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THA-surfaces of finite type in the Galilean space $\mathbb{G}^3$

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Abstract In this article, we study THA-surfaces in the Galilean space $\mathbb{G}^3$, which has finite type immersion.

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Keywords THA-surfaces; finite type immersion; Laplacian operator.

1 Introduction

The notion of finite type immersion of submanifolds of a Euclidean space has been widely used in classifying and characterizing well known Riemannian submanifolds [5]. An Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ [5]. B.-Y. Chen posed the problem of classifying the finite type submanifolds in the 3-dimensional Euclidean space $\mathbb{E}^3$. These can be regarded as a generalization of minimal submanifolds.

The notion of finite type immersion has played an important role in classifying and characterizing the submanifolds in Euclidean space.

Since then the theory of submanifolds of finite type has been studied by many geometers.

A well known result due to Takahashi [17] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition

$$\Delta r = \lambda r, \ \lambda \in \mathbb{R}.$$

In [9] Ferrandez, Garay and Lucas proved that the surfaces of $\mathbb{E}^3$ satisfying

$$\Delta H = AH, \ A \in Mat(3,3)$$

are either minimal, or an open piece of sphere or of a right circular cylinder.

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In [8] F. Dillen, J. Pas and L. Verstraelen proved that the only surfaces in $\mathbb{E}^3$ satisfying

$$\Delta r = Ar + B,\ A \in Mat(3,3),\ B \in Mat(3,1),$$

are the minimal surfaces, the spheres and the circular cylinders.

In [3], the authors classified the factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces, whose component functions are eigenfunctions of their Laplace operator. The authors in [4] studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition

$$\Delta^{III} r_i = \mu_i r_i, \quad \mu_i \in \mathbb{R},$$

where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III. In [11], the authors proved that every homothetical (factorable) lightlike hypersurface in a semi-Euclidean space is minimal. Aydin et al. [2] investigated the homothetical surfaces in pseudo-Galilean space with null Gaussian and mean curvature.

In this paper, we study THA-surfaces in the Galilean space $\mathbb{G}^3$, which has finite type immersion.

A submanifold $\mathcal{M}^2$ of a 3-dimensional Euclidean space $\mathbb{E}^3$ is said to be of finite type if each component of its position vector field χ can be written as a finite sum of eigenfunctions of the Laplacian Δ of $\mathcal{M}^2$, that is, if

$$\chi = \chi_0 + \sum_{i=1}^{k} \chi_i,$$

where χ_i are $\mathbb{E}^3 -$ valued eigenfunctions of the Laplacian of $(\mathcal{M}^2, r)$ [5]:

$$\Delta\chi_i = \lambda_i \chi_i,$$

where $\lambda_i \in \mathbb{R}, i = 1, 2, .., k$. If λ_i are different, then $\mathcal{M}^2$ is said to be of k-type.

The Laplace-Beltrami operator of a smooth function $\psi : \mathcal{M}^2 \to \mathbb{R}, (s,t) \mapsto \psi(s,t)$ with respect to the first fundamental form of the surface $\mathcal{M}^2$ is the operator Δ, defined in [16] as follows:

$$\Delta\psi = \frac{-1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial s} \left(\frac{G\psi_s - F\psi_t}{\sqrt{|EG - F^2|}} \right) + \frac{\partial}{\partial t} \left(\frac{E\psi_t - F\psi_s}{\sqrt{|EG - F^2|}} \right) \right]. \tag{1.1}$$

The second differential parameter of Beltrami of a function

$$\psi : \mathcal{M}^2 \to \mathbb{R}, (s,t) \longmapsto \psi(s,t)$$

with respect to the second fundamental form of $\mathcal{M}^2$ is the operator Δ^{II} which is defined by [16]

$$\Delta^{II}\psi = \frac{-1}{\sqrt{|LN - M^2|}} \left[\frac{\partial}{\partial s} \left(\frac{N\psi_s - M\psi_t}{\sqrt{|LN - M^2|}} \right) + \frac{\partial}{\partial t} \left(\frac{L\psi_t - M\psi_s}{\sqrt{|LN - M^2|}} \right) \right], \tag{1.2}$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

In this article, we study THA-surfaces in the Galilean space $\mathbb{G}^3$, which has finite type immersion.

2 Preliminaries

Let $\mathbb{G}^3$ be the three-dimensional Galilean space. We recall some well-known formulas for the surfaces in $\mathbb{G}^3$.
The Galilean space $\mathbb{G}^3$ is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$. The absolute figure of the Galilean space consists of an ordered triple $\omega,\ f,\ I$, where ω is the ideal (absolute) plane, f is the line (absolute line) in ω and I is the fixed elliptic involution of points of f.
In Galilean space a vector $\mu = (\mu_1, \mu_2, \mu_3)$ is called a isotropic (non-isotropic) if $\mu_1 = 0$ ($\mu_1 \neq 0$). All unit non-isotropic vectors are of the form $(1, \mu_2, \mu_3)$
Let $V = (v_1, v_2, v_3)$ and $W = (\omega_1, \omega_2, \omega_3)$ be vectors in Galilean space. The Galilean scalar product of V and W is defined by

$$g_{\mathbb{G}^3}(V, W) = \begin{cases} v_1\omega_1 & \text{if } v_1 \neq 0 \text{ or } \omega_1 \neq 0 \\ v_2\omega_2 + v_3\omega_3 & \text{if } v_1 = 0 \text{ and } \omega_1 = 0. \end{cases} \tag{2.1}$$

The Galilean norm of a vector ξ in $\mathbb{G}^3$ is given by $\| \xi \|_{\mathbb{G}_3} = \sqrt{g_{\mathbb{G}^3}(\xi,\ \xi)}$. In particular, if $\| \xi \|_{\mathbb{G}_3} = 1$, the vector ξ is called a unit vector.
The cross product of the vectors $\xi = (\xi_1, \xi_2, \xi_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ is defined as follows:

$$\xi \wedge_{\mathbb{G}^3} \zeta = \begin{vmatrix} 0 & e_2 & e_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix} = (\xi_3\zeta_1 - \xi_1\zeta_3)e_2 + (\xi_1\zeta_2 - \xi_2\zeta_1)e_3,$$

where e_2 and e_3 are Euclidean standard basis.
The group of motions of $\mathbb{G}_3$ is a six-parameter group given by

$$\pi :\ (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1', \alpha_2', \alpha_3') : \begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu & \cos\theta & \sin\theta \\ \nu & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

where $(\alpha_1, \alpha_2, \alpha_3)$ denote the affine coordinates and $\theta,\ \beta_i, \nu, \mu \in \mathbb{R}$. More information about Galilean space can be found in [1, 6, 7, 10, 14, 15].

In this paper, we describe, THA-surfaces (**T**ranslation and **h**omothetical **a**ffine surfaces) in the Galilean space $\mathbb{G}^3$ having constant Gaussian and mean curvatures and completely classify minimal or flat THA-surfaces. Also, we classify linear Weingarten THA-surfaces in the Galilean space $\mathbb{G}^3$.
Let $\mathcal{M}^2$ be a surface given by the parametrization

$$\chi(s, t) = (\chi_1(s, t),\ \chi_2(s, t),\ \chi_3(s, t)),$$

where $\chi_1(s, t),\ \chi_2(s, t),\ \chi_3(s, t) \in \mathcal{C}^3$.
The first fundamental form of the surface is given by

$$\mathrm{d}s^2 = (\xi_1\mathrm{d}u + \xi_2\mathrm{d}v)^2 + \epsilon(\eta_{11}\mathrm{d}u^2 + 2\eta_{12}\mathrm{d}u\mathrm{d}v + \eta_{22}\mathrm{d}v^2),$$

where $\xi_s = \chi_{1,s},\ \eta_{st} = \chi_{2,s}\chi_{2,t} + \chi_{3,s}\chi_{3,t},\ \chi_{k,1} = \frac{\partial\chi_k}{\partial s},\ \chi_{k,2} = \frac{\partial\chi_k}{\partial t},\quad s, t = 1, 2;\quad k = 1, 2, 3,$ and

$$\epsilon = \begin{cases} 0 & \text{if the direction } \mathrm{d}u : \mathrm{d}v \text{ is non-isotropic} \\ 1 & \text{if the direction } \mathrm{d}u : \mathrm{d}v \text{ is isotropic.} \end{cases}$$

A surface is called admissible if it has no Euclidean tangent planes, i.e., $\xi_s \neq 0$ for some $s = 1, 2$. The isotropic normal vector field is given by

$$\mathcal{N}(s,t) = \chi_s \wedge_{\mathbb{G}^3} \chi_t,$$

where χ_s and χ_t are partial differentiations with respect to s and t, respectively.
The isotropic unit normal vector N is defined by

$$N = \frac{\chi_s \wedge_{\mathbb{G}^3} \chi_t}{W},$$

where $W = \| \chi_s \wedge_{\mathbb{G}^3} \chi_t \|_{\mathbb{G}^3}$.
The coefficients of the second fundamental form are given by

$$\sigma_{ij} = g_{\mathbb{G}^3}\left(\frac{\chi_{ij}\chi_{1,1} - \chi_{1,ij}\chi_s}{\chi_{1,1}}, N\right) = g_{\mathbb{G}^3}\left(\frac{\chi_{ij}\chi_{1,2} - \chi_{1,ij}\chi_t}{\chi_{1,2}}, N\right).$$

The mean curvature H and the Gaussian curvature K_G are, respectively, defined by [10]

$$2H = \sum_{i,j=1}^{2} g^{ij}\sigma_{ij} = \frac{(g_2)^2\sigma_{11} - 2g_1g_2\sigma_{12} + (g_1)^2\sigma_{22}}{W^2} \quad \text{and} \quad K_G = \frac{\sigma_{11}\sigma_{22} - \sigma_{12}^2}{W^2},$$

where $g_1 = \chi_{1,1}$, $g_2 = \chi_{1,2}$, $g^{ij} = g^i g^j$, $g^1 = \frac{\chi_{1,2}}{W}$ and $g^2 = -\frac{\chi_{1,1}}{W}$.
We can see that $\| \chi_s \wedge_{\mathbb{G}^3} \chi_t \|_{\mathbb{G}_3}$ is the area of the parallelogram determined by χ_s and χ_t [6].

3 THA-surfaces in $\mathbb{G}^3$

Definition 3.1. *[12, 13] A surface $\mathcal{M}^2$ in the Galilean space $\mathbb{G}^3$ is called a THA-surface if it can be locally written as*

$$\chi(s,\ t) = (s,\ t,\ Af(s)g(as+t) + B(f(s) + g(as+t))) \tag{3.1}$$

or

$$\chi(s,\ t) = (s,\ Af(s)g(as+t) + B(f(s) + g(as+t)),\ t) \tag{3.2}$$

or

$$\chi(s,\ t) = (Af(s)g(as+t) + B(f(s) + g(as+t)),\ s,\ t), \tag{3.3}$$

where $f, g \in \mathcal{C}^3$ and A, B are non-zero real numbers.

The surfaces given by (3.1), (3.2) and (3.3) are called the THA-surfaces of the first, the second and the third type, respectively.

We see that the first type (3.1) and the second type (3.2) THA-surfaces have up to a sign similar first fundamental form, so we will only discuss first type and the third type THA-surfaces.

3.1 THA-surfaces of first type in $\mathbb{G}^3$

By a transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

From (3.1) and (3.1) we have

$$\chi(x,\ y) = (x,\ y - ax,\ Af(x)g(y) + B(f(x) + g(y))), \tag{3.4}$$

The coefficients of the first fundamental form of $\mathcal{M}^2$ are given by

$$E = 1, \quad F = 0,\ G = A^{-2}((\alpha\gamma')^2 + A^2), \tag{3.5}$$

where $\alpha = Af + B$ and $\gamma = Ag + B$.

The unit normal vector is given by

$$\boldsymbol{N} = \frac{1}{WA}(0,\ -\alpha\gamma',\ A),$$

where $W^2 = A^{-2}((\alpha\gamma')^2 + A^2)$.
The coefficients of the second fundamental form are given by

$$\sigma_{11} = L = \frac{\gamma\alpha''}{AW}, \quad \sigma_{12} = \sigma_{21} = M = \frac{\alpha'\gamma'}{AW}, \quad \sigma_{22} = N = \frac{\alpha\gamma''}{AW}. \tag{3.6}$$

A surface in Galilean space $\mathbb{G}^3$ parameterized by (3.4) has Gaussian curvature

$$K_G = \frac{\alpha\gamma\alpha''\gamma'' - \gamma'^2\alpha'^2}{A^2W^4}. \tag{3.7}$$

The expression of mean curvature H is

$$H = \frac{\alpha\gamma''}{2AW^3}. \tag{3.8}$$

Then $\mathcal{M}^2$ is a minimal surface if and only if $\alpha\gamma'' = 0$.

3.2 THA-surfaces of third type in $\mathbb{G}^3$

By a transformation

$$x = s, \quad y = as + t. \tag{3.9}$$

From (3.9) and (3.3) we have

$$\chi(x,\ y) = (A^{-1}(\alpha\gamma - B^2),\ x,\ y - ax), \tag{3.10}$$

The coefficients of the first fundamental form of M^2 are given by

$$E = \frac{\alpha'^2\gamma^2}{A^2}, \quad F = \frac{\alpha\gamma\alpha'\gamma'}{A^2},\ G = \frac{\alpha^2\gamma'^2}{A^2}. \tag{3.11}$$

The unit normal vector is given by

$$\boldsymbol{N} = \frac{1}{WA}(0,\ -(\alpha'\gamma + a\gamma'\alpha),\ -\gamma'\alpha),$$

where $W^2 = A^{-2}((\gamma'\alpha)^2 + (\alpha'\gamma + a\gamma'\alpha)^2)$, $\alpha\gamma \neq 0$.
The coefficients of the second fundamental form are given by

$$\sigma_{11} = L = \frac{\gamma\alpha''}{AW},\quad \sigma_{12} = \sigma_{21} = M = \frac{\alpha'\gamma'}{AW},\ \sigma_{22} = N = \frac{\alpha\gamma''}{AW}. \tag{3.12}$$

A surface in Galilean space $\mathbb{G}^3$ parameterized by (3.10) has Gaussian curvature

$$K_G = \frac{\alpha\gamma\alpha''\gamma'' - \gamma'^2\alpha'^2}{A^2W^4}. \tag{3.13}$$

The expression of mean curvature H is

$$H = \frac{\alpha''\alpha^2\gamma'^2\gamma + \gamma''\gamma^2\alpha'^2\alpha - 2\alpha\gamma\alpha'^2\gamma'^2}{2A^3W^3}.$$

4 THA-surfaces satisfying $\Delta\chi_i = \lambda_i\chi_i$ in $\mathbb{G}^3$

4.1 THA-surfaces of first type satisfying $\Delta\chi_i = \lambda_i\chi_i$ in $\mathbb{G}^3$

In this section we explore the classification of the THA-surfaces of first type $\mathcal{M}^2$ of $\mathbb{G}^3$ satisfying the condition

$$\Delta\chi_i = \lambda_i\chi_i, \tag{4.1}$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$.

Lemma 4.1. *The Laplacian Δ of $\mathcal{M}^2$ can be expressed as follows:*

$$\Delta\psi = -\varphi_{xx} - Q_1(x,y)\psi_{yy} - Q_2(x,y)\psi_x + Q_3(x,y)\psi_y, \tag{4.2}$$

where

$$Q_1(x,y) = \frac{1}{W^2},\quad Q_2(x,y) = \frac{\alpha\alpha'\gamma'^2}{A^2W^2},\quad Q_3(x,y) = \frac{\gamma'\gamma''\alpha^2}{A^2W^4}.$$

Applying (4.2) on the coordinate functions $\chi_1(x,y) = x$, $\chi_2(x,y) = y - ax$ and $\chi_3(x,y) = A^{-1}(\alpha\gamma - B^2)$ of the position vector χ we find

$$\begin{cases} \Delta(x) = -\frac{\alpha\alpha'\gamma'^2}{A^2W^2} \\ \Delta(y - ax) = \frac{a\alpha\alpha'\gamma'^2}{A^2W^2} + \frac{\gamma'\gamma''\alpha^2}{A^2W^4} \\ \Delta(\frac{\alpha\gamma - B^2}{A}) = -\frac{\alpha''\gamma}{A} - \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} - \frac{\gamma''\alpha}{AW^4}. \end{cases} \tag{4.3}$$

By using (4.1) and (4.3) we have the following equations

$$\begin{cases} -\frac{\alpha\alpha'\gamma'^2}{A^2W^2} = \lambda_1 x \\ \frac{aW^2\alpha\alpha'\gamma'^2+\gamma'\gamma''\alpha^2}{A^2W^4} = \lambda_2(y-ax) \\ -\frac{\alpha''\gamma}{A} - \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} - \frac{\gamma''\alpha}{AW^4} = \lambda_3\left(\frac{\alpha\gamma-B^2}{A}\right). \end{cases} \tag{4.4}$$

Therefore, the problem of classifying the THA-surfaces $\mathcal{M}^2$ satisfying (4.1) is reduced to the integration of this system of ordinary differential equations.

Case 1. Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$. From (4.37), we obtain

$$\alpha\alpha'\gamma'^2 = 0 \tag{4.5}$$

$$\gamma'\gamma''\alpha^2 = 0 \tag{4.6}$$

$$\frac{\alpha''\gamma}{A} + \frac{\gamma''\alpha}{AW^4} = 0. \tag{4.7}$$

In this case we have four possibilities:

a) If $\alpha = 0$. Then $K_G = H = 0$.

b) If $\alpha' = 0$ ($\alpha \neq 0$) equations (4.6) and (4.7) imply that $\gamma'' = 0$. Then $K_G = H = 0$.

c) If $\gamma = 0$. Then $K_G = H = 0$.

d) If $\gamma' = 0$ ($\gamma \neq 0$), then from (4.7) we obtain $\alpha'' = 0$. Then $K_G = H = 0$.

Case 2. Let $\lambda_2 \neq 0$ and $\lambda_1 = \lambda_3 = 0$. From (4.37), we obtain

$$\alpha\alpha'\gamma'^2 = 0 \tag{4.8}$$

$$\frac{aW^2\alpha\alpha'\gamma'^2 + \gamma'\gamma''\alpha^2}{A^2W^4} = \lambda_2(y-ax) \tag{4.9}$$

$$\frac{\alpha''\gamma}{A} + \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} + \frac{\gamma''\alpha}{AW^4} = 0. \tag{4.10}$$

We discuss all possibilities.

a) If $\alpha\gamma' = 0$. Then (4.9) gives $\lambda_2 = 0$, a contradiction.

b) If $\alpha' = 0$ ($\alpha \neq 0$). Then (4.10) gives $\gamma'' = 0$. From (4.9), we get $\lambda_2 = 0$, a contradiction.

Case 3. Let $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = 0$. From (4.37), we obtain

$$-\frac{\alpha\alpha'\gamma'^2}{A^2W^2} = \lambda_1 x \tag{4.11}$$

$$\alpha\gamma'(aW^2\alpha'\gamma' + \gamma''\alpha) = 0 \tag{4.12}$$

$$\frac{\alpha''\gamma}{A} + \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} + \frac{\gamma''\alpha}{AW^4} = 0. \tag{4.13}$$

We discuss all possibilities.

a) If $\alpha\gamma' = 0$. Then (4.11) gives $\lambda_1 = 0$, a contradiction.

b) If $\alpha\gamma' \neq 0$. Then from (4.12) we have

$$aW^2\alpha'\gamma' + \gamma''\alpha = 0. \tag{4.14}$$

Since $\alpha\gamma' \neq 0$, we divide (4.14) by $(\alpha\gamma')$ and differentiating with respect to x, we obtain

$$-\frac{A^2}{2\alpha'^2}\left(\frac{\alpha'}{\alpha}\right)_{,x} = \gamma'^2.$$

Since α and γ are functions of two independent variables, the above equation can be written as

$$-\frac{A^2}{2\alpha'^2}\left(\frac{\alpha'}{\alpha}\right)_{,x} = k, \quad \gamma'^2 = k, \tag{4.15}$$

where $k \in \mathbb{R}$.
Then from the second part of (4.15) we obtain $\gamma'' = 0$. From (4.14), we have $\alpha' = 0$, then it is a contradiction to $\lambda_1 \neq 0$.
Case 4. Let $\lambda_3 \neq 0$ and $\lambda_1 = \lambda_2 = 0$. From (4.37), we obtain

$$\alpha\alpha'\gamma' = 0 \tag{4.16}$$
$$\alpha\gamma'(aW^2\alpha'\gamma' + \gamma''\alpha) = 0 \tag{4.17}$$
$$-\frac{\alpha''\gamma}{A} - \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} - \frac{\gamma''\alpha}{AW^4} = \lambda_3\left(\frac{\alpha\gamma - B^2}{A}\right). \tag{4.18}$$

We discuss all possibilities.
a) If $\alpha = 0$. Then (4.18) gives $\lambda_3 = 0$, a contradiction.
b) If $\alpha' = 0$ ($\alpha = b \neq 0$). Then (4.17) gives $\gamma'\gamma'' = 0$. From (4.18), we get $\lambda_3 = 0$, a contradiction.
c) If $\gamma = 0$. Then (4.18) gives $\lambda_3 = 0$, a contradiction.
d) If $\gamma' = 0$ ($\gamma = b \neq 0$). Then (4.18) gives

$$\alpha'' + \lambda_3\alpha - \frac{\lambda_3 B^2}{b} = 0. \tag{4.19}$$

We discuss two cases according to constant λ_3.
$(d-i)$: $\lambda_3 < 0$, from (4.19), we obtain

$$\alpha = k_1 e^{rx} + k_2 e^{-rx} + \frac{B^2}{b}; \quad k_1, k_2 \in \mathbb{R},\ r = \sqrt{-\lambda_3}. \tag{4.20}$$

$(d-ii)$: $\lambda_3 > 0$, from (4.19), we obtain

$$\alpha = k_1\cos(\rho x) + k_2\sin(\rho x) + \frac{B^2}{b}; \quad k_1, k_2 \in \mathbb{R},\ \rho = \sqrt{\lambda_3}. \tag{4.21}$$

Case 5. Let $\lambda_1 = 0$ and $\lambda_2\lambda_3 \neq 0$. From (4.37), we obtain

$$\alpha\alpha'\gamma' = 0 \tag{4.22}$$
$$\frac{aW^2\alpha\alpha'\gamma'^2 + \gamma'\gamma''\alpha^2}{A^2W^4} = \lambda_2(y - ax) \tag{4.23}$$
$$-\frac{\alpha''\gamma}{A} - \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} - \frac{\gamma''\alpha}{AW^4} = \lambda_3\left(\frac{\alpha\gamma - B^2}{A}\right). \tag{4.24}$$

We have two sub-cases:
a) If $\alpha = 0$. Then (4.23) gives $\lambda_2 = 0$, a contradiction.
b) If $\alpha' = 0$ ($\alpha = b \neq 0$). Then (4.23) gives

$$\frac{A^2b^2\gamma'\gamma''}{(A^2 + b^2\gamma'^2)^2} = \lambda_2(y - ax). \tag{4.25}$$

Differentiating (4.25) with respect to x, we get $\lambda_2 a = 0$, a contradiction.

Case 6. Let $\lambda_2 = 0$ and $\lambda_1\lambda_3 \neq 0$. From (4.37), we obtain

$$-\frac{\alpha\alpha'\gamma'^2}{A^2W^2} = \lambda_1 x \tag{4.26}$$

$$aW^2\alpha\alpha'\gamma'^2 + \gamma'\gamma''\alpha^2 = 0 \tag{4.27}$$

$$-\frac{\alpha''\gamma}{A} - \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} - \frac{\gamma''\alpha}{AW^4} = \lambda_3\left(\frac{\alpha\gamma - B^2}{A}\right). \tag{4.28}$$

We have two sub-cases:

a) If $\alpha\gamma' = 0$. Then (4.26) gives $\lambda_1 = 0$, a contradiction.

b) If $\alpha\gamma' \neq 0$. Then from (4.27) we have

$$aW^2\alpha'\gamma' + \gamma''\alpha = 0. \tag{4.29}$$

Since $\alpha\gamma' \neq 0$, we divide (4.29) by $(\alpha\gamma')$ and differentiating with respect to x, we obtain

$$-\frac{A^2}{2\alpha'^2}\left(\frac{\alpha'}{\alpha}\right)_{,x} = \gamma'^2.$$

Since α and γ are functions of two independent variables, the above equation can be written as

$$-\frac{A^2}{2\alpha'^2}\left(\frac{\alpha'}{\alpha}\right)_{,x} = k, \quad \gamma'^2 = k, \tag{4.30}$$

where $k \in \mathbb{R}$.

Then from (4.30) we obtain $\gamma'' = 0$. From (4.29), we have $\alpha' = 0$, then it is a contradiction to $\lambda_1 \neq 0$.

Case 7. Let $\lambda_3 = 0$ and $\lambda_1\lambda_2 \neq 0$. From (4.37), we obtain

$$-\frac{\alpha\alpha'\gamma'^2}{A^2W^2} = \lambda_1 x \tag{4.31}$$

$$\frac{aW^2\alpha\alpha'\gamma'^2 + \gamma'\gamma''\alpha^2}{A^2W^4} = \lambda_2(y - ax) \tag{4.32}$$

$$\frac{\alpha''\gamma}{A} + \frac{\alpha\alpha'^2\gamma\gamma'^2}{A^3W^2} + \frac{\gamma''\alpha}{AW^4} = 0. \tag{4.33}$$

We have two sub-cases:

a) If $\alpha\gamma' = 0$. Then (4.32) gives $\lambda_2 = 0$, a contradiction.

b) If $\alpha\gamma' \neq 0$. Equation (4.31) writes as

$$\frac{\alpha\alpha'\gamma'^2}{A^2 + \alpha^2\gamma'^2} = -\lambda_1 x.$$

Integration with respect to x leads to

$$\gamma'^2 = \frac{ke^{-\lambda_1 x^2} - A^2}{\alpha^2}.$$

We use again the fact that each side of this equation depends only on y and only on x respectively, hence there exists $c \in \mathbb{R} - \{0\}$ such that $\gamma' = c$.
Equation (4.32) can be rewritten as

$$\frac{ac^2\alpha\alpha'}{A^2 + c^2\alpha^2} = \lambda_2(y - ax).$$

Differentiating the previous expression with respect to y, we get $\lambda_2 = 0$, a contradiction.
Case 8. Let $\lambda_1\lambda_2\lambda_3 \neq 0$. Following the similar steps as in **Case 7,** we arrive at similar types of contradictions.
Consequently, we have:

Theorem 4.2. *Let $\mathcal{M}^2$ be a THA-surfaces given by (3.1) in $\mathbb{G}^3$. Then $\mathcal{M}^2$ satisfies the equation $\Delta\chi_i = \lambda_i\chi_i$ $(i = 1, 2, 3)$ if and only if the following statement is true:*
1) *$\mathcal{M}^2$ has zero mean curvature everywhere.*
2) *$\mathcal{M}^2$ has zero Gaussian curvature everywhere.*
3) *$\mathcal{M}^2$ is parametrized as*

$$\chi(s,\ t) = (s,\ t,\ z(s,\ t)),$$

where
a) $z(s,\ t) = k_1 e^{\sqrt{-\lambda_3}s} + k_2 e^{-\sqrt{-\lambda_3}s},\ \lambda_3 < 0.$
b) $z(s,\ t) = k_3 \cos(\sqrt{\lambda_3}s) + k_4 \sin(\sqrt{\lambda_3}s),\ \lambda_3 > 0,$
where $k_l \in \mathbb{R}$, $l = 1, 2, 3, 4$.

4.2 THA-surfaces of third type satisfying $\Delta\chi_i = \lambda_i\chi_i$ in $\mathbb{G}^3$

In this section we explore the classification of the THA-surfaces of third type $\mathcal{M}^2$ of $\mathbb{G}^3$ satisfying the condition

$$\Delta\chi_i = \lambda_i\chi_i, \quad \lambda_i \in \mathbb{R}, \quad (i = 1, 2, 3). \tag{4.34}$$

Lemma 4.3. *The Laplacian Δ of $\mathcal{M}^2$ can be expressed as follows:*

$$\Delta\psi = Q_1(x, y)\psi_{xx} + Q_2(x, y)\psi_{yy} + Q_3(x, y)\psi_{xy}, \tag{4.35}$$

where $Q_1(x, y) = -\dfrac{G}{W^2},\ Q_2(x, y) = -\dfrac{E}{W^2},\ \ Q_3(x, y) = \dfrac{2F}{W^2}.$
Applying (4.35) on the coordinate functions $\chi_1(x, y) = A^{-1}(\alpha\gamma - B^2)$, $\chi_2(x, y) = x$ and $\chi_3(x, y) = y - ax$ of the position vector χ we find

$$\begin{cases} \Delta(\frac{\alpha\gamma - B^2}{A}) = -2HW \\ \Delta(x) = 0 \\ \Delta(y - ax) = 0. \end{cases} \tag{4.36}$$

By using (4.34) and (4.36) we have the following equations

$$\begin{cases} -2HW = \lambda\left(\frac{\alpha\gamma - B^2}{A}\right) \\ 0 = \lambda_2 x \\ 0 = \lambda_3(y - ax). \end{cases} \tag{4.37}$$

Then from (4.37), we have

$$-2HW = \lambda\left(\frac{\alpha\gamma - B^2}{A}\right), \lambda \in \mathbb{R}. \tag{4.38}$$

This means that $\mathcal{M}^2$ is at most of 1-type.
Case 1. Let $\lambda = 0$. From (4.38), we obtain $H = 0$.
Case 2. Let $\lambda \neq 0$. From (4.38), we obtain

$$\alpha''\alpha^2\gamma'^2\gamma + \gamma''\gamma^2\alpha'^2\alpha - 2\alpha\gamma\alpha'^2\gamma'^2 = -\lambda(\alpha\gamma - B^2)((\gamma'\alpha)^2 + (\alpha'\gamma + a\gamma'\alpha)^2). \tag{4.39}$$

We distinguish cases.
Case 2-a. Assume $\alpha' = 0$. Then $\alpha = b$, $b \in \mathbb{R} - \{0\}$, and (4.39) gives $\gamma' = 0$.
Case 2-b. Assume $\alpha'' = 0$. Then $\alpha = cx + d$, $c \in \mathbb{R} - \{0\}$. We write (4.39) as

$$(\gamma''\gamma^2 - 2\gamma\gamma'^2)c^2\alpha + \lambda(\alpha\gamma - B^2)(\gamma'^2\alpha^2(1 + a^2) + c^2\gamma^2 + 2ac\alpha\gamma\gamma') = 0. \tag{4.40}$$

This is a polynomial equation on x of degree 3. Then the leader coefficient, namely, $\lambda c^3\gamma\gamma'^2(1 + a^2)$, must vanish. This means $\gamma' = 0$ and (4.39) gives now $c = 0$, a contradiction.
The other THA-surfaces of third type satisfying $\Delta\chi_i = \lambda_i\chi_i$ in $\mathbb{G}^3$ should satisfy the equation (4.39).

Theorem 4.4. *Let $\mathcal{M}^2$ be a THA-surfaces given by (3.10) in $\mathbb{G}^3$. Then $\mathcal{M}^2$ satisfies the equation $\Delta\chi_i = \lambda_i\chi_i$ $(i = 1, 2, 3)$ if and only if the following statement is true:*
1) *$\mathcal{M}^2$ has zero mean curvature everywhere.*
2) *$\mathcal{M}^2$ is parametrized as*

$$\chi(x,\ y) = (A^{-1}(\alpha\gamma - B^2),\ x,\ y - ax),$$

where α and γ satisfying the equation (4.39).

5 $\Lambda-$ Harmonic THA-surfaces in $\mathbb{G}^3$

In this section we classify THA-surfaces in a Galilean 3-space $\mathbb{G}^3$ satisfying the equation $\Delta^\Lambda\chi = 0$, $\Lambda = I, II$. A surface of in the three dimensional space is said to be $\Lambda-$ Harmonic if it satisfies the condition $\Delta^\Lambda\chi = 0$, $\Lambda = I, II$, where Δ^Λ denotes the Laplace operator with respect to the fundamental forms I and II.

5.1 $I-$ Harmonic THA-surfaces of first type in $\mathbb{G}^3$

From (4.3), $\Delta^I\chi = 0$ if and only if $H = 0 = K_G$.

Theorem 5.1. *Let $\chi : \mathcal{M}^2 \to \mathbb{G}^3$ be an isometric immersion given by (3.4). Then $\Delta^I r = 0$ if and only if $H = 0 = K_G$.*

5.2 $II-$ Harmonic THA-surfaces of first type in $\mathbb{G}^3$

In this subsection we are concerned with non-degenerate THA-surfaces of first type $\mathcal{M}^2$ without parabolic points satisfying the condition $\Delta^{II}\chi = 0$.
By a straightforward computation, the Laplacian Δ^{II} of the second fundamental form II on $\mathcal{M}^2$ with the help of (3.12) and (1.2) turns out to be

$$\begin{aligned}\Delta^{II} &= -\frac{1}{AWD^2}\left(\alpha\gamma''\frac{\partial^2}{\partial x^2}+\gamma\alpha''\frac{\partial^2}{\partial y^2}-2\alpha'\gamma'\frac{\partial^2}{\partial x\partial y}\right)\\ &\quad -\frac{1}{2D^4}\left(P(x,\ y)\frac{\partial}{\partial x}+Q(x,\ y)\frac{\partial}{\partial y}\right),\end{aligned}$$

where

$$\begin{aligned}P(x,\ y) &= \frac{1}{AW}(-\alpha'\gamma''(LN+2M^2)+3\gamma'\alpha''MN+\alpha\gamma'''ML-\gamma\alpha'''N^2),\\ Q(x,\ y) &= \frac{1}{AW}(-\alpha''\gamma'(LN+2M^2)+3\alpha'\gamma''ML+\gamma\alpha'''MN-\alpha\gamma'''L^2)\end{aligned}$$

and $D^2 = LN - M^2$.
Accordingly, we get

$$\begin{aligned}\Delta^{II}\chi(x,\ y) &= \begin{pmatrix}\Delta^{II}(\frac{\alpha\gamma}{A}-\frac{B^2}{A})\\ \Delta^{II}(x)\\ \Delta^{II}(y-ax)\end{pmatrix}\\ &= -\frac{1}{2D^4}\begin{pmatrix}4WD^4+\frac{1}{A}(\alpha'\gamma P(x,\ y)+\alpha\gamma'Q(x,\ y))\\ P(x,\ y)\\ Q(x,\ y)-aP(x,\ y)\end{pmatrix}.\end{aligned}$$

$\Delta^{II}\chi = 0$, if and only if

$$\begin{cases}P(x,\ y)=0\\ Q(x,\ y)=0\\ W=0.\end{cases}\tag{5.1}$$

From (5.1) we have $W = 0$, which is a contradiction. Consequently, there are no THA-surfaces in $\mathbb{G}^3$ without parabolic points, satisfying the condition $\Delta^{II}\chi = 0$.
Finally, we mention the following classification:

Theorem 5.2. *Let $\mathcal{M}^2$ be a THA-surface of first type given by (3.4), in $\mathbb{G}^3$. There are no THA-surface in $\mathbb{G}^3$ without parabolic points, satisfying the condition $\Delta^{II}\chi = 0$.*

5.3 $I-$ Harmonic THA-surfaces of third type in $\mathbb{G}^3$

From (4.38), $\Delta^{I}\chi = 0$ if and only if $H = 0$.

Theorem 5.3. *Let $\chi : \mathcal{M}^2 \to \mathbb{G}^3$ be an isometric immersion given by (3.10). Then $\Delta^{I}\chi = 0$ if and only if $H = 0$.*

5.4 $II-$ Harmonic THA-surfaces of third type in $\mathbb{G}^3$

In this subsection we are concerned with non-degenerate THA-surfaces of third type $\mathcal{M}^2$ without parabolic points satisfying the condition $\Delta^{II}\chi = 0$.
By a straightforward computation, the Laplacian Δ^{II} of the second fundamental form II on $\mathcal{M}^2$ with the help of (3.12) and (1.2) turns out to be

$$\begin{aligned}\Delta^{II} &= -\frac{1}{AWD^2}\left(\alpha\gamma''\frac{\partial^2}{\partial x^2}+\gamma\alpha''\frac{\partial^2}{\partial y^2}-2\alpha'\gamma'\frac{\partial^2}{\partial x\partial y}\right)\\ &\quad -\frac{1}{2D^4}\left(P(x,\ y)\frac{\partial}{\partial x}+Q(x,\ y)\frac{\partial}{\partial y}\right),\end{aligned}$$

where

$$\begin{aligned}P(x,\ y) &= \frac{1}{AW}(-\alpha'\gamma''(LN+2M^2)+3\gamma'\alpha''MN+\alpha\gamma'''ML-\gamma\alpha'''N^2),\\ Q(x,\ y) &= \frac{1}{AW}(-\alpha''\gamma'(LN+2M^2)+3\alpha'\gamma''ML+\gamma\alpha'''MN-\alpha\gamma'''L^2)\end{aligned}$$

and $D^2 = LN - M^2$.
Accordingly, we get

$$\begin{aligned}\Delta^{II}\chi(x,\ y) &= \begin{pmatrix}\Delta^{II}(\frac{\alpha\gamma}{A}-\frac{B^2}{A})\\ \Delta^{II}(x)\\ \Delta^{II}(y-ax)\end{pmatrix}\\ &= -\frac{1}{2D^4}\begin{pmatrix}4WD^4+\frac{1}{A}(\alpha'\gamma P(x,\ y)+\alpha\gamma'Q(x,\ y))\\ P(x,\ y)\\ Q(x,\ y)-aP(x,\ y)\end{pmatrix}.\end{aligned}$$

$\Delta^{II}\chi = 0$, if and only if

$$\begin{cases}P(x,\ y)=0\\ Q(x,\ y)=0\\ W=0.\end{cases}\tag{5.2}$$

From (5.2) we have $W = 0$, which is a contradiction. Consequently, there are no THA-surfaces in $\mathbb{G}^3$ without parabolic points, satisfying the condition $\Delta^{II}\chi = 0$.
Finally, we mention the following classification:

Theorem 5.4. *Let $\mathcal{M}^2$ be a THA-surface of third type given by (3.10), in $\mathbb{G}^3$. There are no THA-surface in $\mathbb{G}^3$ without parabolic points, satisfying the condition $\Delta^{II}\chi = 0$.*

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