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DOI: 10.2478/awutm-2022-0003

Annals of West University of Timisoara
Mathematics and Computer Science
58, 1, (2022), 18– 38

Optimal control of a contact problem with slip dependent friction

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Abstract The aim of this paper is to study a frictional contact problem between an elastic body and a foundation. We focus on the optimal control of the model which consists of acting with a control on a portion of the boundary of the body and leading the stress tensor as close as possible to a given target. We state an optimal control problem for which we establish an existence theorem of the solution. We then introduce a sequence of regularized problems depending on a positive parameter α and we study the convergence of the sequence of the solutions when the parameter α tends to zero. Finally, an optimality condition is established for this problem.

AMS Subject Classification (2020) 49J20; 49K20; 74M10; 74M15.
Keywords optimal control; variational inequalities; elastic materials; slip-dependent friction; regularized problem.

1 Introduction

The optimal control of variational inequalities has received considerable attention in the mathematical literature, see for instance [2, 3, 6, 9, 11, 14, 15, 17]. This paper is a continuation and an extension of [1, 4, 12, 13, 19]. In [1], the quasistatic frictional bilateral contact problem for elastic materials has been delivered, the authors obtained existence results and necessary optimality conditions for a class of optimal control problems. The bilateral frictional contact with a rigid foundation where the friction is described by a nonlocal version of Coulomb's law was considered in [4]; existence of an optimal control was proven and necessary optimality conditions were derived. In contrast to [1, 4], the contact between an elastic body and a deformable foundation, modelling the contact by means of a normal compliance condition was treated in [13]. In [12], the friction

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bound is slip dependent but the authors have studied a control optimal problem for an antiplane problem which models the deformation of a linear elastic cylindrical body in static frictional contact with a rigid fondation.

From the mathematical point of view, the main difficulties are generated by some features of the functionals employed, like non-convexity, non-coercivity or nondifferentiability. Moreover, the most characteristic property of variational inequalities is that their solution does not depend smoothly, in general, on the control.

The novelty of the present paper consists in dealing with a static frictional contact problem for a linear elastic material which leads to a quasi-variational inequality. In modelling the contact, we impose a non-positive normal stress $-S$ and we consider the Tresca model in which the friction bound prescribed on the contact zone depends on the slip. Our model of slip-dependence is considered in geophysics and solid mechanics corresponding to a dependence of the friction coefficient on the slip u_τ, *i.e.*, $\mu = \mu(|u_\tau|)$, see [7,8,18]. More precisely, the model of the contact discussed below reads as follows:

$$\begin{cases} \sigma_\nu = -S, \\ |\sigma_\tau| \leq S\mu\left(|u_\tau|\right), \\ \sigma_\tau = -S\mu\left(|u_\tau|\right)\dfrac{u_\tau}{|u_\tau|} \ if \ u_\tau \neq 0. \end{cases}$$

The rest of this paper is structured as follows. In Section 2 we introduce some notation and some preliminaries. Section 3 is dedicated to describe the mechanical problem, derive its variational formulation and under certain conditions we establish its weak solvability in Section 4. In Section 5, we state the optimal control problem and prove that it has at least one solution. In Section 6, we show that the solutions of the regularized optimal control problem converge to a solution of the optimal control problem. In Section 7 we provide a necessary optimality condition for the regularized optimal control problem.

2 Notation and preliminaries

Here we introduce the notation we shall use and some preliminary materials. For further details we refer the reader to [5, 18]. We use the notation $\mathbb{N}^*$ for the set of positive integers. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, $(d{=}2, 3)$, and we define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v{=}\sum_{i=1}^{d} u_i v_i, \ \ |u| = \sqrt[2]{u \cdot u}, \ \forall u, \ v \ \in \mathbb{R}^d;$$

$$\sigma \cdot \xi = \sum_{1 \leq i,j \leq d} \sigma_{ij}\xi_{ij}, \ \ |\sigma| = \sqrt[2]{\sigma \cdot \sigma}, \ \forall \sigma, \ \xi \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$, $(d{=}2, 3)$, be a bounded domain with a Lipschitz boundary Γ, let ν be the unit outer normal on Γ and let $x \in \bar{\Omega}$ be the spatial variable. We introduce the spaces

$$H = L^2(\Omega; \mathbb{R}^d), \ \mathcal{Q} = L^2(\Omega; \mathbb{S}^d),$$

$$H_1 = \{u \in H,\ \varepsilon(u) \in \mathcal{Q}\},$$
$$\mathcal{Q}_1 = \{\sigma \in \mathcal{Q},\ Div\sigma \in H\}.$$

Here and below $\varepsilon : H_1 \to \mathcal{Q}$ is the deformation operator defined by

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right),$$

where $(\nabla u)^T$ is the transpose of the matrix ∇u defined by

$$\nabla u = \left(\frac{\partial u_i}{\partial x_j}\right)_{1\le i,j\le d},$$

and $Div : \mathcal{Q}_1 \to H$ is the divergence operator defined by

$$Div\sigma = ((Div\sigma)_i)_{1\le i\le d} = \left(\sum_{j=1}^{d}\frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1\le i\le d}.$$

Note that $H, \mathcal{Q}, H_1, \mathcal{Q}_1$ are Hilbert spaces equipped with the respective canonical inner products

$$(u,v)_H = \int_\Omega u\cdot v dx,\ (\sigma,\tau)_{\mathcal{Q}} = \int_\Omega \sigma\cdot\tau dx,$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{Q}},$$
$$(\sigma,\tau)_{\mathcal{Q}_1} = (Div\sigma, Div\tau)_H + (\sigma,\tau)_{\mathcal{Q}},$$

where the associated norms are denoted by $\|.\|_H$, $\|.\|_{\mathcal{Q}}$, $\|.\|_{H_1}$ and $\|.\|_{\mathcal{Q}_1}$.

Let $H_\Gamma = H^{1/2}(\Gamma;\mathbb{R}^d)$ and let $\tilde{\gamma} : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$ we also use the notation v to denote the trace $\tilde{\gamma}(v)$ of v on Γ and for all $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ:

$$v_\nu = v\cdot\nu,\ v_\tau = v - v_\nu\nu \text{ on } \Gamma.$$

In a similar manner, the normal and tangential components of a regular (say C^1) tensor field σ are defined by

$$\sigma_\nu = \sigma\nu\cdot\nu,\ \sigma_\tau = \sigma\nu - \sigma_\nu\nu \text{ on } \Gamma.$$

Moreover, the following Green formula holds:

$$(Div\sigma, v)_H + (\sigma,\varepsilon(v))_{\mathcal{Q}} = \int_\Gamma \sigma\nu\cdot v da,\ \forall\ v \in H_1, \tag{2.1}$$

where da is the surface measure element.

For any Banach space X we denote by X', $\|.\|_X$ and $\langle,\rangle_{X'\times X}$ the dual of X, the norm on the spaces X and the duality pairing between X and X', respectively. Let us finally recall a standard result that can be found in [9].

Proposition 2.1. *Let Z be a Banach space; let X and Y be two reflexive Banach spaces. Let also be given two $\mathcal{C}^1$ functions*

$$\mathcal{F} :\ Z\times X \to Y,$$
$$\mathcal{L} :\ Z\times X \to \mathbb{R}.$$

We suppose that for all $\varphi \in Z$

i) There exists a unique $\tilde{u}(\varphi)$ *such that* $\mathcal{F}(\varphi, \tilde{u}(\varphi)) = 0$;

ii) $\partial_2\mathcal{F}(\varphi, \tilde{u}(\varphi))$ *is an isomorphism from* X *onto* Y.

Then $J(\varphi) = \mathcal{L}(\varphi, \tilde{u}(\varphi))$ *is differentiable such that*

$$\frac{dJ(\varphi)}{d\varphi}\psi = \partial_1\mathcal{L}(\varphi, \tilde{u}(\varphi))\psi - \langle l(\varphi), \partial_1\mathcal{F}(\varphi, \tilde{u}(\varphi))\psi\rangle_{Y',Y}, \ \forall\psi \in Z,$$

where $l(\varphi) \in Y$ *is the adjoint state, unique solution of*

$$[\partial_2\mathcal{F}(\varphi, \tilde{u}(\varphi))]^* l(\varphi) = \partial_2\mathcal{L}(\varphi, \tilde{u}(\varphi)) \ \text{in } X'.$$

Here and below, by ∂_i, $i \in \{1, 2\}$, we denote a partial derivative.

3 Problem statement. Assumptions. Variational formulation

The physical setting is as follows. A deformable body occupies the reference configuration $\Omega \subset \mathbb{R}^{d=2,3}$ which is a bounded domain with smooth enough boundary Γ. The body is assumed to have elastic law and the process is static. We assume that Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that meas $(\Gamma_1) > 0$. The body is clamped on Γ_1 and therefore the displacement field vanishes there, while volume forces of density f_0 act in Ω and surface tractions of density f_2 act on Γ_2. On Γ_3 the body is in frictional contact with a foundation such that the normal stress is prescribed on the contact surface and the coefficient of friction depends on the slip. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Under the above assumptions, the classical formulation of our problem is the following.

Problem 3.1. Find a displacement field $u : \Omega \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \to \mathbb{S}^d$ such that

$$\sigma = \mathcal{A}\varepsilon(u), \ \text{in } \Omega, \tag{3.1}$$

$$Div\sigma + f_0 = 0, \ \text{in } \Omega, \tag{3.2}$$

$$u = 0, \ \text{on } \Gamma_1, \tag{3.3}$$

$$\sigma\nu = f_2, \ \text{on } \Gamma_2, \tag{3.4}$$

$$\sigma_\nu = -S, \ \text{on } \Gamma_3, \tag{3.5}$$

$$\begin{cases} |\sigma_\tau| \le S\mu(|u_\tau|), \\ \sigma_\tau = -S\mu(|u_\tau|)\dfrac{u_\tau}{|u_\tau|} \ if \ u_\tau \ne 0 \end{cases} \quad \text{on } \Gamma_3. \tag{3.6}$$

Equation (3.1) represents the elastic constitutive law in which $\varepsilon(u)$ denotes the small strain tensor, $\mathcal{A}$ is the elasticity operator. Equation (3.2) represents the equilibrium equation posed on the domain Ω. Conditions (3.3)-(3.4) are the displacement-traction

boundary conditions where $\sigma\nu$ represents the Cauchy stress vector. Conditions (3.5)-(3.6) characterize the contact boundary conditions. We consider here the Tresca model in which the friction bound prescribed on the contact zone depends on the slip and the normal stress is given, i.e. $\sigma_\nu = -S$ on Γ_3. Condition (3.6) states that, if there is contact, the tangential traction σ_τ is bounded by the normal stress S multiplied by the value of the friction coefficient $\mu(|u_\tau|)$. If the strict inequality is satisfied, sliding does not occur, and when equality is satisfied, the friction stress is opposed to the slip.

To obtain a variational formulation of the mechanical Problem (3.1)-(3.6), we need additional notation. We consider the space of fourth-order tensor fields

$$\mathbf{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega),\ \forall i,j,k,l \in \{1,...,d\}\},$$

which is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1\le i,j,k,l\le d} \|\mathcal{E}_{jikl}\|_{L^\infty(\Omega)}. \tag{3.7}$$

Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1,\ v = 0\ on\ \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, the following Korn's inequality holds:

$$C_K \|v\|_{H_1} \le \|\varepsilon(v)\|_{\mathcal{Q}},\ \forall v \in V, \tag{3.8}$$

where $C_K > 0$ is a positive constant depending only on Ω and Γ_1. A proof of Korn's inequality can be found, for instance, in [16], p. 79. Over the space V, we consider the inner product given by

$$(w,v)_V = (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}},\ \forall w, v \in V,$$

and let $\|.\|_V$ be the associated norm. It follows from Korn's inequality (3.8) that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V. Therefore $(V, (.,.)_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem, there exists a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le c_0 \|v\|_V,\ \forall v \in V. \tag{3.9}$$

In the study of the mechanical Problem (3.1)-(3.6) we assume that $\mathcal{A} = (\mathcal{A}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \mathcal{A}\varepsilon \cdot \varepsilon \ge m_{\mathcal{A}} |\varepsilon|^2,\ \forall\ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.e.\ x \in \Omega; \\ \\ \text{(b) } \mathcal{A} \in \mathbf{Q}_\infty. \end{cases} \tag{3.10}$$

We assume that the coefficient of friction $\mu : \Gamma_3 \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$\begin{cases} \text{(a) There exists } L_\mu > 0 \text{ such that} \\ |\mu(x, r_1) - \mu(x, r_2)| \le L_\mu |r_1 - r_2|,\ \forall\ r_1, r_2 \in \mathbb{R}^+,\ a.e.\ x \in \Gamma_3; \\ \\ \text{(b) The mapping } x \to \mu(x,r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \forall\ r \in \mathbb{R}^+; \\ \\ \text{(c) The mapping } x \to \mu(x,0) \in L^2(\Gamma_3). \end{cases} \tag{3.11}$$

We assume that

$$S \in L^{\infty}(\Gamma_3) \text{ and } S \geq 0 \text{ } a.e. \text{ } on \text{ } \Gamma_3. \tag{3.12}$$

$$(i) \ f_0 \ \in H, \ (ii) \ f_2 \in L^2(\Gamma_2;\mathbb{R}^d). \tag{3.13}$$

Next, we use the functional $j_\tau : V \times V \longrightarrow \mathbb{R}$ defined by

$$j_\tau(w,v) = \int_{\Gamma_3} S\mu\left(x,|w_\tau|\right)|v_\tau| \, da, \ \forall w, \ v \in V. \tag{3.14}$$

Finally, we use the Riesz representation theorem to define the element $f \in V$ by

$$(f,w)_V = \int_\Omega f_0 \cdot w dx - \int_{\Gamma_3} S w_\nu da + \int_{\Gamma_2} f_2 \cdot w da, \ \forall w \in V. \tag{3.15}$$

Now, assume u and σ are smooth functions satisfying (3.1)-(3.6) and use Green formula (2.1) to obtain the following variational formulation in the term of displacement field only.

Problem 3.2 ($\mathcal{PV}$). Find a displacement field $u \in V$ such that

$$(\mathcal{A}\varepsilon(u), \varepsilon\left(w-u\right))_{\mathcal{Q}} + j_\tau(u,w) - j_\tau\left(u,u\right) \geq \left(f, w-u\right)_V, \ \forall \ w \in V. \tag{3.16}$$

4 Existence and uniqueness of a weak solution

Our main existence and uniqueness result in this section is the following.

Theorem 4.1. *Assume that* (3.10)-(3.13) *are fulfilled. Then, there exists* $L_0 > 0$ *such that for* $L_\mu \|S\|_{L^\infty(\Gamma_3)} < L_0$, *Problem* $\mathcal{PV}$ *has a unique solution* $u \in V$.

Proof. Let $A : V \to V$ be the operator defined by

$$(Aw,v)_V = \left(\mathcal{A}\varepsilon\left(w\right), \varepsilon\left(v\right)\right)_{\mathcal{Q}}, \ \forall w, \ v \in V. \tag{4.1}$$

Let $\eta \in V$. Consider the following problem. Find $u_\eta \in V$ such that

$$\left(Au_\eta, v-u_\eta\right)_V + j_\tau(\eta,v) - j_\tau(\eta,u_\eta) \geq \left(f, v-u_\eta\right)_V, \text{ for all } v \in \ V. \tag{4.2}$$

It follows from (3.10) and (4.1) that the operator A satisfies

$$m_{\mathcal{A}} \|w_1 - w_2\|_V^2 \leq \left(Aw_1 - Aw_2, w_1 - w_2\right)_V, \ \forall w_1, w_2 \in V. \tag{4.3}$$

Moreover, there exists $L_{\mathcal{A}} > 0$ such that

$$\|\mathcal{A}\varepsilon_1 - \mathcal{A}\varepsilon_2\|_{\mathcal{Q}} \leq L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|_{\mathcal{Q}}, \ \forall \varepsilon_1, \varepsilon_2 \in \mathcal{Q}, \tag{4.4}$$

which with (4.1) gives

$$\|Aw_1 - Aw_2\|_V \leq L_{\mathcal{A}} \|w_1 - w_2\|_V, \ \forall w_1, w_2 \in V. \tag{4.5}$$

Thus, A is a strongly monotone Lipschitz continuous operator. Using the following inequality

$$|\lambda w + (1-\lambda)\, v| \leq \lambda\, |w| + (1-\lambda)\, |v|\,,\ \forall w,\, v \in \mathbb{R}^d,\ \forall \lambda \in [0,1]\,,$$

it follows from (3.11)-(3.12) and (3.14), that the functional $\Theta_\eta : V \to \mathbb{R}$ defined by

$$\Theta_\eta(v) = j_\tau(\eta, v),\ \forall v \in V, \tag{4.6}$$

is proper convex and continuous. Therefore, using a standard result on elliptic variational inequalities of the second kind, we deduce that the problem (4.2) has a unique solution $u_\eta \in V$. To continue, we define the operator $\Lambda : V \to V$ by

$$\Lambda\,(\eta) = u_\eta. \tag{4.7}$$

Let $\eta_1,\ \eta_2 \in V$, using the notation $u_1 = u_{\eta_1}$ and $u_2 = u_{\eta_2}$, then by (4.2) we get

$$(Au_1 - Au_2, u_1 - u_2)_V \leq j_\tau(\eta_1, u_2) - j_\tau(\eta_1, u_1) + j_\tau(\eta_2, u_1) - j_\tau(\eta_2, u_2),$$

and keeping in mind (4.3), 3.9, (3.11)-(3.12) and (3.14), we obtain

$$m_{\mathcal{A}}\, \|u_1 - u_2\|_V^2 \leq c_0^2 L_\mu\, \|S\|_{L^\infty(\Gamma_3)}\, \|\eta_1 - \eta_2\|_V\, \|u_1 - u_2\|_V\,,$$

which together with (4.7) gives

$$\|\Lambda\eta_2 - \Lambda\eta_1\|_V \leq \frac{c_0^2 L_\mu\, \|S\|_{L^\infty(\Gamma_3)}}{m_{\mathcal{A}}}\, \|\eta_1 - \eta_2\|_V\,.$$

This last inequality implies that, for $L_\mu\, \|S\|_{L^\infty(\Gamma_3)} < L_0 = \dfrac{m_{\mathcal{A}}}{c_0^2}$, Λ is a contraction in the Banach space V. Therefore, there exists a unique $\eta^* \in V$, such that $\eta^* = \Lambda\eta^* = u_{\eta^*}$. We have now all the ingredients to prove Theorem 4.1. Let η^* be the unique fixed point of Λ and let u_{η^*} be the unique solution of the problem (4.2) for $\eta = \eta^*$, then u_{η^*} is a solution of Problem $\mathcal{PV}$. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ and of the uniqueness of the solution of Problem (4.2). □

In the rest of this paper, we assume that

$$L_\mu\, \|S\|_{L^\infty(\Gamma_3)} < \frac{m_{\mathcal{A}}}{c_0^2}. \tag{4.8}$$

5 The optimal control problem

Now, we would like to act a surface traction on Γ_2 so that the resulting stress $\sigma = \mathcal{A}\varepsilon(u)$ be as close as possible to a given target

$$\sigma_g = \mathcal{A}\varepsilon(u_g). \tag{5.1}$$

Here in (5.1) and everywhere in this paper u_g is a given function in V. Thanks to (4.4), we conclude that

$$\|\sigma - \sigma_g\|_{\mathcal{Q}} \leq L_{\mathcal{A}}\, \|u - u_g\|_V\,.$$

Thus, σ closes to σ_g in the sense of $\mathcal{Q}-$norm if the function u closes to the target u_g in the sense of $V-$norm. As an example, we can consider the particular case $u_g = 0$. Then, if we act through a good choice a control f_2 on Γ_2, we obtain a small stress σ in the sense of $\mathcal{Q}-$norm, even if the volume force does not vanish in Ω.

Let S and f_0 be as in (3.12) and (i) (3.13). Using the Riesz representation theorem, we deduce that there exists a unique element $h \in V$ which satisfies

$$(h, w)_V = \int_\Omega f_0 \cdot w dx - \int_{\Gamma_3} S w_\nu da, \; \forall w \in V. \tag{5.2}$$

Also, we can introduce the operator $F : L^2\left(\Gamma_2; \mathbb{R}^d\right) \to V$ defined by

$$\left(F\left(\varphi\right), w\right)_V = \int_{\Gamma_2} \varphi \cdot w da, \; \forall \left(\varphi, w\right) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V. \tag{5.3}$$

Now, consider the following state problem.

Problem 5.1 $(\mathcal{P}_S)$. Let $f_2 \in L^2\left(\Gamma_3; \mathbb{R}^d\right)$ be a given control. Find a displacement field $u \in V$ such that

$$\left(Au, w - u\right)_V + j_\tau(u, w) - j_\tau\left(u, u\right) \geq \left(h + F\left(f_2\right), w - u\right)_V, \text{ for all } w \in V. \tag{5.4}$$

Using Theorem 4.1, we deduce that for all $f_2 \in L^2\left(\Gamma_3; \mathbb{R}^d\right)$ the state Problem $\mathcal{P}_S$ has a unique solution $u \in V$. Next, we introduce the functional $\mathcal{L} : L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V \to \mathbb{R}$ defined by

$$\mathcal{L}\left(\varphi, w\right) = \frac{\beta}{2} \left\|w - u_g\right\|_V^2 + \frac{\delta}{2} \left\|\varphi\right\|_{L^2\left(\Gamma_2; \mathbb{R}^d\right)}^2, \; \forall \left(\varphi, w\right) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V, \tag{5.5}$$

where β, $\delta > 0$ are two positive constants. We consider the following optimal control problem.

Problem 5.2 $(\mathcal{P}_{OC})$. Find $(\varphi^*, u^*) \in \mathcal{W}$ such that

$$\mathcal{L}\left(\varphi^*, u^*\right) = \min_{(\varphi, u) \in \mathcal{W}} \left\{\mathcal{L}\left(\varphi, u\right) \right\}, \tag{5.6}$$

where

$$\mathcal{W} = \left\{\left(f_2, u\right) / \left(f_2, u\right) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V, \text{ such that (5.4) holds}\right\}. \tag{5.7}$$

We have the following result.

Theorem 5.3. *Problem $\mathcal{P}_{OC}$ has at least one solution $(\varphi^*, u^*) \in \mathcal{W}$.*

Proof. From (5.5), we deduce that there exists a sequence $\left\{(\varphi_n, u_n)\right\}_{n \in \mathbb{N}^*} \subset \mathcal{W}$ such that

$$0 \leq \inf_{(\varphi, u) \in \mathcal{W}} \left\{\mathcal{L}\left(\varphi, u\right)\right\} \leq \mathcal{L}\left(\varphi_n, u_n\right) \leq \inf_{(\varphi, u) \in \mathcal{W}} \left\{\mathcal{L}\left(\varphi, u\right)\right\} + \frac{1}{n}, \tag{5.8}$$

which implies that $\left\{\varphi_n\right\}_{n \in \mathbb{N}^*}$ and $\left\{u_n\right\}_{n \in \mathbb{N}^*}$ are bounded, respectively, in $L^2\left(\Gamma_2; \mathbb{R}^d\right)$ and V. Moreover, we have

$$\lim_{n \to +\infty} \mathcal{L}\left(\varphi_n, u_n\right) = \inf_{(\varphi, u) \in \mathcal{W}} \left\{\mathcal{L}\left(\varphi, u\right)\right\}. \tag{5.9}$$

Thus, from standard compactness arguments, it follows that there exists $(\varphi^*, u^*) \in L^2\left(\Gamma_2;\mathbb{R}^d\right) \times V$ and a subsequence of $\{(\varphi_n, u_n)\}_{n\in\mathbb{N}^*}$, again denoted by $\{(\varphi_n, u_n)\}_{n\in\mathbb{N}^*}$, such that the following convergences hold

$$u_n \rightharpoonup u^* \text{ weakly in } V \text{ as } n \to +\infty, \tag{5.10}$$
$$u_n \to u^* \text{ strongly in } L^2\left(\Gamma_2;\mathbb{R}^d\right) \text{ as } n \to +\infty, \tag{5.11}$$
$$u_n \to u^* \text{ strongly in } L^2\left(\Gamma_3;\mathbb{R}^d\right) \text{ as } n \to +\infty, \tag{5.12}$$
$$\varphi_n \rightharpoonup \varphi^* \text{ weakly in } L^2\left(\Gamma_2;\mathbb{R}^d\right) \text{ as } n \to +\infty. \tag{5.13}$$

Using (5.5), (5.10) and (5.13) we obtain

$$\mathcal{L}\left(\varphi^*, u^*\right) \leq \liminf_{n\to+\infty} \mathcal{L}\left(\varphi_n, u_n\right), \tag{5.14}$$

which together with (5.9) gives

$$\mathcal{L}\left(\varphi^*, u^*\right) \leq \inf_{(\varphi,u)\in\mathcal{W}} \left\{\mathcal{L}\left(\varphi, u\right)\right\}. \tag{5.15}$$

Moreover, using standard semicontinuity arguments, we get

$$\liminf_{n\to+\infty} \left(Au_n, u_n - w\right)_V \geq \left(Au^*, u^* - w\right)_V. \tag{5.16}$$

On the other hand, from (3.11), (3.12), (3.14) and (5.12), it follows that

$$\lim_{n\to+\infty} \left[j_\tau(u_n, w) - j_\tau\left(u_n, u_n\right)\right] = j_\tau(u^*, w) - j_\tau\left(u^*, u^*\right). \tag{5.17}$$

Using (5.2)-(5.3), (5.10)-(5.11) and (5.13) we deduce that

$$\lim_{n\to+\infty} \left(h + F\left(\varphi_n\right), w - u_n\right)_V = \left(h + F\left(\varphi^*\right), w - u^*\right)_V. \tag{5.18}$$

Now, since $\{(\varphi_n, u_n)\}_{n\in\mathbb{N}^*} \subset \mathcal{W}$ we have

$$\left(Au_n, w - u_n\right)_V + j_\tau(u_n, w) - j_\tau\left(u_n, u_n\right) \geq \left(h + F\left(\varphi_n\right), w - u_n\right)_V, \text{ for all } w \in V. \tag{5.19}$$

Keeping in mind (5.15)-(5.18) and passing to the lim sup as $n \to +\infty$ in (5.19), we deduce that (φ^*, u^*) is a solution of Problem $\mathcal{P}_{OC}$. □

Next, a solution of Problem $\mathcal{P}_{OC}$ will be called an optimal pair and the first component of the optimal pair is called an optimal control.

6 Regularized problems and convergence results

We want to apply a regularization technique to problem $\mathcal{P}_S$ and $\mathcal{P}_{OC}$ in order to obtain smooth problems. These problems have some interest on their own, and provide an approximation for the solution of the original problem.

Let us fix $\alpha > 0$, we introduce the regularized functional $j_\alpha : V \times V \to \mathbb{R}$ such that, for all $z, w \in V$,

$$j_\alpha(z,w) = \int_{\Gamma_3} S\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) \times \left(\sqrt[2]{|w_\tau|^2 + \alpha^2} - \alpha\right) da, \tag{6.1}$$

and for all $z \in V$, let $j'_\alpha(z; w, .) : V \to \mathbb{R}$ be the Gâteaux derivative of $j_\alpha(z, .)$ at the point $w \in V$. Since

$$\frac{\sqrt[2]{|w_\tau + \lambda v_\tau|^2 + \alpha^2} - \sqrt[2]{|w_\tau|^2 + \alpha^2}}{\lambda} = \frac{|\lambda v_\tau|^2 + 2\lambda w_\tau \cdot v_\tau}{\lambda\left(\sqrt[2]{|w_\tau + \lambda v_\tau|^2 + \alpha^2} + \sqrt[2]{|w_\tau|^2 + \alpha^2}\right)}, \quad \forall\ w,\ v \in V,\ \forall \lambda \in \mathbb{R}, \tag{6.2}$$

we obtain

$$j'_\alpha(z; w, v) = \int_{\Gamma_3} S\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) \frac{w_\tau}{\sqrt[2]{|w_\tau|^2 + \alpha^2}} \cdot v_\tau da,\ \forall\ v \in V. \tag{6.3}$$

On the other hand, we have

$$\begin{aligned}
&\left|\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) \times \left(\sqrt[2]{|w_\tau|^2 + \alpha^2} - \alpha\right) - \mu\left(x, |z_\tau|\right)|w_\tau|\right| \\
&\qquad \leq \left|\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) - \mu\left(x, |z_\tau|\right)\right| \left(\sqrt[2]{|w_\tau|^2 + \alpha^2} - \alpha\right) \\
&\qquad\quad + \left|\sqrt[2]{|w_\tau|^2 + \alpha^2} - \alpha - |w_\tau|\right| \mu\left(x, |z_\tau|\right),
\end{aligned}$$

which gives

$$\begin{aligned}
&\left|\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) \times \left(\sqrt[2]{|w_\tau|^2 + \alpha^2} - \alpha\right) - \mu\left(x, |z_\tau|\right)|w_\tau|\right| \\
&\qquad \leq L_\mu \left(\left(\sqrt[2]{|z_\tau|^2 + \alpha^2} - |z_\tau|\right) + \alpha\right) \left(\sqrt[2]{|w_\tau|^2 + \alpha^2} - \alpha\right) \\
&\qquad\quad + \left(\left(\sqrt[2]{|w_\tau|^2 + \alpha^2} - |w_\tau|\right) + \alpha\right) \left(L_\mu |z_\tau| + \mu\left(x, 0\right)\right),
\end{aligned}$$

and using the fact that

$$\sqrt[2]{|w_\tau|^2 + \alpha^2} \leq |w_\tau| + \alpha,\ \forall\ w \in V,$$

we deduce that there exists $c > 0$ such that

$$|j_\alpha(z,w) - j_\tau(z,w)| \leq c\alpha\left[\|w\|_V + \|z\|_V + 1\right],\ \forall z, w \in V. \tag{6.4}$$

Now, we state the following regularized problem.

Problem 6.1 ($\mathcal{P}_S^\alpha$). Let $\alpha > 0$. Let $f_2 \in L^2\left(\Gamma_3;\mathbb{R}^d\right)$ be a given control. Find a displacement field $u \in V$ such that

$$(Au, w-u)_V + j_\alpha(u,w) - j_\alpha(u,u) \geq (h + F(f_2), w-u)_V, \text{ for all } w \in V. \tag{6.5}$$

Theorem 6.2. *Problem $\mathcal{P}_S^\alpha$ has a unique solution $u \in V$.*

Proof. The proof can be obtained by arguments similar to those used in the proof of Theorem 4.1. □

Let $\mathcal{L}$ be the functional given by (5.5). We introduce an optimal control problem as follows

Problem 6.3 ($\mathcal{P}_{OC}^\alpha$). Find $(\overline{\varphi}, \overline{u}) \in \mathcal{W}^\alpha$ such that

$$\mathcal{L}(\overline{\varphi}, \overline{u}) = \min_{(\varphi,u)\in\mathcal{W}^\alpha} \{\mathcal{L}(\varphi,u)\ \}, \tag{6.6}$$

where

$$\mathcal{W}^\alpha = \left\{(f_2,u) /\ (f_2,u) \in L^2\left(\Gamma_2;\mathbb{R}^d\right) \times V \text{ such that (6.5) holds}\right\}. \tag{6.7}$$

The following result holds.

Theorem 6.4. *Problem $\mathcal{P}_{OC}^\alpha$ has at least one solution $(\overline{\varphi}, \overline{u})$.*

Proof. The proof can be obtained by arguments similar to those used in the proof of Theorem 5.3. □

Next, a solution of Problem $\mathcal{P}_{OC}^\alpha$ will be called a regularized optimal pair and the first component of the optimal pair is called a regularized optimal control.

Remark 6.1. Let (φ^*, u^*) be a solution of Problem $\mathcal{P}_{OC}$. Then, φ^* minimizes the functional $\mathcal{J} : L^2\left(\Gamma_2;\mathbb{R}^d\right) \to \mathbb{R}$ defined by

$$\mathcal{J}(\varphi) = \frac{\beta}{2}\|u - u_g\|_V^2 + \frac{\delta}{2}\|\varphi\|^2_{L^2(\Gamma_2;\mathbb{R}^d)}, \tag{6.8}$$

where $(\varphi, u) \in \mathcal{W}$. Also, let $\alpha > 0$ and let $(\overline{\varphi}, \overline{u})$ be a solution of Problem $\mathcal{P}_{OC}^\alpha$. Then, $\overline{\varphi}$ minimizes the functional $\mathcal{J}_\alpha : L^2\left(\Gamma_2;\mathbb{R}^d\right) \to \mathbb{R}$ defined by

$$\mathcal{J}_\alpha(\varphi) = \frac{\beta}{2}\|u - u_g\|_V^2 + \frac{\delta}{2}\|\varphi\|^2_{L^2(\Gamma_2;\mathbb{R}^d)}, \tag{6.9}$$

where $(\varphi, u) \in \mathcal{W}^\alpha$.

We now study the relation between the solutions of Problem $\mathcal{P}_S^\alpha$ and Problem $\mathcal{P}_S$ when α tends to zero. More precisely, we are going to prove the following result.

Theorem 6.5. *Let $(f_2^\alpha, u^\alpha) \in \mathcal{W}^\alpha$ and let $(f_2, u) \in \mathcal{W}$. If*

$$f_2^\alpha \rightharpoonup f_2 \text{ weakly in } L^2\left(\Gamma_2;\mathbb{R}^d\right) \text{ as } \alpha \to 0, \tag{6.10}$$

then,

$$u^\alpha \to u \text{ strongly in } V \text{ as } \alpha \to 0. \tag{6.11}$$

Proof. Choosing in (6.5) $w = 0_V$, we deduce that

$$\|u^\alpha\|_V \leq \frac{1}{m_{\mathcal{A}}} \left(\|h\|_V + \|F\left(f_2^\alpha\right)\|_V\right), \tag{6.12}$$

which together with (6.10) and (5.3) implies that $\{u^\alpha\}_\alpha$ is a bounded sequence in V. Thus, using standard compactness arguments, we deduce that there exists $u^* \in V$ and a subsequence of $\{u^\alpha\}_\alpha$, again denoted by $\{u^\alpha\}_\alpha$, such that the following convergences hold

$$u^\alpha \rightharpoonup u^* \text{ weakly in } V \text{ as } \alpha \to 0. \tag{6.13}$$

$$u^\alpha \to u^* \text{ strongly in } L^2\left(\Gamma_2;\mathbb{R}^d\right) \text{ as } \alpha \to 0. \tag{6.14}$$

$$u^\alpha \to u^* \text{ strongly in } L^2\left(\Gamma_3;\mathbb{R}^d\right) \text{ as } \alpha \to 0. \tag{6.15}$$

Since, (f_2^α, u^α) satisfies

$$\left(Au^\alpha, w - u^\alpha\right)_V + j_\alpha(u^\alpha, w) - j_\alpha\left(u^\alpha, u^\alpha\right) \geq \left(h + F\left(f_2^\alpha\right), w - u^\alpha\right)_V, \text{ for all } w \in V, \tag{6.16}$$

then, from (6.10), (6.13)-(6.15) and using arguments similar to those used in (5.16)-(5.18), it follows that $(f_2, u^*) \in \mathcal{W}$, which implies that $u^* = u$. To continue, using (5.4) and (6.16), we get

$$\begin{aligned}\left(Au^\alpha - Au, u^\alpha - u\right)_V \leq & j_\alpha(u^\alpha, u) - j_\tau(u, u) + j_\tau\left(u, u^\alpha\right) - j_\alpha\left(u^\alpha, u^\alpha\right) \\ & + \left(F\left(f_2^\alpha - f_2\right), u^\alpha - u\right)_V,\end{aligned}$$

which gives

$$\begin{aligned}\left(Au^\alpha - Au, u^\alpha - u\right)_V \leq & j_\tau\left(u, u^\alpha\right) - j_\tau(u, u) + j_\tau(u^\alpha, u) - j_\tau\left(u^\alpha, u^\alpha\right) + j_\alpha(u^\alpha, u) \\ & - j_\tau(u^\alpha, u) + j_\tau\left(u^\alpha, u^\alpha\right) - j_\alpha\left(u^\alpha, u^\alpha\right) + \left(F\left(f_2^\alpha - f_2\right), u^\alpha - u\right)_V.\end{aligned}$$

Therefore, using (4.3), (3.9), (3.11)-(3.12) and (3.14), one has

$$\begin{aligned}m_{\mathcal{A}} \|u^\alpha - u\|_V^2 \leq & c_0^2 L_\mu \|S\|_{L^\infty(\Gamma_3)} \|u^\alpha - u\|_V^2 + j_\alpha(u^\alpha, u) - j_\tau(u^\alpha, u) + j_\tau\left(u^\alpha, u^\alpha\right) \\ & - j_\alpha\left(u^\alpha, u^\alpha\right) + \left(F\left(f_2^\alpha - f_2\right), u^\alpha - u\right)_V,\end{aligned}$$

from which we get

$$\begin{aligned}\left(m_{\mathcal{A}} - c_0^2 L_\mu \|S\|_{L^\infty(\Gamma_3)}\right) \|u^\alpha - u\|_V^2 \leq & \left|j_\alpha(u^\alpha, u) - j_\tau(u^\alpha, u)\right| + \left|j_\tau(u^\alpha, u^\alpha) - j_\alpha(u^\alpha, u^\alpha)\right| \\ & + \left(F\left(f_2^\alpha - f_2\right), u^\alpha - u\right)_V.\end{aligned}$$

Keeping in mind (6.4), (5.3), (6.10) and passing to the limit as $\alpha \to 0$ in the last inequality we obtain

$$\lim_{\alpha \to 0} \|u^\alpha - u\|_V = 0. \tag{6.17}$$

Finally, since Problem $\mathcal{P}_S$ has a unique solution, the above arguments applied to any subsequences of $\{u^\alpha\}_\alpha$ again bring us to the same limit u. Thus, we deduce that the whole sequence $\{u^\alpha\}_\alpha$ satisfies (6.11). □

Next, we prove convergence results involving the solutions of the Problems $\mathcal{P}_{OC}^{\alpha}$ and $\mathcal{P}_{OC}$.

Theorem 6.6. *For each $\alpha > 0$, let $\left(\overline{f}_2^{\alpha}, \overline{u}^{\alpha} \right)$ be a solution of Problem $\mathcal{P}_{OC}^{\alpha}$. Then, there exists a subsequence of $\left\{ \left(\overline{f}_2^{\alpha}, \overline{u}^{\alpha} \right) \right\}_{\alpha>0}$, again denoted by $\left\{ \left(\overline{f}_2^{\alpha}, \overline{u}^{\alpha} \right) \right\}_{\alpha>0}$, and there exists, $\left(\overline{f}_2, \overline{u} \right) \in \mathcal{W}$ a solution of the Problem $\mathcal{P}_{OC}$, such that the following convergences hold*

$$\overline{u}^{\alpha} \rightarrow \overline{u} \text{ strongly in } V \text{ as } \alpha \rightarrow 0, \tag{6.18}$$

$$\overline{f}_2^{\alpha} \rightharpoonup \overline{f}_2 \text{ weakly in } L^2\left(\Gamma_2; \mathbb{R}^d\right) \text{ as } \alpha \rightarrow 0. \tag{6.19}$$

Proof. Indeed, let $\left(\overline{f}_2^{\alpha}, \overline{u}^{\alpha} \right)$ be a solution of Problem $\mathcal{P}_{OC}^{\alpha}$ and let $(\varphi_2, u^{\alpha}) \in \mathcal{W}^{\alpha}$, then, from Remark 6.1, we get

$$\mathcal{J}_{\alpha}\left(\overline{f}_2^{\alpha} \right) \leq \mathcal{J}_{\alpha}\left(\varphi_2 \right). \tag{6.20}$$

Moreover, choosing in (6.5) $w = 0_V$, we deduce that

$$\|u^{\alpha}\|_V \leq \frac{1}{m_{\mathcal{A}}} \left(\|h\|_V + \|F\left(\varphi_2\right)\|_V \right),$$

which together with (6.20) and (6.9) implies that $\left\{ \mathcal{J}_{\alpha}\left(\overline{f}_2^{\alpha} \right) \right\}_{\alpha}$ is a bounded sequence. Consequently, $\left\{ \overline{f}_2^{\alpha} \right\}_{\alpha}$ and $\{\overline{u}^{\alpha}\}_{\alpha}$ are bounded sequences in $L^2\left(\Gamma_2; \mathbb{R}^d\right)$ and V, respectively. Thus, from standard compactness arguments, it follows that there exists $\left(\overline{f}_2, \overline{u} \right) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V$ and a subsequence of $\left\{ \left(\overline{f}_2^{\alpha}, \overline{u}^{\alpha} \right) \right\}_{\alpha}$, again denoted by $\left\{ \left(\overline{f}_2^{\alpha}, \overline{u}^{\alpha} \right) \right\}_{\alpha}$, such that the following convergences hold

$$\overline{u}^{\alpha} \rightharpoonup \overline{u} \text{ weakly in } V \text{ as } \alpha \rightarrow 0,$$
$$\overline{f}_2^{\alpha} \rightharpoonup \overline{f}_2 \text{ weakly in } L^2\left(\Gamma_2; \mathbb{R}^d\right) \text{ as } \alpha \rightarrow 0.$$

We use now arguments similar to those used in (5.16)-(5.18) to see that $\left(\overline{f}_2, \overline{u} \right) \in \mathcal{W}$. Moreover, using Theorem 6.5, we have

$$\overline{u}^{\alpha} \rightarrow \overline{u} \text{ strongly in } V \text{ as } \alpha \rightarrow 0. \tag{6.21}$$

On the other hand, let $(\overline{\varphi}_2, \overline{w})$ be a solution of Problem $\mathcal{P}_{OC}$, then

$$\mathcal{L}\left(\overline{\varphi}_2, \overline{w} \right) \leq \mathcal{L}\left(\overline{f}_2, \overline{u} \right), \tag{6.22}$$

Now, using convexity arguments, we obtain

$$\mathcal{L}\left(\overline{f}_2, \overline{u} \right) \leq \liminf_{\alpha \rightarrow 0} \mathcal{L}\left(\overline{f}_2^{\alpha}, \overline{u}^{\alpha} \right). \tag{6.23}$$

and let us consider the sequence $\{(\overline{\varphi}_2, u^{\alpha})\}_{\alpha} \subset \mathcal{W}^{\alpha}$, then, using arguments similar to those used in (5.16)-(5.18), we deduce that

$$\lim_{\alpha \rightarrow 0} \|u^{\alpha} - \overline{w}\|_V = 0.$$

Moreover, we have

$$\begin{aligned}\limsup_{\alpha\to 0}\mathcal{L}\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right) &\leq \limsup_{\alpha\to 0}\mathcal{L}\left(\overline{\varphi}_2,u^{\alpha}\right)\\ &\leq \mathcal{L}\left(\overline{\varphi}_2,\overline{w}\right).\end{aligned} \tag{6.24}$$

Thus, from (6.22)-(6.24), it follows that

$$\mathcal{L}\left(\overline{\varphi}_2,\overline{w}\right)=\mathcal{L}\left(\overline{f}_2,\overline{u}\right).$$

From this we deduce that $\left(\overline{f}_2,\overline{u}\right)$ is a solution of Problem $\mathcal{P}_{OC}$. □

Corollary 6.7. *Let* $\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right)$ *be a solution of Problem* $\mathcal{P}_{OC}^{\alpha}$ *and let* $(f_2^*,u^*)\in\mathcal{W}$. *If*

$$\overline{f}_2^{\alpha}\rightharpoonup f_2^* \text{ weakly in } L^2\left(\Gamma_2;\mathbb{R}^d\right) \text{ as } \alpha\to 0. \tag{6.25}$$

Then (f_2^*,u^*) *is a solution of Problem* $\mathcal{P}_{OC}$. *Moreover, we have*

$$\overline{u}^{\alpha} \to u^* \text{ strongly in } V \text{ as } \alpha\to 0, \tag{6.26}$$

$$\overline{f}_2^{\alpha} \to f_2^* \text{ stronly in } L^2\left(\Gamma_2;\mathbb{R}^d\right) \text{ as } \alpha\to 0. \tag{6.27}$$

Proof. Let $\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right)$ be a solution of Problem $\mathcal{P}_{OC}^{\alpha}$. From Theorem 6.6, there exists $\left(\overline{f}_2,\overline{u}\right)\in\mathcal{W}$, which is a solution of the Problem $\mathcal{P}_{OC}$, and there exists a subsequence of $\left\{\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right)\right\}_{\alpha}$, again denoted by $\left\{\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right)\right\}_{\alpha}$, such that the following convergences hold

$$\overline{u}^{\alpha} \to \overline{u} \text{ strongly in } V \text{ as } \alpha\to 0, \tag{6.28}$$

$$\overline{f}_2^{\alpha} \rightharpoonup \overline{f}_2 \text{ weakly in } L^2\left(\Gamma_2;\mathbb{R}^d\right) \text{ as } \alpha\to 0, \tag{6.29}$$

Using (6.25) and (6.28)-(6.29) we find that $\left(\overline{f}_2,\overline{u}\right)=(f_2^*,u^*)$. Now, since Problem $\mathcal{P}_S$ has a unique solution, we deduce that the whole sequence $\{\overline{u}^{\alpha}\}_{\alpha}$ satisfies (6.26). To continue, using (6.25) and convexity arguments, we deduce that

$$\mathcal{L}\left(f_2^*,u^*\right)\leq\liminf_{\alpha\to 0}\mathcal{L}\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right). \tag{6.30}$$

On the other hand, let $\{(f_2^*,u^{\alpha})\}_{\alpha}\subset\mathcal{W}^{\alpha}$, then we get

$$\lim_{\alpha\to 0}\|u^{\alpha}-u^*\|_V=0.$$

Moreover, we have

$$\begin{aligned}\limsup_{\alpha\to 0}\mathcal{L}\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right) &\leq \limsup_{\alpha\to 0}\mathcal{L}\left(f_2^*,u^{\alpha}\right)\\ &\leq \mathcal{L}\left(f_2^*,u^*\right).\end{aligned} \tag{6.31}$$

Thus, from (6.30)-(6.31), it follows that

$$\mathcal{L}\left(f_2^*,u^*\right)=\lim_{\alpha\to 0}\mathcal{L}\left(\overline{f}_2^{\alpha},\overline{u}^{\alpha}\right).$$

From this and (6.26) we get

$$\lim_{\alpha\to 0}\left\|\overline{f}_2^{\alpha}\right\|_{L^2\left(\Gamma_2;\mathbb{R}^d\right)}=\|f_2^*\|_{L^2\left(\Gamma_2;\mathbb{R}^d\right)} \tag{6.32}$$

Now, (6.27) is a consequence of (6.25) and (6.32). □

7 An optimality condition

In this section we obtain a necessary optimality condition for the regularized Problems. Our main interest lies in the differentiation of $\mathcal{J}_\alpha$ defined by (6.9) and use the fact that at the extremum the differential of $\mathcal{J}_\alpha$ must be null. To this end, we replace the hypotheses in (3.13), with the following stronger ones

$$\begin{cases} \text{(a) The mapping } r \to \mu(x,r) \text{ is } \mathcal{C}^1, \ a.e.\ x \in \Gamma_3. \\ \text{(b) The mapping } x \to \mu(x,r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \text{for any } r \in \mathbb{R}^+; \\ \text{(c) There exists } L_\mu > 0 \text{ such that } |\partial_2 \mu(x,r)| \leq L_\mu \\ \forall r \in \mathbb{R}^+, \ a.e.\ x \in \Gamma_3. \\ \text{(d) There exists } M_\mu > 0 \text{ such that } |\mu(x,r)| \leq M_\mu. \\ \forall r \in \mathbb{R}^+, \ a.e.\ x \in \Gamma_3. \end{cases} \tag{7.1}$$

We introduce the functions $\psi_\alpha : V \to V$ and $\psi'_\alpha : V \times V \to V$ defined by

$$(\psi_\alpha(z), v)_V = j'_\alpha(z; z, v), \ \forall z, \ v \in V, \tag{7.2}$$

$$(\psi'_\alpha(z,w), v)_V = \int_{\Gamma_3} I_\alpha(x,z,w) \cdot v_\tau \, da, \ \forall z, \ v, \ w \in V, \tag{7.3}$$

where $j'_\alpha(z;z,v)$ is defined by (6.3) and $I_\alpha(x,z,w)$ is given by

$$\begin{aligned} I_\alpha(x,z,w) =& S \frac{\partial_2 \mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) z_\tau \cdot w_\tau}{|z_\tau|^2 + \alpha^2} z_\tau \\ &+ S\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) \frac{\left(|z_\tau|^2 + \alpha^2\right) w_\tau + (z_\tau \cdot w_\tau) z_\tau}{\left(|z_\tau|^2 + \alpha^2\right)^{3/2}}. \end{aligned} \tag{7.4}$$

Denote

$$J_\alpha(x,z) = S\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) \frac{z_\tau}{\sqrt[2]{|z_\tau|^2 + \alpha^2}}. \tag{7.5}$$

Then, we have

$$\begin{aligned} \frac{J_\alpha(x, z + \lambda w) - J_\alpha(x,z)}{\lambda} =& S\left(J_{\lambda,1} \times J_{\lambda,2} + J_{\lambda,3}\right)(z_\tau + \lambda w_\tau) \\ &+ S\mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right) \frac{\left(|z_\tau|^2 + \alpha^2\right)}{\left(|z_\tau|^2 + \alpha^2\right)^{3/2}} w_\tau, \ \forall \lambda \in \mathbb{R}, \end{aligned} \tag{7.6}$$

where

$$J_{\lambda,1} = \frac{\mu\left(x, \sqrt[2]{|z_\tau + \lambda w_\tau|^2 + \alpha^2} - \alpha\right) - \mu\left(x, \sqrt[2]{|z_\tau|^2 + \alpha^2} - \alpha\right)}{\sqrt[2]{|z_\tau + \lambda w_\tau|^2 + \alpha^2} - \sqrt[2]{|z_\tau|^2 + \alpha^2}}, \tag{7.7}$$

$$J_{\lambda,2}=\frac{1}{\lambda}\frac{\sqrt[2]{|z_\tau+\lambda w_\tau|^2+\alpha^2}-\sqrt[2]{|z_\tau|^2+\alpha^2}}{\sqrt[2]{|z_\tau+\lambda w_\tau|^2+\alpha^2}}, \tag{7.8}$$

$$J_{\lambda,3}=\frac{1}{\lambda}\frac{\sqrt[2]{|z_\tau|^2+\alpha^2}-\sqrt[2]{|z_\tau+\lambda w_\tau|^2+\alpha^2}}{\sqrt[2]{|z_\tau|^2+\alpha^2}\sqrt[2]{|z_\tau+\lambda w_\tau|^2+\alpha^2}}\mu\left(x,\sqrt[2]{|z_\tau|^2+\alpha^2}-\alpha\right). \tag{7.9}$$

Let $\lambda\in\left]0,1\right]$, after some algebraic manipulations, we obtain

$$\left|\frac{J_\alpha(x,z+\lambda w)-J_\alpha(x,z)}{\lambda}-I_\alpha(x,z,w)\right|\leq c\left(|w|^2+|z|\,|w|+|w|\right),$$

which gives

$$\left|\frac{J_\alpha(x,z+\lambda w)-J_\alpha(x,z)}{\lambda}-I_\alpha(x,z,w)\right|^2\leq c\left(|w|^4+|z|^4+|w|^2\right),\ \forall z,w\in V,\ x\in\Gamma_3,$$

with $c=c\left(S,\mu,\alpha\right)>0$. On the other hand, keeping in mind (6.2), (7.4) and (7.5)-(7.9), we get

$$\lim_{\lambda\to0}\frac{J_\alpha(x,z+\lambda w)-J_\alpha(x,z)}{\lambda}=I_\alpha(x,z,w),\ \forall z,w\in V,\ x\in\Gamma_3. \tag{7.10}$$

To continue, taking into account (6.3), (7.2), (7.3), (7.4)), (7.5) and (3.9), we obtain

$$\left\|\frac{\psi_\alpha(z+\lambda w)-\psi_\alpha(z)}{\lambda}-\psi'_\alpha(z,w)\right\|_V^2\leq(c_0)^2\int_{\Gamma_3}\left|\frac{J_\alpha(x,z+\lambda w)-J_\alpha(x,z)}{\lambda}-I_\alpha(x,z,w)\right|^2da,$$

for all z, $w\in V$. Now, passing to the limit as $\lambda\to0$ in the last inequality, by using the Lebesgue-dominated convergence theorem, we find that

$$\lim_{\lambda\to0}\frac{\psi_\alpha(z+\lambda w)-\psi_\alpha(z)}{\lambda}=\psi'_\alpha(z,w),\ \forall z,\ w\in V. \tag{7.11}$$

Consider the following problem.

Problem 7.1. Let $\alpha>0$. Let $f_2\in L^2\left(\Gamma_3;\mathbb{R}^d\right)$, find a displacement field $u\in V$ such that

$$\mathcal{F}\left(f_2,u\right)=0, \tag{7.12}$$

where $\mathcal{F}:L^2\left(\Gamma_2;\mathbb{R}^d\right)\times V\to V$ is defined by

$$\mathcal{F}\left(\varphi,u\right)=Au+\psi_\alpha(u)-h-F\left(\varphi\right),\text{ for all }\left(\varphi,u\right)\in L^2\left(\Gamma_2;\mathbb{R}^d\right)\times V. \tag{7.13}$$

Lemma 7.2. *Problem* $\mathcal{P}_S^\alpha$ *is equivalent to problem* (7.12)-(7.13).

Proof. Setting in (6.5) $w=u\pm\lambda v$, we obtain

$$\left(Au,\pm v\right)_V+\frac{j_\alpha\left(u,u\pm\lambda v\right)-j_\alpha\left(u,u\right)}{\lambda}\geq\left(h+F\left(f_2\right),\pm v\right)_V,\text{ for all }v\in V,$$

which gives

$$\left(Au,\pm v\right)_V+\left(\psi_\alpha(u),\pm v\right)_V\geq\left(h+F\left(f_2\right),\pm v\right)_V,\text{ for all }v\in V,$$

which implies that u satisfies (7.12). On the other hand, since j_α is convex with respect to the second variable, it follows, (see, e.g., [10]), that

$$(\psi_\alpha(u), w-u)_V = j'_\alpha(u; u, w-u) \leq j_\alpha(u, w) - j_\alpha(u, u),\ \forall w \in V,$$

which together with (7.12)-(7.13) implies that u is also satisfies (6.5). □

Next, we have to perform the following calculus. For all $(f_2, u) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V$, we have

$$\begin{aligned}\lim_{\lambda\to 0} \frac{\mathcal{F}(f_2+\lambda q, w) - \mathcal{F}(f_2, w)}{\lambda} &= \lim_{\lambda\to 0} \frac{F(f_2) - F(f_2+\lambda q)}{\lambda} \\ &= -F(q),\ \forall q \in L^2\left(\Gamma_2; \mathbb{R}^d\right).\end{aligned}$$

Thus, we deduce that the function $\partial_1\mathcal{F}(f_2, u) : L^2\left(\Gamma_2; \mathbb{R}^d\right) \to V$ is defined by

$$\partial_1\mathcal{F}(f_2, u)q = -F(q),\ \forall (f_2, q, u) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V. \tag{7.14}$$

Also, we have

$$\lim_{\lambda\to 0} \frac{\mathcal{F}(f_2, u+\lambda w) - \mathcal{F}(f_2, u)}{\lambda} = Aw + \psi'_\alpha(u, w),\ \forall (f_2, w, u) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V \times V.$$

Therefore, the functional $\partial_2\mathcal{F}(f_2, u) : V \to V$ is given by

$$\partial_2\mathcal{F}(f_2, u)w = Aw + \psi'_\alpha(u, w),\ \forall (f_2, w, u) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V \times V. \tag{7.15}$$

We have the following result.

Lemma 7.3. *Let $(f_2, u) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V$. Then, $\partial_2\mathcal{F}(f_2, u)$: $V \to V$ is an isomorphism.*

Proof. Let $(f_2, u) \in L^2\left(\Gamma_2; \mathbb{R}^d\right) \times V$ and let $\overline{h} \in V$. We have to prove that there exists a unique element $\zeta \in V$ such that

$$\left(\partial_2\mathcal{F}(f_2, u)\zeta, v\right)_V = \left(\overline{h}, v\right)_V,\ \forall v \in V. \tag{7.16}$$

Let $b_{(f_2,u)} : V \times V \to \mathbb{R}$ be the bilinear form defined by

$$b_{(f_2,u)}(w, v) = \left(\partial_2\mathcal{F}(f_2, u)w, v\right)_V,\ \forall w, v \in V. \tag{7.17}$$

Using the fact that

$$\frac{(u_\tau \cdot w_\tau)^2}{|u_\tau|^2 + \alpha^2} \leq |w_\tau|^2,$$

and keeping in mind (7.1) and (3.9), we obtain

$$-c_0^2 L_\mu \|S\|_{L^\infty(\Gamma_3)} \|w\|_V^2 \leq \int_{\Gamma_3} S \frac{\partial_2\mu\left(x, \sqrt[2]{|u_\tau|^2 + \alpha^2} - \alpha\right)(u_\tau \cdot w_\tau)^2}{|u_\tau|^2 + \alpha^2} da.$$

Moreover, since

$$\begin{aligned} 0 &\leq \alpha^2 \left|w_\tau\right|^2 \\ &\leq \left|w_\tau\right|^2 \left(\left|u_\tau\right|^2 + \alpha^2\right) - \left|u_\tau\right|^2 \left|w_\tau\right|^2 \\ &\leq \left|w_\tau\right|^2 \left(\left|u_\tau\right|^2 + \alpha^2\right) - \left(u_\tau \cdot w_\tau\right)^2, \end{aligned}$$

we deduce that

$$-c_0^2 L_\mu \left\|S\right\|_{L^\infty(\Gamma_3)} \left\|w\right\|_V^2 \leq \left(\psi'_\alpha(u,w), w\right)_V, \ \forall w \in V. \tag{7.18}$$

Furthermore, taking into account (7.11) and (7.1), we can write

$$\begin{aligned} \left|\left(\psi'_\alpha(u,w), v\right)_V\right| \leq\ & L_\mu \left\|S\right\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \frac{\left|u_\tau\right|^2}{\left|u_\tau\right|^2 + \alpha^2} \left|w\right| \left|v\right| da \\ &+\ M_\mu \left\|S\right\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \frac{\left(\left|u_\tau\right|^2 + \alpha^2\right)}{\left(\left|u_\tau\right|^2 + \alpha^2\right)^{3/2}} \left|w\right| \left|v\right| da \\ &+ M_\mu \left\|S\right\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \frac{\left|u_\tau\right|^2}{\left(\left|u_\tau\right|^2 + \alpha^2\right)^{3/2}} \left|w\right| \left|v\right| da, \end{aligned}$$

and using (3.9), we deduce that there exists $L > 0$ such that

$$\left|\left(\psi'_\alpha(u,w), v\right)_V\right| \leq L \left\|w\right\|_V \times \left\|v\right\|_V, \ \forall w,\ v \in V. \tag{7.19}$$

Consequently, we deduce that $b_{(f_2,u)}$ is a continuous bilinear form and V-elliptic. Thus, the existence of a unique element $\zeta \in V$ such that (7.16) holds is obtained from Lax–Milgram's lemma. □

The main result of this Section is the following theorem.

Theorem 7.4. *Let $\alpha > 0$ and let $\left(\overline{f}_2, \overline{u}\right)$ be a solution of Problem $\mathcal{P}_{OC}^{\alpha}$. Then, $\overline{f}_2$ satisfies*

$$\int_{\Gamma_2} \overline{f}_2 \cdot q da = -\frac{1}{\delta} \int_{\Gamma_2} l(\overline{f}_2) \cdot q da, \ \forall q \in L^2\left(\Gamma_2; \mathbb{R}^d\right), \tag{7.20}$$

where $l(\overline{f}_2)$ is the unique solution of the variational equation

$$\beta \left(\bar{u} - u_g, w\right)_V = \left(l(\overline{f}_2), Aw + \psi'_\alpha(\bar{u}, w)\right)_V, \ \forall w \in V. \tag{7.21}$$

Proof. We apply Proposition 2.1 to $Z = L^2\left(\Gamma_2; \mathbb{R}^d\right)$, $X = Y = V = V'$. Using (5.5), we get

$$\begin{aligned} \lim_{\lambda \to 0} \frac{\mathcal{L}\left(f_2 + \lambda q, u\right) - \mathcal{L}\left(f_2, u\right)}{\lambda} &= \lim_{\lambda \to 0} \frac{\dfrac{\delta}{2} \left\|f_2 + \lambda q\right\|^2_{L^2\left(\Gamma_2, \mathbb{R}^d\right)} - \dfrac{\delta}{2} \left\|f_2\right\|^2_{L^2\left(\Gamma_2, \mathbb{R}^d\right)}}{\lambda} \\ &= \delta \left(f_2, q\right)_{L^2\left(\Gamma_2; \mathbb{R}^d\right)}, \ \forall q \in L^2\left(\Gamma_2; \mathbb{R}^d\right), \end{aligned}$$

for all $(f_2, u) \in L^2\left(\Gamma_2;\mathbb{R}^d\right) \times V$. Thus, we deduce that the functional $\partial_1\mathcal{L}\left(f_2,u\right) : L^2\left(\Gamma_2;\mathbb{R}^d\right) \to \mathbb{R}$ satisfies

$$\partial_1\mathcal{L}\left(f_2,u\right)q = \delta\left(f_2,q\right)_{L^2\left(\Gamma_2;\mathbb{R}^d\right)},\ \forall q \in L^2\left(\Gamma_2;\mathbb{R}^d\right). \tag{7.22}$$

On the other hand, for all $(f_2, u) \in L^2\left(\Gamma_2;\mathbb{R}^d\right) \times V$, we have $\partial_2\mathcal{L}\left(f_2,u\right) : V \to \mathbb{R}$

$$\partial_2\mathcal{L}\left(f_2,u\right)w = \beta\left(u-u_g,w\right)_V,\ \forall w \in V. \tag{7.23}$$

It is straightforward to show that $\mathcal{F}$ and $\mathcal{L}$ are two $\mathcal{C}^1$ functions on $L^2\left(\Gamma_2;\mathbb{R}^d\right)\times V$. Now, from Proposition 2.1, we deduce that for all $q \in L^2\left(\Gamma_2;\mathbb{R}^d\right)$,

$$\frac{d\mathcal{J}_\alpha}{d\varphi}(\bar{f}_2)q = \partial_1\mathcal{L}\left(\bar{f}_2,\bar{u}\right)q - \left(l(\bar{f}_2),\partial_1\mathcal{F}(\bar{f}_2,\bar{u})q\right)_{V',V}, \tag{7.24}$$

where $l(\bar{f}_2) \in V$ is the unique solution of the following equation,

$$\left(\left[\partial_2\mathcal{F}(\bar{f}_2,\bar{u})\right]^* l(\bar{f}_2),w\right)_{V',V} = \partial_2\mathcal{L}\left(\bar{f}_2,\bar{u}\right)w,\ \forall w \in V. \tag{7.25}$$

Thus, $l(\bar{f}_2)$ is the unique solution of the following variational equation,

$$\left(l(\bar{f}_2),\left[\partial_2\mathcal{F}(\bar{f}_2,\bar{u})\right]w\right)_V = \partial_2\mathcal{L}\left(\bar{f}_2,\bar{u}\right)w,\ \forall w \in V. \tag{7.26}$$

Since $\overline{f}_2$ is a minimizer of $\mathcal{J}_\alpha$, we obtain that

$$\frac{d\mathcal{J}_\alpha}{d\varphi}(\overline{f}_2)q = 0, \forall q \in L^2(\Gamma_2;\mathbb{R}^d). \tag{7.27}$$

Finally, from (7.22)-(7.27), we get the following optimality condition

$$\int_{\Gamma_2}\overline{f}_2\cdot q da = -\frac{1}{\delta}\int_{\Gamma_2} l(\overline{f}_2)\cdot q da,\ \forall q \in L^2\left(\Gamma_2;\mathbb{R}^d\right).$$

□

The replacement of the functional j_τ by the Gâteaux differentiable function j_α has enabled us to deduce the optimality condition (7.20)-(7.21). How to deduce an optimality condition for problem $\mathcal{P}_S$ is an open question.

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