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# On uniform polynomial splitting of variational nonautonomous difference equations in Banach spaces

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*Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary*

**Abstract** In this paper we consider a concept of uniform polynomial splitting for a discrete cocycle over a discrete semiflow in Banach spaces. We obtain some characterizations of Datko type and also in terms of Lyapunov functions. The study is made from the point of view of the invariant projectors and also strongly invariant projectors.

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## 1 Introduction

For infinite-dimensional theory of dynamical systems the notion of skew-product semiflow plays a central role. A considerable number of papers have been written dealing with various asymptotic behaviours for skew-product semiflow. Starting with [20] this problem was also studied in [6], [7], [14]. Important contributions in this direction were obtained by [16], [17] and [18] dealing with exponential stability respectively exponential instability concepts. Along this line, [13] deals with the exponential dichotomy concept. See also [19] and [12] for a fairly complete discussion with various applications.

Then, when we consider a dynamical system defined only on the positive integers it is natural that this one to be non-invertible. In this line, it is worth mentioning [1] and [15] which consider the splitting concept instead of exponential dichotomy. These articles deal with the case of difference equations. For the skew-product semiflow framework we refer to [4] and [5]. Together with the exponential concepts, polynomial ones have been considered. Various results have been obtained in [2], [3], [9], [10], [11]. In this line, the

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present work considers the concept of polynomial splitting for a cocycle over a discrete semiflow.

The paper is organized as follows. Section 2 is devoted to the preliminaries. Here, by using an example, the difference between invariant and strongly invariant projectors is established. In Section 3 we give some necessary and sufficient conditions for uniform polynomial splitting, first with the hypothesis that the projectors are invariant for the discrete cocycles and then for a strongly invariant family of projectors. In Section 4 the connection between uniform polynomial splitting and uniform polynomial dichotomy is proved. Next, Section 5 presents a logarithmic criteria for uniform polynomial dichotomy. The results are obtained from the point of view of the invariant projectors and also strongly invariant projectors. Finally, in Section 6 we obtain some results of Datko type for uniform polynomial splitting of cocycles (see [8]) and in Section 7 it is proved a characterization of uniform polynomial splitting in terms of Lyapunov functions.

## 2 Preliminaries

Let us denote by $X$ a metric space, by $V$ a Banach space and by $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators on $V$. The norm on $V$ and on $\mathcal{B}(V)$ will be denoted by $\|\cdot\|$. Let $I$ be the identity operator on $V$ and $Y = X \times V$.

**Definition 2.1.** A mapping $S : \mathbb{N} \times X \to X$ is called a *discrete semiflow* on $X$ if

$(ds_1)$ $S(0, x) = x$, for every $x \in X$;

$(ds_2)$ $S(m, S(n, x)) = S(m + n, x)$,

for all $(m, n, x) \in \mathbb{N}^2 \times X$.

**Definition 2.2.** A mapping $C : \mathbb{N} \times X \to \mathcal{B}(V)$ is called a *discrete cocycle* over the discrete semiflow $S : \mathbb{N} \times X \to X$ on the space $Y = X \times V$ if

$(dc_1)$ $C(0, x) = I$, for every $x \in X$;

$(dc_1)$ $C(m, S(n, x))C(n, x) = C(m + n, x)$;

for all $(m, n, x) \in \mathbb{N}^2 \times X$.

**Definition 2.3.** A mapping $P : X \to \mathcal{B}(V)$ is called a *family of projectors* on the Banach space $V$ if

$$P^2(x) = P(x), \text{ for every } x \in X.$$

**Definition 2.4.** A family of projectors $P : X \to \mathcal{B}(V)$ is calles *invariant* for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$ if

$$C(n, x)P(x) = P(S(n, x))C(n, x), \text{ for all } (n, x) \in \mathbb{N} \times X.$$

**Remark 2.1.** If the family of projectors $P : X \to \mathcal{B}(V)$ is invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$, then

$$Q : X \to \mathcal{B}(V) \quad \text{defined by} \quad Q(x) = I - P(x)$$

is a family of projectors (called the *complementary family of* $P$), which is also invariant for $C$.

**Definition 2.5.** A family of projectors $P : X \to \mathcal{B}(V)$ is *strongly invariant* for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$ if for all $(n, x) \in \mathbb{N} \times X$ the operator $C(n, x) : Q(x)X \to Q(S(n, x))$ is an isomorphism.

**Remark 2.2.** If the family of projectors $P : X \to \mathcal{B}(V)$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$ over the discrete semiflow $S : \mathbb{N} \times X \to X$, then there exists $D : \mathbb{N} \times X \to \mathcal{B}(V)$ such that for all $(n, x) \in \mathbb{N} \times X$ the bounded linear operator $D(n, x) : Q(S(n, x)) \to Q(x)X$ is an isomorphism. Moreover, the following relations hold:

$(D_1)$ $C(n, x)D(n, x)Q(S(n, x)) = Q(S(n, x))$

$(D_2)$ $D(n, x)C(n, x)Q(x) = Q(x)$

$(D_3)$ $Q(x)D(n, x)Q(S(n, x)) = D(n, x)Q(S(n, x))$,

for all $(n, x) \in \mathbb{N} \times X$.

**Proposition 2.1.** If the family of projectors $P : X \to \mathcal{B}(V)$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$, then the isomorphism $D : \mathbb{N} \times X \to \mathcal{B}(V)$ satisfies the evolution property:

$(D_4)$ $D(n + m, x)Q(S(n + m, x)) = D(m, x)D(n, S(m, x))Q(S(n + m, x))$,

for all $(n, m, x) \in \mathbb{N}^2 \times X$.

*Proof.* See [4] □

**Example 2.1.** Let $X = \mathbb{R}_+$ (with the euclidian metric), $V = \mathbb{R}^3$ with the norm

$$\|v\| = |v_1| + |v_2| + |v_3|, \ v = (v_1, v_2, v_3).$$

Then, $C : \mathbb{N} \times X \to \mathcal{B}(V)$ given by

$$C(n, x)v = \begin{cases} (v_1, v_2, v_3), & n = x = 0, \quad n = 0, x > 0 \quad or \quad n > 0, x > 0 \\ (v_1, 0, v_3), & x = 0 < n \end{cases}$$

is a discrete cocycle over the discrete semiflow $S : \mathbb{N} \times X \to X$, $S(n, x) = n + x$.
First, we have to prove that

$$C(m + n, x) = C(m, S(n, x))C(n, x) \Leftrightarrow C(m + n, x) = C(m, n + x)C(n, x).$$

For $m = n = 0$ the relation is obvious.
For $m = 0$ and $n \neq 0$ we obtain

$$C(n, x) = I \cdot C(n, x) = C(n, x).$$

For $m \neq 0$ and $n = 0$ we have that

$$C(m, x) = C(m, x) \cdot I = C(m, x).$$

Now, we suppose that $m \neq 0$ and $n \neq 0$. If $x = 0$ then

$$C(m + n, 0)v = (v_1, 0, v_3)$$

and

$$C(m, n)C(n, 0)v = C(m, n)(v_1, 0, v_3) = (v_1, 0, v_3).$$

If $x \neq 0$ then

$$C(m + n, x)v = (v_1, v_2, v_3) = v$$

and

$$C(m, n + x)C(n, x)v = C(m, n + x)v = v.$$

Hence, $C : \mathbb{N} \times X \to \mathcal{B}(V)$ is a discrete cocycle over the discret semiflow $S : \mathbb{N} \times X \to X$. We consider the family of projectors $P : X \to \mathcal{B}(V)$, defined by

$$P(x)v = \begin{cases} (v_1, v_2, 0), & x = 0 \\ (v_1 + v_2, 0, 0), & x > 0. \end{cases}$$

Now we check that

$$C(n, x)P(x) = P(S(n, x))C(n, x) \Leftrightarrow C(n, x)P(x) = P(n + x)C(n, x).$$

For $n = 0$ this equality is equivalent with $P(x) = P(x)$. Let $n \in \mathbb{N}^*$. For $x = 0$ we have to prove that

$$C(n, 0)P(0)v = P(n)C(n, 0)v.$$

It is obvious that

$$C(n, 0)P(0)v = C(n, 0)(v_1, v_2, 0) = (v_1, 0, 0)$$

and that

$$P(n)C(n, 0)v = P(n)(v_1, 0, v_3) = (v_1, 0, 0).$$

For $n > 0$ and $x > 0$ we have that

$$C(n, x)P(x)v = C(n, x)(v_1 + v_2, 0, 0) = (v_1 + v_2, 0, 0)$$

and

$$P(n + x)C(n, x)v = P(n + x)(v_1, v_2, v_3) = (v_1 + v_2, 0, 0).$$

Hence, we obtain that

$$C(n, x)P(x) = P(S(n, x))C(n, x).$$

We observe that

$$C(n,0)(0,0,v_3) = (0,0,v_3) \quad \text{and} \quad Q(S(n,0)) = Q(n).$$

Moreover,

$$Q(n)(w_1,w_2,w_3) = \begin{cases} (0,0,w_3), & n = 0 \\ (-w_2,w_2,w_3), & n > 0. \end{cases}$$

Hence, for $n > 0$,

$$C(n,0)(Q(0)V) = \{C(n,0)(0,0,w_3) : w_3 \in \mathbb{R}\} = \{(0,0,w_3) : w_3 \in \mathbb{R}\},$$

and

$$Q(n)V = \{(-w_2,w_2,w_3) : w_2,w_3 \in \mathbb{R}\}.$$

It follows that for $n > 0$ we have that $C(n,0) : Q(0)V \to Q(S(n,0))V = Q(n)V$ is not an isomorphism. Therefore the family of projectors $P : X \to \mathcal{B}(V)$ is not strongly invariant for the cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$.

## 3 Uniform polynomial splitting

Let $C : \mathbb{N} \times X \to \mathcal{B}(V)$ be a discrete cocycle over the discrete semiflow $S : \mathbb{N} \times X \to X$ and $P : X \to \mathcal{B}(V)$ a family of projectors which is invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$.

**Definition 3.1.** We say that the pair $(C,P)$ has *uniform polynomial splitting* if there exist three constants $N \geq 1$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that

$(ups_1)$ $\|C(n,x)P(x)v\| \leq N(n+1)^{\alpha}\|P(x)v\|$

$(ups_2)$ $(n+1)^{\beta}\|Q(x)v\| \leq N\|C(n,x)Q(x)v\|,$

for all $(n,x,v) \in \mathbb{N} \times Y$.

**Remark 3.1.** The constants $\alpha$ and $\beta$ are called *splitting rates*.

**Remark 3.2.** In particular, if $\alpha < 0 < \beta$, then we say that $(C,P)$ is *uniformly polynomially dichotomic*.

**Proposition 3.1.** The pair $(C,P)$ has uniform polynomial splitting if and only if there exist the constants $N \geq 1$ and $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ such that

$(ups_1')$ $\|C(m+n,x)P(x)v\| \leq N(m+1)^{\alpha}\|C(n,x)P(x)v\|$

$(ups_2')$ $(m+1)^{\beta}\|C(n,x)Q(x)v\| \leq N\|C(m+n,x)Q(x)v\|,$

for all $(m,n,x,v) \in \mathbb{N}^2 \times Y$.

*Proof. Necessity.* From $(ups_1)$ and $(ups_2)$ for $n = m$, $x = S(n, x)$ and $v = C(n, x)v$ we obtain that

$$\begin{aligned}
&\|C(m, S(n, x))P(S(n, x))C(n, x)v\| \leq N(m + 1)^{\alpha}\|P(S(n, x))C(n, x)v\| \Leftrightarrow\\
&\|C(m, S(n, x))C(n, x)P(x)v\| \leq N(m + 1)^{\alpha}\|C(n, x)P(x)v\| \Leftrightarrow\\
&\|C(m + n, x)P(x)v\| \leq N(m + 1)^{\alpha}\|C(n, x)P(x)v\|
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(m + 1)^{\beta}\|Q(S(n, x))C(n, x)v\| \leq N\|C(m, S(n, x))Q(S(n, x))C(n, x)v\| \Leftrightarrow\\
&(m + 1)^{\beta}\|C(n, x)Q(x)v\| \leq N\|C(m, S(n, x))C(n, x)Q(x)v\| \Leftrightarrow\\
&(m + 1)^{\beta}\|C(n, x)Q(x)v\| \leq N\|C(m + n, x)Q(x)v\|
\end{aligned}$$

*Sufficiency.* It follows from $(ups_1^{'})$ and $(ups_2^{'})$ for $n = 0$. □

**Proposition 3.2.** Let $P : X \to \mathcal{B}(V)$ be a family of projectors which is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. Then the pair $(C, P)$ has uniform polynomial splitting if and only if there exist $N \geq 1$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that

$(ups_1^{''})$ $\|C(n, x)P(x)v\| \leq N(n + 1)^{\alpha}\|P(x)v\|$

$(ups_2^{''})$ $(n + 1)^{\beta}\|D(n, x)Q((S(n, x))v\| \leq N\|Q((S(n, x))v\|,$

for all $(n, x, v) \in \mathbb{N} \times Y$.

*Proof.* It is sufficient to prove that $(ups_2)$ and $(ups_2^{''})$ are equivalent. First we prove that $(ups_2)$ implies $(ups_2^{''})$. Let $(n, x, v) \in \mathbb{N} \times Y$. Then we have

$$\begin{aligned}
(n + 1)^{\beta}\|D(n, x)Q((S(n, x))v\| &\leq (n + 1)^{\beta}\|Q(x)D(n, x)Q((S(n, x))v\|\\
&\leq N\|C(n, x)Q(x)D(n, x)Q((S(n, x))v\|\\
&= N\|Q(S(n, x))C(n, x)D(n, x)Q(S(n, x))v\|\\
&= N\|Q(S(n, x))Q(S(n, x))v\|\\
&= N\|Q(S(n, x))v\|
\end{aligned}$$

Now we prove that $(ups_2^{''})$ implies $(ups_2)$. It follows that

$$\begin{aligned}
(n + 1)^{\beta}\|Q(x, v)\| &= (n + 1)^{\beta}\|D(n, x)C(n, x)Q(x)v\|\\
&= (n + 1)^{\beta}\|D(n, x)Q(S(n, x))C(n, x)v\|\\
&\leq N\|Q(S(n, x))C(n, x)v\|\\
&= N\|C(n, x)Q(x)v\|.
\end{aligned}$$

□

## 4 The connection between uniform polynomial splitting and uniform polynomial dichotomy

In this section we show that the splitting rates $\alpha$ and $\beta$ could be replaced by the dichotomy rate $\delta$.

**Proposition 4.1.** The pair $(C, P)$ has uniform polynomial dichotomy if and only if there exist $N \geq 1$ and $\nu > 0$ such that

$(upd_1)$ $\|C(n,x)P(x)v\| \leq N(n+1)^{-\nu}\|P(x)v\|$

$(upd_2)$ $(n+1)^{\nu}\|Q(x)v\| \leq N\|C(n,x)Q(x)v\|,$

for all $(n, x, v) \in \mathbb{N} \times Y$.

*Proof. Necessity.* If the pair $(C, P)$ has uniform polynomial dichotomy then the relations $(ups_1)$ and $(ups_2)$ hold with $\alpha < 0 < \beta$. For $\nu = \min\{-\alpha, \beta\}$ we obtain $(upd_1)$ and $(upd_2)$.
*Sufficiency.* From $(upd_1)$ and $(upd_2)$ we obtain $(ups_1)$ and $(ups_2)$ with $\alpha < 0 < \beta$ if we consider $\beta = \nu = -\alpha$. □

**Proposition 4.2.** Let $P : X \to \mathcal{B}(V)$ be a family of projectors which is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. Then the pair $(C, P)$ has uniform polynomial dichotomy if and only if there exist $N \geq 1$ and $\nu > 0$ such that

$(upd_1^{'})$ $\|C(n,x)P(x)v\| \leq N(n+1)^{-\nu}\|P(x)v\|$

$(upd_2^{'})$ $(n+1)^{\nu}\|D(n,x)Q((S(n,x))v\| \leq N\|Q((S(n,x))v\|,$

for all $(n, x, v) \in \mathbb{N} \times Y$.

*Proof.* It is sufficient to prove that the relations $(upd_2)$ and $(upd_2^{'})$ are equivalent. Let $(n, x, v) \in \mathbb{N} \times Y$. First we prove that $(upd_2)$ implies $(upd_2^{'})$.

$$\begin{aligned}(n+1)^{\nu}\|D(n,x)Q((S(n,x))v\| &= (n+1)^{\nu}\|Q(x)D(n,x)Q((S(n,x))v\| \\ &\leq N\|C(n,x)D(n,x)Q((S(n,x))v\| \\ &= N\|Q((S(n,x))Q((S(n,x))v\| \\ &= N\|Q((S(n,x))v\|.\end{aligned}$$

Next, we prove the implication $(upd_2^{'}) \Rightarrow (upd_2)$. It follows that

$$\begin{aligned}(n+1)^{\nu}\|Q(x)v\| &= (n+1)^{\nu}\|D(n,x)C(n,x)Q(x)v\| \\ &= (n+1)^{\nu}\|D(n,x)Q((S(n,x))C(n,x)v\| \\ &\leq N\|Q((S(n,x))C(n,x)v\| \\ &= N\|C(n,x)Q(x)v\|\end{aligned}$$

□

**Remark 4.1.** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. We consider $\gamma = \frac{\alpha+\beta}{2}$ and $\delta = \frac{\beta-\alpha}{2}$ and we define the application $C^\gamma : \mathbb{N} \times X \to \mathcal{B}(V)$ by

$$C^\gamma(n,x) = (n+1)^{-\gamma} C(n,x), \quad \text{for all} \quad (n,x) \in \mathbb{N} \times X.$$

**Proposition 4.3.** The pair $(C,P)$ has uniform polynomial splitting with the splitting rates $\alpha$ and $\beta$ if and only if the pair $(C^\gamma, P)$ has uniform polynomial dichotomy with the dichotomy rate $\delta$.

*Proof. Necessity.* Suppose that the pair $(C,P)$ has uniform polynomial splitting with the splitting rates $\alpha$ and $\beta$ with $\alpha < \beta$. Then

$$\begin{aligned}\|C^\gamma(n,x)P(x)v\| &= (n+1)^{-\gamma}\|C(n,x)P(x)v\| \\ &\leq N(n+1)^{-\gamma}(n+1)^{\alpha}\|P(x)v\| \\ &= N(n+1)^{\alpha-\gamma}\|P(x)v\| \\ &= N(n+1)^{-\delta}\|P(x)v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Similarly,

$$\begin{aligned}N(n+1)^{\delta}\|Q(x)v\| &\leq N(n+1)^{\delta}(n+1)^{-\beta}\|C(n,x)Q(x)v\| \\ &= N(n+1)^{\delta-\beta}\|C(n,x)Q(x)v\| \\ &= N(n+1)^{-\gamma}\|C(n,x)Q(x)v\| \\ &= N\|C^\gamma(n,x)Q(x)v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$.
*Sufficiency.* Suppose now that the pair $(C^\gamma, P)$ has uniform polynomial dichotomy with the dichotomy rate $\delta$. Then

$$\begin{aligned}\|C(n,x)P(x)v\| &= (n+1)^{\gamma}\|C^\gamma(n,x)P(x)v\| \\ &\leq N(n+1)^{\gamma}(n+1)^{-\delta}\|P(x)v\| \\ &= N(n+1)^{\gamma-\delta}\|P(x)v\| \\ &= N(n+1)^{\alpha}\|P(x)v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Similarly,

$$\begin{aligned}(n+1)^{\beta}\|Q(x)v\| &\leq N(n+1)^{\beta}(n+1)^{-\delta}\|C^\gamma(n,x)Q(x)v\| \\ &= N(n+1)^{\beta-\delta}(n+1)^{-\gamma}\|C(n,x)Q(x)v\| \\ &= N\|C(n,x)Q(x)v\|,\end{aligned}$$

for all $(n,x,v) \in \mathbb{N} \times Y$. □

**Remark 4.2.** We observe that the pair $(C^\gamma, P)$ has uniform polynomial dichotomy with the rate $\delta$ if and only if there exists $N \geq 0$ such that

$(upd_1'')$ $\|C^\gamma(n,x)P(x)v\| \leq N(n+1)^{-\delta}\|P(x)v\|$

$(upd_2'')$ $(n+1)^{\delta}\|Q(x)v\| \leq N\|C^\gamma(n,x)Q(x)v\|,$

for all $(n, x, v) \in \mathbb{N} \times Y$.

**Remark 4.3.** We note that if the family of projectors $P : X \to \mathcal{B}(V)$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$ then the pair $(C^\gamma, P)$ has uniform polynomial dichotomy with the dichotomy rate $\delta$ if and only if there exists $N \geq 1$ such that

$(upd_1''')$ $\|C^\gamma(n, x)P(x)v\| \leq N(n+1)^{-\delta}\|P(x)v\|$

$(upd_2''')$ $(n+1)^\delta\|D^\gamma(n, x)Q(S(n, x))v\| \leq N\|Q(S(n, x))v\|,$

for all $(n, x, v) \in \mathbb{N} \times Y$, where

$$D^\gamma(n, x) = (n+1)^{-\gamma}D(n, x)$$

for all $(n, x) \in \mathbb{N} \times X$.

## 5 Logarithmic criteria

**Remark 5.1.** Let $\theta > 0$ and the function $g : (1, +\infty) \to \mathbb{R}$ defined by $g(t) = \frac{\ln t}{t^\theta}$. Then

$$0 < g(t) \leq \frac{1}{\theta e},$$

for all $t \in (1, +\infty)$. So,

$$0 < \ln t < \frac{t^\theta}{\theta e}$$

or equivalently

$$0 < \theta e \ln t < t^\theta,$$

for all $t \in (1, +\infty)$.

**Definition 5.1.** We say that the pair $(C^\gamma, P)$ is *uniformly logarithmic dichotomic* if there exists $D \geq 1$ such that

$(uld_1)$ $\|C^\gamma(n, x)P(x)v\| \ln(n+1) \leq D\|P(x)v\|$

$(uld_2)$ $\|Q(x)v\| \ln(n+1) \leq D\|C^\gamma(n, x)Q(x)v\|,$

for all $(n, x, v) \in \mathbb{N} \times Y$.

**Proposition 5.1.** The pair $(C^\gamma, P)$ is uniformly polinomial dichotomic with the rate $\delta$ if and only if the pair $(C^\gamma, P)$ is uniformly logarithmic dichotomic.

*Proof. Necessity.* Let $(n, x, v) \in \mathbb{N} \times Y$. From the inequality

$$\|C^\gamma(n, x)P(x)v\| \leq N(n+1)^{-\delta}\|P(x)v\|$$

we deduce that

$$\|C^\gamma(n, x)P(x)v\| \ln(n+1) \leq N\frac{\ln(n+1)}{(n+1)^\delta}\|P(x)v\| \leq \frac{N}{\delta\varepsilon}\|P(x)v\| \leq D\|P(x)v\|.$$

In a similar way, from the inequality

$$\|Q(x)v\| \leq N(n+1)^{-\delta}\|C^{\gamma}(n,x)Q(x)v\|$$

we obtain that

$$\begin{aligned}\|Q(x)v\|\ln(n+1) &\leq N\frac{\ln(n+1)}{(n+1)^{\delta}}\|C^{\gamma}(n,x)Q(x)v\| \\ &\leq \frac{N}{\delta\varepsilon}\|C^{\gamma}(n,x)Q(x)v\| \\ &\leq D\|C^{\gamma}(n,x)Q(x)v\|.\end{aligned}$$

*Sufficiency.* The relations $(upd_1^{''})$ and $(upd_2^{''})$ are obvious for $n=0$. Further, we consider that $n \geq 1$. Thus

$$\begin{aligned}\|C^{\gamma}(n,x)P(x)v\| &\leq \frac{D}{\ln(n+1)}\|P(x)v\| \\ &= \frac{D}{(n+1)^{\delta}\ln(n+1)}(n+1)^{-\delta}\|P(x)v\| \\ &\leq K(n+1)^{-\delta}\|P(x)v\|\end{aligned}$$

and

$$\begin{aligned}\|Q(x)v\| &\leq \frac{D}{\ln(n+1)}\|C^{\gamma}(n,x)Q(x)v\| \\ &= \frac{D}{(n+1)^{\delta}\ln(n+1)}(n+1)^{-\delta}\|C^{\gamma}(n,x)Q(x)v\| \\ &\leq K(n+1)^{-\delta}\|C^{\gamma}(n,x)Q(x)v\|.\end{aligned}$$

□

**Proposition 5.2.** If the family of projectors $P : X \to \mathcal{B}(V)$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$ then the pair $(C^{\gamma}, P)$ has uniform polynomial dichotomy with the rate $\delta$ if and only if there exists $D \geq 1$ such that

$(uld_1^{'})$ $\|C^{\gamma}(n,x)P(x)v\|\ln(n+1) \leq D\|P(x)v\|$

$(uld_2^{'})$ $\|D^{\gamma}(n,x)Q(S(n,x))v\|\ln(n+1) \leq D\|Q(S(n,x))v\|,$

for all $(n,x,v) \in \mathbb{N} \times Y$.

## 6 Theorems of Datko type

We remind that the function $\zeta : (0,+\infty) \to (1,+\infty)$ defined for all $\theta \in (0,+\infty)$ by

$$\zeta(\theta) = \sum_{j=1}^{+\infty}\frac{1}{j^{1+\theta}}$$

is called *the Riemann zeta function.*

**Proposition 6.1.** If the pair $(C^\gamma, P)$ has uniform polynomial splitting then there exist the constants $D \geq 1$ and $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ such that

$$(dups_1) \quad \sum_{k=n}^{+\infty} (k-n+1)^{-\mu-1} \|C(k,x)P(x)v\| \leq D\|C(n,x)P(x)v\|$$

$$(dups_2) \quad \sum_{k=0}^{n} (n-k+1)^{\gamma-1} \|C(k,x)Q(x)v\| \leq D\|C(n,x)Q(x)v\|,$$

for all $(n, x, v) \in \mathbb{N} \times Y$.

*Proof.* Let $N \geq 1$ and $\alpha, \beta \in \mathbb{R}$ such that the relations $(ups_1)$ and $(ups_2)$ from Definition 3.1 hold. Let $(n, x, v) \in \mathbb{N} \times Y$ and $k \in \mathbb{N}$. If $n \leq k$ then from $(ups_1^{'})$ it follows that

$$\|C(k,x)P(x)v\| \leq N(k-n+1)^{\alpha}\|C(n,x)P(x)v\|,$$

and if $n \geq k$ then from $(ups_2^{'})$ we deduce that

$$\|C(k,x)Q(x)v\| \leq N(n-k+1)^{-\beta}\|C(n,x)Q(x)v\|.$$

If $\alpha < \beta$ we can take $\mu, \nu \in \mathbb{R}$ such that $\alpha < \mu < \nu < \beta$. We consider

$$D = N(1 + \zeta(\mu - \alpha) + \zeta(\beta - \nu)).$$

It follows that

$$(dups_1) \quad \sum_{k=n}^{+\infty} (k-n+1)^{-\mu-1} \|C(k,x)P(x)v\| \leq N\|C(n,x)P(x)v\| \sum_{k=n}^{+\infty} \frac{1}{(k-n+1)^{1+\mu-\alpha}} \leq D\|C(n,x)P(x)v\|$$

and

$$(dups_2) \quad \sum_{k=0}^{n} (n-k+1)^{\nu-1} \|C(k,x)Q(x)v\| \leq N\|C(n,x)Q(x)v\| \sum_{k=0}^{n} \frac{1}{(n-k+1)^{1+\beta-\nu}} \leq D\|C(n,x)Q(x)v\|$$

□

**Proposition 6.2.** Suppose that for the pair $(C, P)$ there exist the constants $D \geq 1$ and $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ such that the relations $(dups_1)$ and $(dups_2)$ hold. If the inequality $2 < \nu - \mu$ holds then the pair $(C, P)$ admits uniform polynomial splitting with the splitting constants $\mu + 1$ and $\nu - 1$.

*Proof.* Let $(x, v) \in X \times V$. For $n = 0$ and $k \in \mathbb{N}$ from $(dups_1)$ it follows that

$$\|C(k,x)P(x)v\| \leq D(k+1)^{\mu+1}\|P(x)v\|.$$

Now for $k = 0$ and $n \in \mathbb{N}$ from $(dups_2)$ it follows that

$$(n+1)^{\nu-1}\|Q(x)v\| \leq D\|C(n,x)Q(x)v\|.$$

□

**Theorem 6.1.** The pair $(C, P)$ has uniform polynomial splitting with the splitting rates $\alpha, \beta \in \mathbb{R}$ which satisfy the condition $2 < \beta - \alpha$ if and only if there exist the constants $D \geq 1$ and $\mu, \nu \in \mathbb{R}$ with $2 < \nu - \mu$ such that the relations $(dups_1)$ and $(dups_2)$ hold.

*Proof.* It follows from Proposition 6.1 and Proposition 6.2. □

**Proposition 6.3.** Suppose that the family of projectors $P : X \to \mathcal{B}(V)$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. If the pair $(C, P)$ admits uniform polynomial splitting then there exist the constants $D \geq 1$ and $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ such that

$$(dups_1^{'}) \quad \sum_{k=n}^{+\infty} (k-n+1)^{-\mu-1} \|C(k,x)P(x)v\| \leq D\|C(n,x)P(x)v\|$$

$$(dups_2^{'}) \quad \sum_{k=0}^{n} (n-k+1)^{\nu-1} \|D(n-k,x)S(k,x)Q((S(n,x))x)v\| \leq D\|Q((S(n,x))v\|$$

for all $(n, x, v) \in \mathbb{N} \times Y$.

*Proof.* Replacing $n$ with $n-k$ and $x$ with $S(k,x)$ in $(ups_2^{''})$ we obtain that

$$(n-k+1)^{\beta} \|D(n-k)S(k,n)Q(S(n-k,S(k,x)))v\| \leq N\|Q(S(n-k,S(k,x)))v\|$$

or equivalent

$$(n-k+1)^{\beta} \|D(n-k,S(k,x))Q(S(n,x))v\| \leq N\|Q(S(n,x))v\|.$$

Next we proceed as in the proof of the Proposition 6.1. The details are omitted. □

**Proposition 6.4.** Suppose that the family of projectors $P : X \to \mathcal{B}(V)$ is strongly invariant for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. Also, we suppose that for the pair $(C, P)$ there exist the constants $D \geq 1$ and $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ such that the relations $(dups_1^{'})$ and $(dups_2^{'})$ hold. If the inequality $2 < \nu - \mu$ hold then the pair $(C, P)$ admits uniform polynomial splitting with the splitting constants $\mu + 1$ and $\nu - 1$.

*Proof.* The proof is similarly with the proof of Proposition 6.2. □

**Theorem 6.2.** Let $P : X \to \mathcal{B}(V)$ be a strongly invariant family of projectors for the discrete cocycle $C : \mathbb{N} \times X \to \mathcal{B}(V)$. The pair $(C, P)$ admits uniform polynomial splitting with the rates $\alpha, \beta \in \mathbb{R}$ with $2 < \beta - \alpha$ if and only if there exist the constant $D \geq 1$ and $\mu, \nu \in \mathbb{R}$ with $2 < \nu - \mu$ such that the relations $(dups_1^{'})$ and $(dups_2^{'})$ hold.

*Proof.* It follows from Proposition 6.3 □

## 7 Lyapunov functions

**Definition 7.1.** A mapping $L : \mathbb{N} \times Y \to \mathbb{R}_+$ is called a *Lyapunov function* for the pair $(C, P)$ if there exist two constants $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ such that the following inequalities hold:

$(l_1)$ $L(n,x,P(x)v) + \sum_{k=0}^{n-1}(k+1)^{-c_1-1}\|C(k,x)P(x)v\| \leq L(0,x,P(x)v)$

$(l_2)$ $L(0,x,Q(x)v) + \sum_{k=0}^{n-1}(n-k+1)^{c_2-1}\|C(k,x)Q(x)v\| \leq L(n,x,Q(x)v),$

for all $(n,x,v) \in \mathbb{N}^* \times Y$.

**Proposition 7.1.** If the pair $(C,P)$ has uniform polynomial splitting then there exists a Lyapunov function for the pair $(C,P)$ and a constant $D \geq 1$ such that

(i) $\|L(0,x,P(x)v)\| \leq D\|P(x)v\|$

(ii) $L(n,x,Q(x)v) \leq \|C(n,x)Q(x)v\|,$

for all $(n,x,v) \in \mathbb{N} \times Y$.

*Proof.* Let $\mu$ and $\nu$ be the constants given by Proposition 6.1. We consider $L : \mathbb{N} \times Y \to \mathbb{R}_+$ defined by

$$L(n,x,v) = \sum_{k=n}^{+\infty}(k+1)^{-\mu-1}\|C(k,x)P(x)v\| + \sum_{k=0}^{n}(n-k+1)^{\nu-1}\|C(k,x)Q(x)v\|$$

for all $(n,x,v) \in \mathbb{N} \times Y$. Let $(n,x,v) \in \mathbb{N}^* \times Y$. Then

$$\begin{aligned} L(0,x,P(x)v) &= \sum_{k=0}^{+\infty}(k+1)^{-\mu-1}\|C(k,x)P(x)v\| \\ &= \sum_{k=0}^{n-1}(k+1)^{-\mu-1}\|C(k,x)P(x)v\| + \sum_{k=n}^{+\infty}(k+1)^{-\mu-1}\|C(k,x)P(x)v\| \\ &= \sum_{k=0}^{n-1}(k+1)^{-\mu-1}\|C(k,x)P(x)v\| + L(n,xP(x)x), \end{aligned}$$

so inequality $(l_1)$ holds. In addition,

$$\begin{aligned} L(0,x,P(x)v) &= \sum_{k=0}^{+\infty}(k+1)^{-\mu-1}\|C(k,x)P(x)v\| \\ &\leq N\|P(x)v\|\sum_{k=0}^{+\infty}\frac{1}{(k+1)^{1+\mu-\alpha}} \leq D\|P(x)v\| \end{aligned}$$

so inequality $(i)$ holds. Next we have that

$$\begin{aligned} L(n,x,Q(x)v) - L(0,x,Q(x)v) &= \sum_{k=0}^{n}(n-k+1)^{\gamma-1}\|C(k,x)Q(x)v\| - \|Q(x)v\| \\ &= \sum_{k=0}^{n}(n-k+1)^{\gamma-1}\|C(k,x)Q(x)v\| \\ &\geq \sum_{k=0}^{n-1}(n-k+1)^{\gamma-1}\|C(k,x)Q(x)v\|, \end{aligned}$$

so inequality $(l_2)$ holds. Moreover, from Proposition 6.1 we obtain that

$$L(n,x,Q(x)v)=\sum_{k=0}^{n}(n-k+1)^{\gamma-1}\|C(k,x)Q(x)v\|\leq D\|C(n,x)Q(x)v\|,$$

so inequality $(ii)$ holds. □

**Proposition 7.2.** Suppose that for the pair $(C,P)$ there exists a Lyapunov function for which the constants $c_1,c_2\in\mathbb{R}$ verify $2<c_2-c_1$ and a constant $D\geq 1$ such that the inequalities $(i)$ and $(ii)$ hold. Then the pair $(C,P)$ admits uniform polynomial splitting with the splitting constants $1+c_1$ and $c_2-1$.

*Proof.* Let $(n,x,v)\in\mathbb{N}^*\times Y$. From $(l_1)$ and $(i)$ we deduce that

$$\sum_{k=0}^{n-1}(k+1)^{-c_1-1}\|C(k,x)P(x)v\|\leq L(0,x,P(x)v)\leq D\|P(x)v\|.$$

For $n\to+\infty$ we obtain that

$$\sum_{k=0}^{+\infty}(k+1)^{-c_1-1}\|C(k,x)P(x)v\|\leq D\|P(x)v\|,$$

so

$$\|C(k,x)P(x)v\|\leq D(k+1)^{1+c_1}\|P(x)v\|.$$

Now from $(l_2)$ and $(ii)$ we deduce that

$$\begin{aligned}\sum_{k=0}^{n}(n-k+1)^{c_2-1}\|C(k,x)Q(x)v\|&=\sum_{k=0}^{n-1}(n-k+1)^{c_2-1}\|C(k,x)Q(x)v\|+\|C(n,x)Q(x)v\|\\&\leq L(n,x,Q(x)v)+\|C(n,x)Q(x)v\|\\&\leq(D+1)\|C(n,x)Q(x)v\|.\end{aligned}$$

For $k=0$ we obtain that

$$(n+1)^{c_2-1}\|Q(x)v\|\leq(D+1)\|C(n,x)Q(x)v\|.$$

□

**Theorem 7.1.** The pair $(C,P)$ has uniform polynomial splitting with the splitting rates $\alpha,\beta$ which verify $2<\beta-\alpha$ if and only if there exist $D\geq 1$ and a Lyapunov function with the constants $c_1,c_2\in\mathbb{R}$ which verify $2<c_2-c_1$ such that the inequalities $(i)$ and $(ii)$ hold.

*Proof.* It follows from Proposition 7.1 and Proposition 7.2. □

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