we approach the exponential growth, the polynomial growth and the h- growth for both uniform and nonuniform cases. Connections between concepts are established. Majorization criteria for the uniform behaviors are given.

AMS Subject Classification (2020) 34C11; 34D09.
Keywords evolution operators; growth concepts; majorization criteria.

1 Introduction

In recent years an impressive progress has been made in the study of the asymptotic behaviors for dynamical systems in Banach spaces. Among the most significant concepts in this field are the exponential dichotomy and the polynomial dichotomy. The importance of the role played by these notions in the theory of dynamical systems is illustrated by the appearance of numerous papers in this domain for the uniform case ([3,10,14,18,23,24,30]) as well as for the nonuniform case ([1,2,5,6,16,22,24,29–31]).

Also, there are many directions of studying the dichotomic behaviors. For example, the relationship between dichotomy and admissibility is an issue that has been intensively studied in the last period. The most recent result on this line of research was obtained by D. Dragičević, A.L. Sasu and B. Sasu in [13] where the authors give some new admissibility conditions for uniform and exponential dichotomy and they also provide new characterizations for polynomial dichotomy by means of some double admissibilities. Moreover, the same authors in [12] obtain for the first time a characterization of polynomial dichotomy with respect to a sequence of norms in terms of exponential dichotomy.

© 2022 Rovana Boruga (Toma). This is an open access article licensed under the Creative Commons Attribution-NonCommercial-NoDerivs License (http://creativecommons.org/licenses/by-nc-nd/3.0/).

In a natural manner, generalizations of the behaviors mentioned above were obtained by considering the h- dichotomy where h is a growth rate (i.e. a nondecreasing and bijective function). The idea of introducing the growth rates belongs to M. Pinto [27] who obtained results in the case of stability for a weakly stable system under some perturbation. Also, this concept was approached in other works ([4, 7, 11, 19, 20, 26, 28]).

The main purpose of this paper is to deal with the h-growth concept which has as particular cases the exponential growth and the polynomial growth. We focus on these growth properties because they represent assumptions in many characterizations of the exponential dichotomy, polynomial dichotomy and h- dichotomy, three of the most studied asymptotic properties in the qualitative theory of dynamical systems. For instance, P.V. Hai in [15] presents characterizations for an evolution family with polynomial growth to be polynomially stable using the Perron method. Also, M. Megan, A.L. Sasu and B. Sasu in [25] emphasize the relationship between a nonuniform instability concept and the Perron condition in the hypothesis that the evolution operator has nonuniform exponential growth. The general case of h -growth represents an important tool on proving some necessary and sufficient conditions of Datko and Barbashin type for uniform stability and instability for evolution operators in Banach spaces (see [4, 7]).

In the present work we establish connections between the growth concepts for both uniform and nonuniform cases and we give some majorization criteria for the uniform part.

2 Definitions and notations

In this paper we continue the study from [3] by extending the results obtained in the general case of h- growth.

In what follows, we maintain all the notations and the concepts presented in our previous papers (see [3] and also [4–10]).

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. The identity operator on X is denoted by I. We also consider the set

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}.$$

Definition 2.1. An application $U : \Delta \to \mathcal{B}(X)$ is called an *evolution operator* on X if

(e_1) $U(t, t) = I$ for every $t \geq 0$

(e_2) $U(t, s)U(s, t_0) = U(t, t_0)$ for all $(t, s, t_0) \in T$.

Definition 2.2. An application $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be a *projection family* on X if $P^2(t) = P(t)$, for all $t \geq 0$.

Remark 2.1. If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is a projection family on X, then the mapping $Q : \mathbb{R}_+ \to \mathcal{B}(X), Q(t) = I - P(t)$ is also a projection family on X, which is called the complementary projection of P.

Definition 2.3. An application $h : \mathbb{R}_+ \to [1, \infty)$ is called *growth rate* if it is bijective and nondecreasing.

Definition 2.4. The pair (U, P) has *uniform h-growth* (u.h.g.) if there are $M > 1$ and $\omega > 0$ such that

(uhg_1) $h(s)^\omega \|U(t,s)P(s)x\| \leq Mh(t)^\omega \|P(s)x\|$

(uhg_2) $h(s)^\omega \|Q(s)x\| \leq Mh(t)^\omega \|U(t,s)Q(s)x\|$,

for all $(t, s, x) \in \Delta \times X$.

Remark 2.2. As particular cases,

- if $h(t) = e^t$ then the pair (U, P) has uniform exponential growth (u.e.g.).
- if $h(t) = t + 1$ then the pair (U, P) has uniform polynomial growth (u.p.g.).

Definition 2.5. The pair (U, P) has *h-growth* (h.g.) if there are $M > 1$, $\omega > 0$ and $\varepsilon \geq 0$ such that

(hg_1) $h(s)^\omega \|U(t,s)P(s)x\| \leq Mh(s)^\varepsilon h(t)^\omega \|P(s)x\|$

(hg_2) $h(s)^\omega \|Q(s)x\| \leq Mh(t)^\varepsilon h(t)^\omega \|U(t,s)Q(s)x\|$,

for all $(t, s, x) \in \Delta \times X$.

Remark 2.3. As particular cases,

- if $h(t) = e^t$ then the pair (U, P) has exponential growth (e.g.).
- if $h(t) = t + 1$ then the pair (U, P) has polynomial growth (p.g.).

Definition 2.6. The pair (U, P) has *nonuniform h-growth* (n.h.g.) if there are a nondecreasing function $M : \mathbb{R}_+ \to [1, \infty)$ and $\omega > 0$ such that

(nhg_1) $h(s)^\omega \|U(t,s)P(s)x\| \leq M(s)h(t)^\omega \|P(s)x\|$

(nhg_2) $h(s)^\omega \|Q(s)x\| \leq M(t)h(t)^\omega \|U(t,s)Q(s)x\|$,

for all $(t, s, x) \in \Delta \times X$.

Remark 2.4. As particular cases,

- if $h(t) = e^t$ then the pair (U, P) has nonuniform exponential growth (n.e.g.).
- if $h(t) = t + 1$ then the pair (U, P) has nonuniform polynomial growth (n.p.g.).

3 Connections between the growth concepts

In this section we obtain new characterizations for the uniform h- growth in order to make the connections between the behavior with growth rates and its particular cases, the exponential growth and the polynomial growth. Also, some examples which describe these connections are given.

Remark 3.1. The following diagram illustrates that the polynomial growth concepts imply the exponential growth concepts and also the uniform cases imply the nonuniform cases.

$$\begin{array}{ccccc} u.p.g. & \Rightarrow & p.g. & \Rightarrow & n.p.g. \\ \Downarrow & & \Downarrow & & \Downarrow \\ u.e.g. & \Rightarrow & e.g. & \Rightarrow & n.e.g. \end{array}$$

In general, the converse implications are not true.

Example 3.1. Let $X = \mathbb{R}^2,\ P(s)x = (x_1, 0),\ Q(s)x = (0, x_2)$ where $x = (x_1, x_2) \in X$. We consider the application

$$u : \mathbb{R}_+ \to [1, \infty)\ u(t) = e^t$$

and the evolution operator

$$U : \Delta \to \mathcal{B}(X),\ U(t, s)x = \left(\frac{u(t)}{u(s)}x_1, \frac{u(s)}{u(t)}x_2\right)$$

Then the pair (U, P):

- has uniform exponential growth, but it does not have uniform polynomial growth
- has exponential growth, but it does not have polynomial growth
- has nouniform exponential growth, but it does not have nonuniform polynomial growth

Indeed, it is easy to see that (U, P) has u.e.g. for $M = 1$ and $\omega = 2$, which implies form Remark 3.1 that (U, P) has e.g. and n.e.g.

If we suppose that (U, P) has u.p.g., it follows from Remark 3.1 that (U, P) has p.g. which means that there are $M > 1,\ \omega > 0$ and $\varepsilon \geq 0$ such that the relations (hg_1) and (hg_2) are satisfied for the particular case $h(t) = t + 1$, for all $t \geq s \geq 0$.

For $s = 0$ and $t \to \infty$ we obtain $\infty \leq M$, absurd.

Similarly, it follows that (U, P) does not have n.p.g.

Example 3.2. Let $X = \mathbb{R}^2,\ P(s)x = (x_1, 0),\ Q(s)x = (0, x_2)$ where $x = (x_1, x_2) \in X$. We consider the application

$$u : \mathbb{R}_+ \to [1, \infty), u(t) = \begin{cases} e^{t^2}\ , & \text{if } t \in \mathbb{N} \\ 1\ , & \text{if } t \notin \mathbb{N} \end{cases}$$

and the evolution operator

$$U : \Delta \to \mathcal{B}(X),\ U(t,s)x = \left(\frac{u(s)}{u(t)} \cdot e^{t-s}x_1,\ \frac{u(t)}{u(s)} \cdot e^{s-t}x_2\right).$$

Then the pair (U, P) has n.e.g., but it does not have e.g.

Indeed, it is easy to see that (U, P) has n.e.g. for $\omega = 1$ and $M(t) = u(t)$.

If we suppose that (U, P) has e.g. it follows that there are $M > 1,\ \omega > 0$ and $\varepsilon \geq 0$ such that (hg_1) and (hg_2) are satisfied for $h(t) = e^t$, for all $(t, s) \in \Delta$. In particular, for $s = n,\ t = n + \dfrac{1}{n}$ we obtain a contradiction.

Remark 3.2. An example of a pair (U, P) which has e.g., but it does not have u.e.g. is given in [17].

Remark 3.3. An example of a pair (U, P) which has p.g., but it does not have u.p.g., respectively a pair (U, P) that has n.p.g., but it does not have p.g. is given in [8].

Remark 3.4. Using Remark 3.1 and the examples from above, it is obvious that if (U, P) has uniform h -growth, then it also has h- growth and nonuniform h - growth, but the converse implications are not valid.

The connections for the uniform case between the h- growth and its particular cases, the exponential growth, respectively the polynomial growth, are given in the next two theorems. Moreover, as consequence of each theorem we obtain two corollaries which express the connections between the uniform exponential growth and the uniform polynomial growth.

Theorem 3.1. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \to \mathcal{B}(X)$ a projection family associated to U. Then the pair (U, P) has uniform h- growth if and only if the pair (U_h, P_h) has uniform exponential growth where*

$$U_h : \Delta \to \mathcal{B}(X),\ U_h(t,s) = U(h^{-1}(e^t), h^{-1}(e^s)),$$

$$P_h : \mathbb{R}_+ \to \mathcal{B}(X),\ P_h(s) = P(h^{-1}(e^s)).$$

Proof. Necessity. We suppose that (U, P) has uniform h- growth. We need to prove that the two relations from the Definition 2.4 are satisfied for the particular case when $h(t) = e^t$. The first one follows in a similar manner as the necessity proved in Theorem III.1 from [9]. For the second relation, if (U, P) has u.h.g and $(t, s) \in \Delta$ then there are $M > 1$ and $\omega > 0$ such that

$$\begin{aligned} Me^{\omega t}\|U_h(t,s)Q_h(s)x\| &= e^{\omega t}\|U(h^{-1}(e^t), h^{-1}(e^s))Q(h^{-1}(e^s))x\| \\ &\leq Me^{\omega t}\left(\frac{h(h^{-1}(e^s))}{h(h^{-1}(e^t))}\right)^{\omega}\|Q(h^{-1}(e^s))x\| \\ &= Me^{\omega s}\|Q_h(s)x\|. \end{aligned}$$

It follows that (U_h, P_h) has u.e.g.

Sufficiency. We suppose that (U_h, P_h) has u.e.g. and we need to prove that (uhg_1) and (uhg_2) are satisfied for the pair (U, P).

Let $(t,s)\in\Delta$. The relation (uhg_1) follows by using similar arguments as in the sufficiency of Theorem III.1 from [9]. For (uhg_2) we have

$$\begin{aligned}Mh(t)^{\omega}\|U(t,s)Q(s)x\| &= Mh(t)^{\omega}\|U(h^{-1}h(t)),h^{-1}(h(s))Q(h^{-1}(h(s)))x\|\\ &= Mh(t)^{\omega}\|U(h^{-1}(e^{\ln h(t)}),h^{-1}(e^{\ln h(s)}))Q(h^{-1}(e^{\ln h(s)}))x\|\\ &= Mh(t)^{\omega}\|U_h(\ln h(t),\ln h(s))Q_h(\ln h(s))x\|\\ &\geq Mh(t)^{\omega}e^{\omega\ln\frac{h(s)}{h(t)}}\|Q_h(\ln h(s))x\|\\ &= Mh(s)^{\omega}\|Q(s)x\|.\end{aligned}$$

It follows that (U,P) has u.h.g. □

Corollary 3.2. *Let $U:\Delta\to\mathcal{B}(X)$ be an evolution operator and $P:\mathbb{R}_+\to\mathcal{B}(X)$ a projection family associated to U. Then the pair (U,P) has uniform polynomial growth if and only if the pair (U_1,P_1) has uniform exponential growth, where*

$$U_1:\Delta\to\mathcal{B}(X),\ U_1(t,s)=U(e^t-1,e^s-1),$$

$$P_1:\mathbb{R}_+\to\mathcal{B}(X),\ P_1(s)=P(e^s-1).$$

Proof. It follows from Theorem 3.1 for $h(t)=t+1$. □

Theorem 3.3. *Let $U:\Delta\to\mathcal{B}(X)$ be an evolution operator and $P:\mathbb{R}_+\to\mathcal{B}(X)$ a projection family associated to U. Then the pair (U,P) has uniform h- growth if and only if the pair (U^h,P^h) has uniform polynomial growth, where*

$$U^h:\Delta\to\mathcal{B}(X),\ U^h(t,s)=U(h^{-1}(t+1),h^{-1}(s+1)),$$

$$P^h:\mathbb{R}_+\to\mathcal{B}(X),\ P^h(s)=P(h^{-1}(s+1)).$$

Proof. Necessity. We suppose that (U,P) has uniform h- growth. We have to prove that (uhg_1) and (uhg_2) are true in the particular case when $h(t)=t+1$ for the pair (U^h,P^h). For the first statement, see the necessity of the Theorem III.2 from [9].

For the second relation, if (U,P) has u.h.g. and $(t,s)\in\Delta$, then there are $M>1$ and $\omega>0$ such that

$$\begin{aligned}M(t+1)^{\omega}\|U^h(t,s)Q^h(s)x\| &= M(t+1)^{\omega}\|U(h^{-1}(t+1),h^{-1}(s+1))Q(h^{-1}(s+1))x\|\\ &\leq (t+1)^{\omega}\left(\frac{h(h^{-1}(s+1))}{h(h^{-1}(t+1))}\right)^{\omega}\|Q(h^{-1}(s+1))x\|\\ &= (s+1)^{\omega}\|Q^h(s)x\|.\end{aligned}$$

It follows that, (U^h,P^h) has u.p.g.

Sufficiency. We suppose that (U^h,P^h) has u.p.g. and we have to prove that (uhg_1) and (uhg_2) are satisfied for the pair (U,P).

Let $(t,s)\in\Delta$. Then, there are $N>1$ and $\nu>0$ such that

$$(s+1)^{\omega}\|Q^h(s)x\|\leq M(t+1)^{\omega}\|U^h(t,s)Q^h(s)x\|,$$

which is equivalent to

$$(s+1)^{\omega}\|Q(h^{-1}(s+1))x\| \leq M(t+1)^{\omega}\|U(h^{-1}(t+1),h^{-1}(s+1))Q(h^{-1}(s+1))x\|. \quad (3.1)$$

If we denote by

$$u = h^{-1}(t+1) \text{ and } v = h^{-1}(s+1),$$

we obtain that the relation (3.1) is equivalent to

$$h(v)^{\omega}\|Q(v)x\| \leq Mh(u)^{\omega}\|U(u,v)Q(v)x\|,$$

which implies that (uhg_2) is satisfied.

For (uhg_1) see the sufficiency of Theorem III.2 from [9]. □

Corollary 3.4. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \to \mathcal{B}(X)$ a projection family associated to U. Then the pair (U, P) has uniform exponential growth if and only if the pair (U_2, P_2) has uniform polynomial growth, where*

$$U_2 : \Delta \to \mathcal{B}(X),\ U_2(t,s) = U(\ln(t+1), \ln(s+1)),$$

$$P_2 : \mathbb{R}_+ \to \mathcal{B}(X),\ P_2(s) = P(\ln(s+1)).$$

Proof. It follows from Theorem 3.3 for $h(t) = e^t$. □

4 Majorization criteria

In this section we will give a majorization criterion for each of the three growth concepts approached in the paper for the uniform case: uniform exponential growth, uniform polynomial growth, and uniform h- growth.

Theorem 4.1. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \to \mathcal{B}(X)$ a family of projections associated to U. Then the pair (U, P) has uniform exponential growth if and only if there exists a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi(t) = \infty$ such that*

$(umeg_1)$ $\|U(t,s)P(s)x\| \leq \varphi(t-s)\|P(s)x\|$

$(umeg_2)$ $\|Q(s)x\| \leq \varphi(t-s)\|U(t,s)Q(s)x\|,$

for all $(t, s, x) \in \Delta \times X$.

Proof. See [21]. □

Theorem 4.2. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \to \mathcal{B}(X)$ a family of projections associated to U. Then the pair (U, P) has uniform polynomial growth if and only if there exists a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi(t) = \infty$ such that*

$(umpg_1)$ $\|U(t,s)P(s)x\| \leq \varphi\left(\frac{t+1}{s+1}\right)\|P(s)x\|$

$(umpg_2)$ $\|Q(s)x\| \leq \varphi\left(\frac{t+1}{s+1}\right)\|U(t,s)Q(s)x\|,$

for all $(t,s,x) \in \Delta \times X$.

Proof. Necessity. We suppose that (U,P) has u.p.g. Then there are $M > 1$ and $\omega > 0$ such that the conditions (uhg_1) and (uhg_2) are satisfied for the particular case $h(t) = t+1$, for all $(t,s) \in \Delta$.

If we denote by $\frac{t+1}{s+1} = u$, then we obtain

$$\|U(t,s)P(s)x\| \leq Me^{\omega u}\|P(s)x\| = \varphi(u)\|P(s)x\| = \varphi\left(\frac{t+1}{s+1}\right)\|P(s)x\|,$$

respectively

$$\|Q(s)x\| \leq Me^{\omega u}\|U(t,s)Q(s)x\| = \varphi(u)\|U(t,s)Q(s)x\| = \varphi\left(\frac{t+1}{s+1}\right)\|U(t,s)Q(s)x\|$$

and the necessity is proved.

Sufficiency. We suppose that there exists a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi(t) = \infty$ such that the relations $(umpg_1)$ and $(umpg_2)$ are satisfied.

We have to prove that the pair (U,P) has u.p.g., which from Corollary 3.2 is equivalent to (U_1,P_1) has u.e.g.

We consider the application $\varphi_1(x) = \varphi(e^x)$ and let $t = e^u - 1,\ s = e^v - 1$.

Then $u = \ln(t+1), v = \ln(s+1)$ and

- $\varphi\left(\frac{t+1}{s+1}\right) = \varphi\left(\frac{e^u}{e^v}\right) = \varphi(e^{u-v}) = \varphi_1(u-v)$
- $\|U(t,s)P(s)x\| = \|U(e^u-1,e^v-1)P(e^v-1)x\| = \|U_1(u,v)P_1(v)x\|$
- $\|U(t,s)Q(s)x\| = \|U(e^u-1,e^v-1)Q(e^v-1)x\| = \|U_1(u,v)Q_1(v)x\|.$

Using the hypothesis we obtain

$$\|U_1(u,v)P_1(v)x\| \leq \varphi_1(u-v)\|P(e^v-1)x\| = \varphi_1(u-v)\|P_1(v)x\|,$$

respectively

$$\|Q_1(v)x\| = \|Q(e^v-1)\| \leq \varphi_1(u-v)\|U_1(u,v)Q_1(v)x\|.$$

From the majorization criteria for the exponential case given in Theorem 4.1 it follows that (U_1,P_1) has u.e.g. and using Corollary 3.2 the proof is complete.

□

Theorem 4.3. *Let* $U : \Delta \to \mathcal{B}(X)$ *be an evolution operator and* $P : \mathbb{R}_+ \to \mathcal{B}(X)$ *a projection family associated to* U. *Then the pair* (U,P) *has uniform h- growth if and only if there exists a nondecreasing function* $\varphi : [1,\infty) \to \mathbb{R}_+$ *with* $\lim_{t\to\infty} \varphi(t) = \infty$ *such that*

$(umhg_1)$ $\|U(t,s)P(s)x\| \leq \varphi\left(\frac{h(t)}{h(s)}\right)\|P(s)x\|$

$(umhg_2)$ $\|Q(s)x\| \leq \varphi\left(\frac{h(t)}{h(s)}\right)\|U(t,s)Q(s)x\|,$

for all $(t,s,x) \in \Delta \times X$.

Proof. Necessity. It follows from Definition 2.4 if we consider $\varphi(t) = Mt^{\omega}$.
Sufficiency. We suppose that there exists a nondecreasing function $\varphi : [1,\infty) \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi(t) = \infty$ such that the relations $(umhg_1)$ and $(umhg_2)$ are satisfied for all $(t,s) \in \Delta$.

We have to prove that (U,P) has u.h.g. which according to Theorem 3.3 is equivalent to (U^h,P^h) has u.p.g.

If we denote by $t = h^{-1}(u+1)$ and $s = h^{-1}(v+1)$ and we use the hypothesis we obtain

$$\|U^h(u,v)P^h(v)x\| = \|U(h^{-1}(u+1),h^{-1}(v+1))P(h^{-1}(v+1))x\| \leq \varphi\left(\frac{u+1}{v+1}\right)\|P^h(v)x\|,$$

respectively

$$\|Q^h(v)x\| = \|Q(h^{-1}(v+1))x\| \leq \varphi\left(\frac{u+1}{v+1}\right)\|U^h(u,v)Q^h(v)x\|.$$

From the majorization criteria for the uniform polynomial growth proved in Theorem 4.2 we obtain that (U^h,P^h) has uniform polynomial growth and from Theorem 3.3 it follows that (U,P) has uniform h- growth, so the proof is complete.

□

Acknowledgement

The author gratefully acknowledges helpful suggestions and support from Professor Emeritus Mihail Megan.

References

[1] L. Barreira, C. Valls, *Polynomial growth rates*, Nonlinear Anal. **71** (2009), 5208–5219.

[2] A.J.G. Bento, C. Silva, *Stable manifolds for nonuniform polynomial dichotomies*, J. Funct. Anal. **257** (2009), 122–148.

[3] R. Boruga (Toma), M. Megan, *On some characterizations for uniform dichotomy of evolution operators in Banach spaces*, Mathematics **10** (2022), 3704.

[4] R. Boruga, M. Megan, D. M.-M. Toth, *On uniform instability with growth rates in Banach spaces*, Carpathian J. Math. **38** (3) (2022), 789–796.

[5] R. Boruga (Toma), M. Megan, *Nonuniform polynomial dichotomy with Lyapunov type norms*, Ann. Acad. Rom. Sci. Ser. Math. Appl. **12** (1-2) (2020), 329–343.

[6] R. Boruga (Toma), M. Megan, *Datko type characterizations for nonuniform polynomial dichotomy*, Carpathian J. Math. **37** (2021), 45–51.

[7] R. Boruga (Toma), M. Megan, D. M.-M. Toth, *Integral characterizations for uniform stability with growth rates in Banach spaces*, Axioms **10** (2021), 235.

[8] R. Boruga, M. Megan, *Nonuniform polynomial dichotomy with invariant projection families*, The 15th ICMA Conf. Proc., Politehnica University of Timisoara (2018), 30–46.

[9] R. Boruga (Toma), D. I. Borlea (Pătraşcu), D. M.-M. Toth, *On uniform stability with growth rates in Banach spaces*, 2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics (SACI) (2021), pp. 000393–000396.

[10] R. Boruga, M. Megan, *On uniform polynomial dichotomy in Banach spaces*, Bul. Ştiinţ. Univ. Politeh. Timiş., Ser. Mat.-Fiz. **63(77)** (2018), 32–40.

[11] V. Crai, M. Megan, *(h,k)-dichotomy and Lyapunov type norms*, An. Univ. Vest Timiş. Ser. Mat.-Inform. **56** (2) (2018), 115–130.

[12] D. Dragičević, A. L. Sasu, B. Sasu, *On polynomial dichotomies of discrete nonautonomous systems on the half-line*, Carpathian J. Math. **38** (2022), 663–680.

[13] D. Dragičević, A. L. Sasu, B. Sasu, *Admissibility and polynomial dichotomy of discrete nonautonomous systems*, Carpathian J. Math. **38** (2022), 737–762.

[14] A. Găină, M. Megan, C.F. Popa, *Uniform dichotomy concepts for discrete-time skew evolution cocycles in Banach spaces,* Mathematics **9** (2021), 2177.

[15] P. V. Hai, *On the polynomial stability of evolution families*, Appl. Anal. **95** (2016), 1239–1255.

[16] P. V. Hai, *Polynomial stability and polynomial instability for skew-evolution semiflows*, Results Math. **175** (2019), 1–19.

[17] N. Lupa, *Comportări asimptotice exponenţiale pentru operatori de evoluţie n spaţii Banach*, Editura Politehnica, Timişoara, 2014.

[18] C. L. Mihiţ, M. Lăpădat, *On uniform polynomial dichotomy of skew-evolution semiflows on the half-line*, Bul. Ştiinţ. Univ. Politeh. Timiş., Ser. Mat.-Fiz. **62** (2017), 54–61.

[19] M. Megan, *On H- stability of evolution operators*, Qualitative Problems for Differential Equations and Control Theory, World Scientific Publishing (1995), 33–40.

[20] M. Megan, *On (h,k)-dichotomy of evolution operators in Banach spaces*, Dyn. Syst. Appl. **5** (2) (1996), 189–195.

[21] M. Megan, C. Stoica, *On uniform exponential trichotomy of evolution operators in Banach spaces*, Integral Equations Operator Theory **60** (4) (2008), 499–506.

[22] M. Megan, B. Sasu, A.L. Sasu, *On nonuniform exponential dichotomy of evolution operators in Banach spaces*, Integral Equations and Operator Theory **44** (2002), 71–78.

[23] M. Megan, A.L. Sasu, B. Sasu, *On uniform exponential dichotomy of linear skew product semiflows*, Bull. Belg. Math. Soc. Simon Stevin **10** (2003), 1–21.

[24] M. Megan, A. L. Sasu, B. Sasu, *The Asymptotic Behavior of Evolution Families*, Ed. Mirton 2003.

[25] M. Megan, A. L. Sasu, B. Sasu, *Nonuniform exponential unstability of evolution operators in Banach spaces*, Glas. Mat. **56** (2001), 287–296.

[26] C. L. Mihiţ, *On uniform h-stability of evolution operators in Banach spaces*, Theory Appl. Math. Comput. Sci. **6** (2016), 19–27.

[27] M. Pinto, *Perturbations of asymptotically stable differential systems*, Analysis **4** (1-2) (1984), 161–176.

[28] M. Pinto, *Asymptotic integration of a system resulting from the perturbation of an h-system*, J. Math. Anal. Appl. **131** (1) (1988), 194–216.

[29] M.L. Rămneanţu, T. Ceauşu, M. Megan, *On nonuniform polynomial dichotomy of evolution operators in Banach spaces*, Int. J. Pure Appl. Math. **75** (3) (2012), 305–318.

[30] M. L. Rămneanţu, M. Megan, T. Ceauşu, *Polynomial instability of evolution operators in Banach spaces*, Carpathian J. Math. **29** (2013), 77–83.

[31] A. L. Sasu, M. G. Babutia, B. Sasu, *Admissibility and nonuniform exponential dichotomy on the half-line*, Bull. Sci. Math. **137** (2013), 466–484.

Rovana Boruga (Toma)
Department of Mathematics
West University of Timişoara
V. Pârvan Blvd., No. 4
300223 Timişoara
Romania
E-mail: rovanaboruga@gmail.com