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On uniform logarithmic dichotomy of discrete skew-evolution semiflows

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Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary

Abstract The paper considers two notions of logarithmic dichotomy for discrete skew-evolution semiflows in Banach spaces. We establish the relation between them, we give a characterization for the uniform logarithmic dichotomy of Zabczyk type and a sufficient criteria for the uniform logarithmic dichotomy.

AMS Subject Classification (2020) 34D05; 34D09
Keywords uniform logarithmic dichotomy; discrete skew-evolution semiflows; Lyapunov function

1 Introduction

One of the most important topic studied for dynamical systems is represented by the property of dichotomy, approached from uniform point of view in: [2], [5], [19], respectively nonuniform in [3], [4], [6], [8], [10], [17], [20], [22], [23].

Among the most significant results from the stability theory of C_0-semigroups is the characterization proved by J. Zabczyk ([21]), a result which was extended by K. M. Przyluski and S. Rolewicz ([14]) for linear discrete-time systems. Recent studies in this direction are made for exponential stability ([7], [15]), exponential dichotomy ([18]) and exponential trichotomy ([16]), respectively trichotomy with growth rates ([12]).

An important line of research is represented by the dichotomic behavior of skew-evolution semiflows in discrete case. In this sense, we mention the contributions of A. Găină, M. Megan, C. F. Popa ([5]), M. Megan, C. Stoica ([9]) and C. Stoica ([20]).

In [13], the authors prove discrete conditions for a splitting concept in the case of skew-evolution semiflows and in [1], discrete characterizations for the splitting property of discrete cocycles are obtained. Also, in [11], the polynomial trichotomy is approached for skew-evolution semiflows as a generalization of the dichotomy notion.

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In this article we approach the uniform logarithmic dichotomy and the uniform logarithmic dichotomy of Zabczyk type, using invariant families of projectors. The connection between them is obtained and the main result of the paper is the characterization of the uniform logarithmic dichotomy of Zabczyk type in terms of Lyapunov functions.

2 Preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $||\cdot||$.
We consider Θ a metric space, $\Gamma = \Theta \times X$ and we define the sets

$$\Delta_d = \{(m,n) \in \mathbb{N}^2 : m \geq n\} \quad \text{and} \quad T_d = \{(m,n,p) \in \mathbb{N}^3 : m \geq n \geq p\}.$$

Definition 2.1. A mapping $\varphi : \Delta_d \times \Theta \to \Theta$ is called *discrete evolution semiflow* on Θ if:

(des_1) $\varphi(n,n,\theta) = \theta$, for all $(n,\theta) \in \mathbb{N} \times \Theta$;
(des_2) $\varphi(m,n,\varphi(n,p,\theta)) = \varphi(m,p,\theta)$, for all $(m,n,p,\theta) \in T_d \times \Theta$.

Definition 2.2. An application $\Phi : \Delta_d \times \Theta \to \mathcal{B}(X)$ is a *discrete evolution cocycle* over the discrete evolution semiflow φ if the following conditions hold:

(dec_1) $\Phi(n,n,\theta) = I$ (the identity operator on X), for all $(n,\theta) \in \mathbb{N} \times \Theta$;
(dec_2) $\Phi(m,n,\varphi(n,p,\theta))\Phi(n,p,\theta) = \Phi(m,p,\theta)$, for all $(m,n,p,\theta) \in T_d \times \Theta$.

Definition 2.3. If φ is a discrete evolution semiflow on Θ and Φ is a discrete evolution cocycle over φ, then the pair $C = (\varphi, \Phi)$ is called *discrete skew-evolution semiflow* on Γ.

Further, an application $P : \mathbb{N} \times \Theta \to \mathcal{B}(X)$, with the property that $P^2(n,\theta) = P(n,\theta)$, for all $(n,\theta) \in \mathbb{N} \times \Theta$, represents a *family of projectors* on X.

Definition 2.4. A family of projectors $P : \mathbb{N} \times \Theta \to \mathcal{B}(X)$ is said to be *invariant* to the discrete skew-evolution semiflow $C = (\varphi, \Phi)$ if

$$P(m,\varphi(m,n,\theta))\Phi(m,n,\theta) = \Phi(m,n,\theta)P(n,\theta), \quad \text{for all } (m,n,\theta) \in \Delta_d \times \Theta.$$

3 The main results

In what follows, we consider $C = (\varphi, \Phi)$ a discrete skew-evolution semiflow, $P : \mathbb{N} \times \Theta \to \mathcal{B}(X)$ an invariant family of projectors to C and $Q : \mathbb{N} \times \Theta \to \mathcal{B}(X)$, $Q(n,\theta) = I - P(n,\theta)$ (the complementary family of projectors of P).

Definition 3.1. We say that the pair (C, P) admits a *uniform logarithmic dichotomy* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$(uld_1) \quad (\ln(m+1))^{\nu}||\Phi(m,p,\theta)P(p,\theta)x|| \leq N(\ln(n+1))^{\nu}||\Phi(n,p,\theta)P(p,\theta)x||;$$
$$(uld_2) \quad (\ln(m+1))^{\nu}||\Phi(n,p,\theta)Q(p,\theta)x|| \leq N(\ln(n+1))^{\nu}||\Phi(m,p,\theta)Q(p,\theta)x||,$$

for all $(m, n, p, \theta, x) \in T_d \times \Gamma$.

Remark 3.1. The pair (C, P) is *uniformly logarithmic dichotomic* if and only if there are the constants $N \geq 1$ and $\nu > 0$ with:

$$(uld_1') \quad (\ln(m+1))^{\nu}||\Phi(m,n,\theta)P(n,\theta)x|| \leq N(\ln(n+1))^{\nu}||P(n,\theta)x||;$$
$$(uld_2') \quad (\ln(m+1))^{\nu}||Q(n,\theta)x|| \leq N(\ln(n+1))^{\nu}||\Phi(m,n,\theta)Q(n,\theta)x||,$$

for all $(m, n, \theta, x) \in \Delta_d \times \Gamma$.

Example 3.1. We consider $X = \mathbb{R}^2$ endowed with the norm

$$||x|| = |x_1| + |x_2|,$$

Θ a metric space and the discrete evolution cocycle $\Phi : \Delta_d \times \Theta \to \mathcal{B}(X)$, defined by

$$\Phi(m,n,\theta)x = \left(\frac{\ln(n+1)}{\ln(m+1)}x_1, \frac{\ln(m+1)}{\ln(n+1)}x_2\right),$$

where φ is a discrete evolution semiflow.

Also, the family of projectors $P : \mathbb{N} \times \Theta \to \mathcal{B}(X)$ is given by $P(n,\theta)x = (x_1, 0)$ and the complementary family of projectors $Q(n,\theta)x = (0, x_2)$, for all $(n, \theta) \in \mathbb{N} \times X$, $x = (x_1, x_2) \in \mathbb{R}^2$.

After some computations, we obtain that (C, P) is uniformly logarithmic dichotomic for $\nu = 1$ and $N \geq 1$.

Further, we denote by $\mathcal{Z}$ the set of all continuous nondecreasing functions $Z : \mathbb{R}_+ \to \mathbb{R}_+$ with $Z(t) > 0$, for all $t > 0$.

Definition 3.2. The pair (C, P) has a *uniform logarithmic dichotomy of Zabczyk type* if there exist $Z \in \mathcal{Z}$, $M \geq 1$ and $\mu > 0$ such that:

$$(uldZ_1) \quad \sum_{j=n}^{+\infty} Z\left(\left(\frac{\ln(j+1)}{\ln(n+1)}\right)^{\mu}||\Phi(j,p,\theta)P(p,\theta)x||\right) \leq MZ(||\Phi(n,p,\theta)P(p,\theta)x||);$$
$$(uldZ_2) \quad \sum_{j=p}^{m} Z\left(\left(\frac{\ln(m+1)}{\ln(j+1)}\right)^{\mu}||\Phi(j,p,\theta)Q(p,\theta)x||\right) \leq MZ(||\Phi(m,p,\theta)Q(p,\theta)x||),$$

for all $(m, n, p, \theta, x) \in T_d \times \Gamma$.

Proposition 3.1. *If the pair (C, P) admits a uniform logarithmic dichotomy of Zabczyk type, then (C, P) is uniformly logarithmic dichotomic.*

Proof. We consider $Z : \mathbb{R}_+ \to \mathbb{R}_+$, $Z(t) = t$.
(uld_1) From the condition $(uldZ_1)$, for $j = m$ we deduce

$$\left(\frac{\ln(m+1)}{\ln(n+1)}\right)^{\mu} ||\Phi(m,p,\theta)P(p,\theta)x|| \leq M||\Phi(n,p,\theta)P(p,\theta)x||,$$

for all $(m,n,p,\theta,x) \in T_d \times \Gamma$.
(uld_2) Similarly, by $(uldZ_2)$, taking $j = n$ we obtain

$$\left(\frac{\ln(m+1)}{\ln(n+1)}\right)^{\mu} ||\Phi(n,p,\theta)Q(p,\theta)x|| \leq M||\Phi(m,p,\theta)Q(p,\theta)x||,$$

for all $(m,n,p,\theta,x) \in T_d \times \Gamma$.

It yields that the pair (C,P) is uniformly logarithmic dichotomic. □

Definition 3.3. The application $L : T_d \times \Gamma \to \mathbb{R}_+$ is a *Lyapunov function* for the pair (C,P) if there exist $Z \in \mathcal{Z}$ and $\mu > 0$ such that:

$$(L_1)\quad L(m,n,p,\theta,P(p,\theta)x) + \sum_{j=n}^{m-1} Z\left(\left(\frac{\ln(j+1)}{\ln(n+1)}\right)^{\mu} ||\Phi(j,p,\theta)P(p,\theta)x||\right) \leq$$
$$\leq L(n,n,p,\theta,P(p,\theta)x);$$
$$(L_2)\quad L(n,n,p,\theta,Q(p,\theta)x) + \sum_{j=n}^{m-1} Z\left(\left(\frac{\ln(m+1)}{\ln(j+1)}\right)^{\mu} ||\Phi(j,p,\theta)Q(p,\theta)x||\right) \leq$$
$$\leq L(m,n,p,\theta,Q(p,\theta)x),$$

for all $(m,n,p,\theta,x) \in T_d \times \Gamma$, $m > n$.

Theorem 3.2. *The pair (C,P) admits a uniform logarithmic dichotomy of Zabczyk type if and only if there are $Z \in \mathcal{Z}$, a Lyapunov function $L : T_d \times \Gamma \to \mathbb{R}_+$ for (C,P) and $K \geq 1$ with:*

(i) $L(n,n,p,\theta,P(p,\theta)x) \leq KZ(||\Phi(n,p,\theta)P(p,\theta)x||)$;
(ii) $L(m,n,p,\theta,Q(p,\theta)x) \leq KZ(||\Phi(m,p,\theta)Q(p,\theta)x||)$,

for all $(m,n,p,\theta,x) \in T_d \times \Gamma$.

Proof. Necessity. Let $L : T_d \times \Gamma \to \mathbb{R}_+$ be defined by

$$L(m,n,p,\theta,x) = \sum_{j=m}^{+\infty} Z\left(\left(\frac{\ln(j+1)}{\ln(n+1)}\right)^{\mu} ||\Phi(j,p,\theta)P(p,\theta)x||\right)$$
$$+ \sum_{j=n}^{m} Z\left(\left(\frac{\ln(m+1)}{\ln(j+1)}\right)^{\mu} ||\Phi(j,p,\theta)Q(p,\theta)x||\right),$$

where $Z \in \mathcal{Z}$ and $\mu > 0$ are given by Definition 3.2. It is immediate to see that the relations (L_1) and (L_2) from Definition 3.3 hold.

Thus,

$$L(n,n,p,\theta,P(p,\theta)x) = \sum_{j=n}^{+\infty} Z\left(\left(\frac{\ln(j+1)}{\ln(n+1)}\right)^{\mu} ||\Phi(j,p,\theta)P(p,\theta)x||\right)$$
$$\leq MZ(||\Phi(n,p,\theta)P(p,\theta)x||),$$

for all $(n,p,\theta,x) \in \Delta_d \times \Gamma$.

Also,

$$L(m,n,p,\theta,Q(p,\theta)x) = \sum_{j=n}^{m} Z\left(\left(\frac{\ln(m+1)}{\ln(j+1)}\right)^{\mu} ||\Phi(j,p,\theta)Q(p,\theta)x||\right)$$
$$\leq MZ(||\Phi(m,p,\theta)Q(p,\theta)x||),$$

for all $(m,n,p,\theta,x) \in T_d \times \Gamma$. Hence, we have that L is a Lyapunov function for the pair (C,P).

Sufficiency.
From Definition 3.3, (L_1), we have

$$\sum_{j=n}^{m-1} Z\left(\left(\frac{\ln(j+1)}{\ln(n+1)}\right)^{\mu} ||\Phi(j,p,\theta)P(p,\theta)x||\right) \leq L(n,n,p,\theta,P(p,\theta)x)$$
$$\leq KZ(||\Phi(n,p,\theta)P(p,\theta)x||), \text{ for all } (m,n,p,\theta,x) \in T_d \times \Gamma,\ m > n$$

and for $m \to +\infty$ it follows the condition $(uldZ_1)$.

Similarly, from (L_2), we deduce

$$\sum_{j=n}^{m-1} Z\left(\left(\frac{\ln(m+1)}{\ln(j+1)}\right)^{\mu} ||\Phi(j,p,\theta)Q(p,\theta)x||\right) \leq L(m,n,p,\theta,Q(p,\theta)x)$$
$$\leq KZ(||\Phi(m,p,\theta)Q(p,\theta)x||),$$

which implies

$$\sum_{j=n}^{m} Z\left(\left(\frac{\ln(m+1)}{\ln(j+1)}\right)^{\mu} ||\Phi(j,p,\theta)Q(p,\theta)x||\right) \leq (K+1)Z(||\Phi(m,p,\theta)Q(p,\theta)x||),$$

for all $(m,n,p,\theta,x) \in T_d \times \Gamma$.

For $n = p$, we obtain

$$\sum_{j=p}^{m} Z\left(\left(\frac{\ln(m+1)}{\ln(j+1)}\right)^{\mu} ||\Phi(j,p,\theta)Q(p,\theta)x||\right) \leq (K+1)Z(||\Phi(m,p,\theta)Q(p,\theta)x||).$$

Hence, the condition $(uldZ_2)$ is satisfied.

We conclude that (C,P) has a uniform logarithmic dichotomy of Zabczyk type. □

Corollary 3.3. *If there exist $Z \in \mathcal{Z}$, a Lyapunov function $L : T_d \times \Gamma \to \mathbb{R}_+$ for (C,P) and $K \geq 1$ such that the conditions (i) and (ii) from Theorem 3.2 hold, then (C,P) is uniformly logarithmic dichotomic.*

Proof. It is immediate from Theorem 3.2 and Proposition 3.1. □

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