sciendo

DOI: 10.2478/awutm-2022-0010

Annals of West University of Timisoara
Mathematics and Computer Science
58, 1, (2022), 126– 136

An explicit second order uniformly convergent difference algorithm for an initial value problem associated to second order differential equations

Pramod Kumar Pandey, Ishver Rani Manchanda, and Archana Yadav

Abstract We consider the development of the direct method for the numerical solution of second order differential equations and corresponding initial value problems. Our technique based on the method of finite difference approximations. We obtain a quadratic order accurate an explicit method using the set of initial conditions in a natural way and approximations for the approximate numerical solution after the discretization of the continuous problem under appropriate conditions. We discuss the development of the method and the periodic property of the solution of the problem. In the numerical experiment, we consider both linear and nonlinear model problems to test the efficiency and accuracy of the method. The tabulated numerical results in computational experiments approve the quadratic order accuracy and efficiency of the method.

AMS Subject Classification (2020) 65L10; 65L12
Keywords Explicit method; Finite difference method; Nonlinear problem; Periodic property; Quadratic order Phase-lag.

1 Introduction

To describe a problem in natural sciences and technology, in general differential equations a mathematical term used. It is important and interesting to explain and solve the modelled problem under given conditions.

For the development and discussion of the explicit finite difference method, in this article we have considered following second order nonlinear differential equation, which occurs during the study of deflection of an elastic string / beam under effect of weight

$$u''(x) = f(x, u, u'), \quad x \in [a, b] \tag{1.1}$$

© 2022 Pramod Kumar Pandey et al. This is an open access article licensed under the Creative Commons Attribution-NonCommercial-NoDerivs License (http://creativecommons.org/licenses/by-nc-nd/3.0/).

subject to the initial conditions

$$u(0) = \alpha, \quad u'(0) = \alpha_1 \tag{1.2}$$

where α, and α_1 are known finite constants.

We have assumed the uniform continuity of the function $f(x, u, u')$ in $[a, b] \times \mathrm{R} \times \mathrm{R}$ to ensure the existence and uniqueness of the problem (1.1). In detail the discussion on the concept of existence and uniqueness of analytical solution of the problem (1.1) can be found in [3, 5, 8]. Further, without any other assumption we presume the existence and uniqueness of the solution of the problem (1.1).

In general, it is difficult/impossible to achieve the acceptable analytical solution of these important problems. We face this problem, usually rely on approximate numerical solution adopting available other mathematical techniques are shooting method [13, 14], multi-step methods [10–12], Runge-Kutta [7], decomposition method [1], interpolating polynomial method [18, 19]. Though, these methods are efficient and accurate but not developed as direct finite difference method for the approximate solution of reference problem (1.1).

The emphasis in this article will be on the explicit direct method for the approximate solution of the reference problem (1.1) using finite differences. There are standard methods reported in the literature for the problem (1.1) by finite differences by reducing problem (1.1) to a system of equations of the first order. So, there are not enrich the work reported in the literature on direct method for the approximate numerical solution of problem (1.1). In this article, we develop a direct explicit finite difference method that exactly satisfies all initial conditions for the accurate numerical solution of problem (1.1).

The work in this article presented as: In the section 2, there we provided a hybrid finite difference method for the approximate numerical solution of the problem (1.1). We have discussed in detail the derivation of the method in the Section 3 and the phase-lag analysis of the proposed method under appropriate conditions in the Section 4. The numerical experiment result discussion of the proposed method applied to test the efficiency and accuracy in the Section 5. A concluding discussion on the performance of the proposed method presented in Section 6.

2 The Explicit Difference Method

Considered problem (1.1) is continuous in the domain $[a,b]$. For the approximate numerical solution of the problem (1.1), we transform it into discrete problem. So, we generate N discrete nodal points in the domain $[a,b]$ i.e. $a = x_0 < x_1 < .. < x_{N-1} < x_N = b$. Let us define uniform step length h between two successive nodes $x_i = a+ih, \quad i = 0, 1, 2, ..., N$ i.e. $h = \frac{b-a}{N}$. We wish to determine the numerical approximation of the theoretical solution $u(x)$ of the problem (1.1) at these discrete nodal points. Let us denote u_i and f_i respectively the numerical approximation of $u(x)$ and $f(x, u, u')$ at the $x = x_i$ nodal point. Let us convert continuous boundary value problem (1.1) into a discrete problem at defined node $x = x_i$ as,

$$u_i'' = f_i, \quad a \le x_i \le b, \tag{2.1}$$

and initial conditions

$$u_0 = \alpha, \quad \text{and} \qquad u_0' = \alpha_1 \tag{2.2}$$

We presumed the property of existence and uniqueness of the solution of continuous problem (1.1) necessarily transfer to the discrete problem (1.2) as work reported in the literature of [2,9]. Let us define nodes $x_{i-\frac{1}{2}} = x_i - \frac{h}{2}$ and following approximations,

$$\overline{u}_{i-\frac{1}{2}} = u_{i-1} + \frac{h}{2}u'_{i-1} + \frac{h^2}{8}f_{i-1}, \qquad i = 1 \tag{2.3}$$

$$\overline{u}'_{i-\frac{1}{2}} = u'_{i-1} + \frac{h}{2}f_{i-1}, \qquad i = 1 \tag{2.4}$$

$$\overline{u}'_{i-\frac{1}{2}} = \frac{1}{h}(u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}}), \quad 2 \le i \le n-1 \tag{2.5}$$

$$\overline{f}_{i-\frac{1}{2}} = \begin{cases} f(x_{i-\frac{1}{2}}, \overline{u}_{i-\frac{1}{2}}, \overline{u}'_{i-\frac{1}{2}}), & i = 1 \\ f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}, \overline{u}'_{i-\frac{1}{2}}), & 2 \le i \le n-1 \end{cases} \tag{2.6}$$

$$\overline{\overline{u}}'_{i-\frac{1}{2}} = \overline{u}'_{i-\frac{1}{2}} + \frac{h}{2}\overline{f}_{i-\frac{1}{2}}, \quad 2 \le i \le n-1 \tag{2.7}$$

$$\overline{\overline{f}}_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}, \overline{\overline{u}}'_{i-\frac{1}{2}}), \quad 2 \le i \le n-1 \tag{2.8}$$

$$\widehat{u}_{i+\frac{1}{2}} = 2u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}} + h^2\overline{\overline{f}}_{i-\frac{1}{2}}, \quad 2 \le i \le N-1 \tag{2.9}$$

$$\widehat{u}'_{i+\frac{1}{2}} = \overline{u}'_{i-\frac{1}{2}} + \frac{3h}{2}\overline{f}_{i-\frac{1}{2}}, \quad 2 \le i \le N-1 \tag{2.10}$$

$$\widehat{f}_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, \widehat{u}_{i+\frac{1}{2}}\widehat{u}'_{i+\frac{1}{2}}), \quad 2 \le i \le N-1 \tag{2.11}$$

We pursue the thought in [15–17] and using approximations (2.3)-(2.11), our propose discretization technique for the problem (2.1) in $(a,b]$ at nodes $x_{i+\frac{1}{2}}$ is given as,

$$u_{i+\frac{1}{2}} = \begin{cases} \overline{u}_{i+\frac{1}{2}} + \frac{h^2}{24}(\overline{f}_{i+\frac{1}{2}} - f_i), & i = 0 \\ 3u_{i-\frac{1}{2}} - 2u_{i-1} + \frac{h^2}{4}(4\overline{f}_{i-\frac{1}{2}} - f_{i-1}) & i = 1 \\ 2u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}} + \frac{h^2}{12}(\widehat{f}_{i+\frac{1}{2}} + 10\overline{\overline{f}}_{i-\frac{1}{2}} + \overline{f}_{i-\frac{3}{2}}), & i = 2 \\ 2u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}} + \frac{h^2}{12}(\widehat{f}_{i+\frac{1}{2}} + 10\overline{\overline{f}}_{i-\frac{1}{2}} + \overline{\overline{f}}_{i-\frac{3}{2}}), & 3 \le i \le N-1 \end{cases} \tag{2.12}$$

and truncation error in (2.12): $T_{i+\frac{1}{2}}$ is

$$T_{i+\frac{1}{2}} =$$

$$= -\frac{h^4}{1152}\begin{cases} (6u^{(4)}_{i+\frac{1}{2}} + (AA_0)_{i+\frac{1}{2}}), & i = 0 \\ 6(15u^{(4)}_{i+\frac{1}{2}} + 4h(AA_0)_{i-\frac{1}{2}}), & i = 1 \\ 2(4(h^2(u^{(4)} - (AA_1)_{i-\frac{1}{2}})\frac{\partial f}{\partial u} - AA_2)_{i+\frac{1}{2}} - 40(AA_1)_{i-\frac{1}{2}} + (AA_0)_{i-\frac{3}{2}}), & i = 2 \\ 8((h^2(u^{(4)} - (AA_1)_{i-\frac{1}{2}})\frac{\partial f}{\partial u} - AA_2)_{i+\frac{1}{2}} - 10(AA_1)_{i-\frac{1}{2}} - (AA_1)_{i-\frac{3}{2}}), & 3 \le i \le N-1 \end{cases} \tag{2.13}$$

where

$$AA_0 = hu^{(3)}\frac{\partial f}{\partial u} + (6u^{(3)} + hu^{(4)})\frac{\partial f}{\partial u'}$$
$$AA_1 = (2u^{(3)} + (-3u'' + hu^{(3)})\frac{\partial f}{\partial u'})\frac{\partial f}{\partial u'}$$
$$AA_2 = (-2u^{(3)} + hu^{(4)})\frac{\partial f}{\partial u'}$$

Thus, we obtained the N linear equations. However, proposed method (2.12) is an explicit method. The solution of equation is the approximate solution of the problem (1.1) at the discrete point of the domain of the problem. We have incorporated initial boundary conditions in a natural way in our proposed method (2.12).

We have computed the approximate solution u_i of the problem (1.1) using following second order accurate difference approximation,

$$u_i = \frac{1}{2}\begin{cases}(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}), & 1 \le i \le N-1\\ (3u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}}), & i = N\end{cases} \tag{2.14}$$

3 Derivation of the Method

We consider the derivation of the proposed explicit finite difference method (2.12). First, we consider the approximation of $u_{i-\frac{1}{2}}$ for $i = 1$, so let

$$\overline{u}_{i-\frac{1}{2}} = a_0 u_{i-1} + a_1 h u'_{i-1} + a_2 h^2 f_{i-1}, \quad i = 1$$

Using Taylor series expansion of each term about nodal point $x_{i-\frac{1}{2}}$ and comparing the coefficients of h^p in so obtained series. We obtained a system of equations in these a_0, a_1 and a_2 constants. For determining the coefficients a_0, a_1 and a_2, we solve the system of equations. It is easy to find,

$$\overline{u}_{i-\frac{1}{2}} = u_{i-1} + \frac{h}{2}u'_{i-1} + \frac{h^2}{8}f_{i-1} + T_{i-\frac{1}{2}}(h) \tag{3.1}$$

and

$$T_{i-\frac{1}{2}}(h) = -\frac{h^3}{48}u^{(3)}_{i-\frac{1}{2}} + \frac{h^4}{128}u^{(4)}_{i-\frac{1}{2}} + O(h^5) \tag{3.2}$$

Thus, $\overline{u}_{i-\frac{1}{2}}$ provides an approximation for $u_{i-\frac{1}{2}}$. Following, similar procedure we obtained

$$\overline{u}'_{i-\frac{1}{2}} = u'_{i-1} + \frac{h}{2}f_{i-1} + T'_{i-\frac{1}{2}}(h) \tag{3.3}$$

and

$$T'_{i-\frac{1}{2}}(h) = -\frac{h^2}{8}u^{(3)}_{i-\frac{1}{2}} + \frac{h^3}{24}u^{(4)}_{i-\frac{1}{2}} + O(h^4) \tag{3.4}$$

Hence we find $\overline{u}'_{i-\frac{1}{2}}$ provides an approximation for $u'_{i-\frac{1}{2}}$. Using (3.3)-(3.4) in (2.6), we obtained

$$\overline{f}_{i-\frac{1}{2}} = f_{i-\frac{1}{2}} + T'_{i-\frac{1}{2}}(h)H_{i-\frac{1}{2}} \tag{3.5}$$

where $H = \frac{\partial f}{\partial u'}$. So, $\overline{f}_{i-\frac{1}{2}}$ provides an approximation for $f_{i-\frac{1}{2}}$. For the discretization of the problem (1.1) at the node $x_{i+\frac{1}{2}}$, we consider following linear combination of solution and source function at some nodal points,

$$u_{i+\frac{1}{2}} = b_0 u_{i-\frac{1}{2}} + b_1 u_{i-1} + \frac{h^2}{4}(b_2 \overline{f}_{i-\frac{1}{2}} + b_3 f_{i-1}), \quad i = 1 \tag{3.6}$$

Using (3.5) and write Taylor series expansion of each term about $x_{i+\frac{1}{2}}$ in (3.6), compare coefficients of h^p in so obtained series. Thus, we obtained

$$u_{i+\frac{1}{2}} = 3u_{i-\frac{1}{2}} - 2u_{i-1} + \frac{h^2}{4}(4f_{i-\frac{1}{2}} - f_{i-1}) + T_{i+\frac{1}{2}}, \quad i = 1 \tag{3.7}$$

and

$$T_{i+\frac{1}{2}} = -\frac{5h^4}{64} u^{(4)}_{i+\frac{1}{2}} + \frac{h^2}{2}(T_{i-\frac{1}{2}} G_{i-\frac{1}{2}} + T'_{i-\frac{1}{2}} H_{i-\frac{1}{2}}) \tag{3.8}$$

where $G = \frac{\partial f}{\partial u}$. We have compared coefficients of $h^p, p = 0, \cdots, 3$ in determining the constants in (3.6) implies that the leading truncating term is at least of $O(h^4)$. Following, above described techniques and procedure, we can easily derive other equations in (2.12). Thus, we conclude the order of accuracy of the proposed method (2.12)is at least quadratic i.e. the discretization of the problem (1.1) at nodes $x_{i+\frac{1}{2}}$, $i = 0, 1, ..., N-1$ is at least of $O(h^2)$ accurate.

4 The Phase Analysis

We will discuss the nature of approximate solution of the following linear initial value problem,

$$u''(x) = f(x, u), \quad x \in [a, b] \tag{4.1}$$

subject to the initial conditions

$$u(0) = \alpha, \quad u'(0) = \alpha_1 \tag{4.2}$$

where α, and α_1 are known finite constants.

The problem (4.1) may obtained in the form

$$u''(x) = -\lambda^2 u(x) \tag{4.3}$$

Applying (2.12) to the problem (4.3) and using (2.9) we get second order difference equation and corresponding to so obtained second order difference equation following stability polynomial,

$$\rho(\xi) = P_1(H)\xi^2 - 2P_2(H)\xi + P_1(H), \tag{4.4}$$

where

$$P_1(H) = 1 + \frac{H^2}{12}, \quad P_2(H) = 1 - \frac{5H^2}{12}, \quad H = h\lambda.$$

The nature of the approximate solution of the problem (4.3) depends on the roots of characteristic equation corresponding to (4.4). Let the roots of characteristic equation (4.4) in form

$$\xi = \exp(\pm i\widetilde{H}) \tag{4.5}$$

and $\widetilde{H}$ is an approximation of H. Thus, we find the relation between the roots of characteristic equation and (4.5),

$$\tan(\widetilde{H}) = \frac{\sqrt{P_1^2(H) - P_2^2(H)}}{P_2(H)} \tag{4.6}$$

Using expansion of $\tan^{-1}(\cdot)$ and from (4.5), we get

$$\widetilde{H} = H + \frac{H^5}{480} - \frac{23H^7}{12096} + \cdots \tag{4.7}$$

We define the phase-lag as defined in [6], the leading coefficient in the expansion of $|\frac{\widetilde{H}-H}{H}|$. Thus, from (4.7) the phase-lag of a proposed numerical method (2.12) is given by

$$P(H) = \frac{H^4}{480} - \frac{23H^6}{12096} + \cdots \tag{4.8}$$

Thus, proposed method (2.12) has at most $\frac{H^4}{480}$ provided $0 < H < \sqrt{6}$. This is phase-lag for $i \geq 2$ in proposed method (2.12).

From (4.5), we find the roots of characteristic equation (4.4) are distinct and complex conjugate. From (4.6), we conclude $\tan(\widetilde{H})$ is real if $H^2 \leq 6$. Thus, the proposed explicit method (2.12) possesses an interval of periodicity with period 2.44948974218.

5 Numerical Experiments

To test the computational efficiency of proposed method (2.12), we have considered linear and non-linear model problems. In each model problem, we took a uniform step size h. In Tables 1-6, we have presented the maximum absolute error MAE and at few points in the domain of the problem the absolute error in the solution $u(x)$ of the considered model problem for different values of h. In computation of MAE and absolute error, we have used following formulas,

$$MAE = \max_{1\leq i\leq N} |U_i - u_i|, \quad \text{and} \quad \text{absolute error} = |U_i - u_i|\,, \quad i = 1, \cdots, N.$$

where U_i and u_i are respectively exact and computed value of solution at node x_i of model problem.

All computations were performed on a Windows 2007 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-8} or the number of iteration reached 10^3.

Problem 1. The linear model problem in [7] is given by

$$u''(x) = -1.5u'(x) - .5u(x), \quad 0 < x \leq 10$$

subject to the boundary conditions

$$u(0) = 1.0, \quad \text{and} \quad u'(0) = 1$$

The analytical solution of the problem is $u(x) = -3\exp(-x) + 4\exp(-\frac{x}{2})$. The MAE computed by method (2.12) for different values of h are presented in Table 1. Also presented an absolute error at few points in domain of the problem in Table 2.

Problem 2. The nonlinear parametric model problem in [4] is given by

$$u''(x) = -(1 + A^2(u'(x))^2), \quad 0 \leq x \leq 1$$

subject to the boundary conditions

$$u(0) = 0, \quad \text{and} \quad u'(0) = .500852079$$

The analytical solution of the problem is $u(x) = \frac{\ln(\frac{\cos(A(x-\frac{1}{2}))}{\cos(\frac{A}{2})})}{A^2}$. We considered $A = \frac{1}{7}$ and computed MAE by method (2.12) for the different values of h, presented in Table 3. Also presented an absolute error at few points in domain of the problem in Table 4.

Problem 3. The non-linear model problem is given by

$$u''(x) = \frac{1}{2}(\exp(2u(x)) + (u'(x))^2) \quad 0 < x \leq 1$$

subject to the boundary conditions

$$u(0) = 0, \quad \text{and} \quad u'(0) = -1$$

The analytical solution of the problem is $u(x) = -\ln(1 + x)$. The MAE computed by method (2.12) for different values of h are presented in Table 5. Also presented an absolute error at few points in domain of the problem in Table 6.

Table 1: Maximum absolute error (Problem 1).

	h				
	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
MAE	.59831910e-4	.15109080e-4	.37960272e-5	.95134328e-6	.23812737e-6

Table 2: Absolute error at x (Problem 1).

x	h				
	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2	5.90699521e-5	1.46314803e-5	3.64107524e-6	9.0818233e-7	2.26784563e-7
4	2.9036436e-5	7.18456356e-6	1.78698366e-6	4.45611318e-7	1.11260456e-7
6	5.23566075e-8	5.01889977e-9	2.75757313e-10	5.15641628e-11	2.87361455e-11
8	7.08030678e-6	1.75869753e-6	4.38269483e-7	1.09392843e-7	2.73269572e-8
10	6.64465064e-6	1.64989407e-6	4.11084391e-7	1.02598641e-7	2.56285621e-8

Table 3: Maximum absolute error (Problem 2).

	h				
	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
MAE	.92016300e-4	.23005061e-4	.57513905e-5	.14378634e-5	.35946779e-6

Table 4: Absolute error at x (Problem 2).

	h				
x	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
.25	3.05533878e-5	7.63868703e-6	1.90971547e-6	4.77434414e-7	1.19359308e-7
.375	3.05223976e-5	7.63111844e-6	1.90784569e-6	4.76969755e-7	1.19243487e-7
.5	3.05108816e-5	7.62841823e-6	1.90719298e-6	4.76809376e-7	1.19203745e-7
.625	3.05188298e-5	7.63058403e-6	1.90775677e-6	4.76953107e-7	1.19240029e-7
.75	3.05462485e-5	7.63761769e-6	1.90953757e-6	4.77401104e-7	1.19352391e-7
.875	3.05931606e-5	7.6495251e-6	1.91253684e-6	4.78153714e-7	1.19540897e-7
1.0	9.20162999e-5	2.30050614e-5	5.75139047e-6	1.43786339e-6	3.59467788e-7

Table 5: Maximum absolute error (Problem 3).

	h				
	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
MAE	.29566459e-4	.75103427e-5	.18924572e-5	.47497512e-6	.11897650e-6

Table 6: Absolute error at x (Problem 3).

	h				
x	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
.25	1.9117009e-5	4.85647409e-6	1.22416517e-6	3.07322071e-7	7.6992187e-8
.375	1.56426837e-5	4.02384353e-6	1.02055254e-6	2.56990329e-7	6.44808905e-8
.50	1.30263901e-5	3.40347281e-6	8.69724713e-7	2.19819816e-7	5.52555254e-8
.625	1.10117466e-5	2.93178844e-6	7.55852893e-7	1.91861366e-7	4.83298177e-8
.75	9.4311129e-6	2.56725173e-6	6.68598549e-7	1.70536092e-7	4.30597622e-8
.875	8.17118952e-6	2.28180959e-6	6.00982241e-7	1.54103368e-7	3.90107251e-8
1.0	2.36051804e-5	5.60336992e-6	1.36292559e-6	3.35948639e-7	8.33865402e-8

In the tables, we presented MAE for the different values of h. Also present absolute error at few point in the domain of considered problems. In absolute error of problem

1, we found the periodic property in numerical solution. For other model problems, we did not observe periodic property in considered small domain. In example 2, we find maximum absolute error occurred at the end point of the domain. However, we observed from the numerical results in tables, the accuracy of the proposed method (2.12) approves the theoretical discussion. It is easy to verify using computational outcome the order of accuracy $O(h^2)$ and stability of proposed method (2.12). Thus, we observed the proposed method (2.12) is efficient, stable and convergent with order of accuracy quadratic.

6 Conclusion

An explicit difference method for the approximate numerical solution of second order initial value problems (1.1) has been developed and discussed in this article. The development of technique is simple and exactly satisfy initial conditions in natural way unlike other reported work in the literature. We obtained an algebraic explicit system of equations (2.12) at discrete nodal points in the domain of continuous problem as the discretization of (1.1). Thus, our proposed method (2.12) is an explicit method for the approximate numerical solution of the model problem (1.1). The computational results are accurate and in good agreement to the theoretical discussion. The approximations together with discretization of the (1.1) is at least quadratic order accurate. In phase-lag analysis, we find phase-lag $\frac{H^4}{480}$ for $N-2$ equations. The phase-lag will differ in those equations in which initial conditions used in approximations. The idea presented in this article leads to the possibility to develop high accuracy explicit finite difference methods for the approximate numerical solution of higher order differential equations and corresponding initial / boundary value problems as in reference problem (1.1), work in this direction in progress.

References

[1] G. Adomian, *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl. **135** (1980), 501–544.

[2] R. P. Agarwal, *On multipoint boundary value problems for discrete equations*, J. Math. Anal. Appl. **96** (1983), 520–534.

[3] R. P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Singapore, 1986.

[4] R. P. Agarwal, H. B. Thompson, C. C. Tisdell, *Difference equations in Banach spaces*, Advances in difference equations, IV, Comput. Math. Appl. **45** (6-9) (2003), 1437–1444.

[5] J. V. Baxley, *Nonlinear Two Point Boundary Value Problems in Ordinary and Partial Differential Equations* (Everrit, W. N., Sleeman , B. D. Eds.), Springer-Verlag, New York, 1981, 46–54.

[6] L. Brusa, L. Nigro, *A one-step method for direct integration of structural dynamic equations*, lnternat. J. Number. Methods Engrg. **15** (1980), 685–699.

[7] M. M. Chawla, S. R. Sharma, *Numerical Solution of General Second-Order Initial-Value-Problems*, ZAMM **60** (1980), 469–478.

[8] L. Collartz, *The Numerical Treatment of Differential Equations*, Springer-Verlag Inc., New York, 1966.

[9] R. Gaines, *Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations*, SIAM J. Numer. Anal. **11** (1974), 411–434.

[10] I. Gladwell, R. M. Thomas, *Damping and phase analysis for some methods for solving second-order ordinary differential equations*, International Journal for Numerical Methods in Engineering, **19** (1983), 495–503.

[11] F. Iavernaro, *Symmetric Boundary Value Methods for Second Order Initial and Boundary Value Problems*, Mediterr. J. Math. **3** (2006), 383–398.

[12] S. N. Jator, J. Li, *An algorithm for second order initial and boundary value problems with an automatic error estimate based on a third derivative method*, Numer. Algor. **59** (2012), 333–346.

[13] H. B. Keller, *Numerical Methods for Two-Point Boundary-Value Problems*, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto,Ont.-London, 1968.

[14] A. Marasco, A. Romano, *Scientific Computing with Mathematica: Mathematical Problems for Ordinary Differential Equations in: Modeling and Simulation in Science, Engineering and Technology*, Birkhauser Boston, Inc., Boston, MA, 2001.

[15] P. K. Pandey, *An Implicit Rational Method for Solution of Second Order Initial Value Problems in Ordinary differential Equation*, International Journal of Computer Applications *57* (2) (2012), 30–33.

[16] P. K. Pandey, *Nonlinear Explicit Method for First Order Initial Value Problems*, Acta Technica Jaurinensis **6** (2) (2013), 118–125.

[17] P. K. Pandey, *Solving two point boundary value problems by Numerov type method with an internal boundary condition*, Journal of Science and Arts **38** (1) (2017), 23–30.

[18] P. K. Pandey, *A new computational algorithm for the solution of second order initial value problems in ordinary differential equations*, Applied Mathematics and Nonlinear Sciences **3** (1) (2018), 167–174.

[19] J. Sunday, M. R. Odekunle, *A New Numerical Integrator for the Solution of Initial Value Problems in Ordinary Differential Equations*, The Pacific Journal of Science and Technology **13** (1) (2012), 221–227.

Pramod Kumar Pandey

Department of Mathematics,
Dyal Singh College (Univ. of Delhi),
Lodhi Road, New Delhi-110003, India

E-mail: pramod_10p@hotmail.com

Ishver Rani Manchanda

Department of Mathematics,
Dyal Singh College (Univ. of Delhi),
Lodhi Road, New Delhi-110003, India

E-mail: manchanda.ishver@gmail.com

Archana Yadav

Department of Mathematics,
Univ. of Delhi,
Delhi-110007, India

E-mail: archanayadav538@gmail.com