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A survey of the maximal and the minimal nullity in terms of omega invariant on graphs

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Abstract. Let $G = (V, E)$ be a simple graph with n vertices and m edges. $\nu(G)$ and $c(G) = m - n + \theta$ be the matching number and cyclomatic number of G, where θ is the number of connected components of G, respectively. Wang and Wong in [18] provided formulae for the upper and the lower bounds of the nullity $\eta(G)$ of G as $\eta(G) = n - 2\nu(G) + 2c(G)$ and $\eta(G) = n - 2\nu(G) - c(G)$, respectively. In this paper, we restate the upper and the lower bounds of nullity $\eta(G)$ of G utilizing omega invariant and inherently vertex degrees of G. Also, in the case of the maximal and the minimal nullity conditions are satisfied for G, we present both two main inequalities and many inequalities in terms of Omega invariant, analogously cyclomatic number, number of connected components and vertex degrees of G.

1 Introduction

Let $G = (V, E)$ be a simple graph (undirected graph without any loops or multiple edges) with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\}$.

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The adjacency matrix $A(G)$ is defined as $n \times n$ symmetric matrix and if v_i and v_j are adjacent vertices then the corresponding entry a_{ij} of $A(G)$ is equal to 1, otherwise a_{ij} is equal to 0. The characteristic polynomial of G is defined to be $\det(x \cdot I - A(G))$, where I is $n \times n$ identity matrix. The roots of the characteristic polynomial of G is called eigenvalues of G. The rank and the nullity of $A(G)$ are defined as the rank and the nullity of G, respectively. In other words, the nullity of G can be expressed to be the multiplicity of zero eigenvalue in the spectrum of G. Let the rank and the nullity of G is denoted by $r(G)$ and $\eta(G)$, respectively. It is well known equality that $r(G)+\eta(G) = n$. If $\eta(G) > 0$, then G is called a singular graph and if $\eta(G) = 0$, then G is called a nonsingular graph. A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Moreover, if some vertices are removed with all edges that are incident to the removing vertices but no more edges, obtained subgraph is called an induced subgraph of G. If there is at least one path between any two vertices of G, then G is called connected. A maximal connected subgraph of G is called a connected component of G. Let $c(G)$ be the cyclomatic number (the dimension of cycle space) of G and it is defined to be $c(G) = m - n + \theta$, where θ is the number of connected components of G. If G is a connected graph, in the cases of $c(G) = 0$ and $c(G) = 1$ then G is a tree and G is a unicyclic graph, respectively. The number of edges that are incident to a vertex v is called the degree of v and it is denoted by $d(v)$. If $d(v) = 1$, then v is called pendant vertex and all edges that are incident to v are called pendant edges. Also, a vertex that is adjacent to a pendant vertex is called quasi pendant vertex. Set of disjoint edges such that none of them have a common vertex is a matching in a graph G and the maximum possible number of edges in a matching set is called a maximum matching. The number of disjoint edges that exist in a maximum matching of G is called the matching number of G. In this paper, the matching number is denoted by $\nu(G)$.

The nullity and inherently the rank are significant notions in the sense of the chemical importance of graph theory. In physical chemistry, there are lots of applications of these notions. One of the most important applications exists in Hückel theory. In the Hückel theory, eigenvectors and eigenvalues of G are considered as molecular orbitals and energies that correspond to the molecular orbitals, respectively. Also, the number of nonbonding molecular orbitals is considered the nullity of G, for details see [18], [2] (Topic 9E). On the basis of Hückel molecular orbital theory, in the case of $\eta(G) > 0$ then the chemical compound whose graph model corresponds to G is very reactive and unstable, or nonexistent, see [3]. In the literature, there are lots of topics in the sense of nullity such that calculating the nullity of G, characterization of graphs

with respect to their nullity, bounds on nullity, and so on. Several preliminary related results can be seen [1, 3, 4, 5, 7, 10, 12, 14, 15, 17].

In physical chemistry, another important notion is topological invariants (indices). Topological invariants associate molecule structures with a diversity of chemical reactivity or physical properties based on a mathematical expression. The first topological index was proposed by Harry Wiener in 1947 and it was named with Wiener index. Wiener index is capable of predicting the boiling points of alkane isomers by characterizing the branching of the graph model of corresponding molecules, more details can be found in [6]. Similar to the Wiener index, there are lots of topological invariants such that Zagreb indices, Hosoya invariant, omega invariant, and so on, see [6, 11, 19]. In the next section, we recall some important results to set the relation between omega invariant and bounds of the nullity $\eta(G)$ of G.

2 Preparation and recalling of some important definitions and results

To arrive at the main theorems, we need some known definitions and results that are given below:
Generally, degree sequences are denoted by multiplicities as $D = \{0^{(a_0)}, 1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$, where a_i is the number of vertices whose degree is i. If the given degree sequence D is equal to the degree sequence of graph G, then G is called a realization of D. However, it is clear that every degree sequence cannot be realizable. To examine realizability, the most important test is Havel-Hakimi's, see [8, 9]. Also, another criterion was touched on by using the handshaking lemma in [6]. Moreover, it is worth noting that we use both $\Omega(G)$ and $\Omega(D)$ notations according to our needs. For some further aspects of omega invariant, see, [6]. Now, we are ready to give the next definition.

Definition 1 (Definition 2.1, [6]) *Let* $D = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$ *be a realizable degree sequence and its realization be the graph* G. *Then omega invariant of* G *is defined as follows:*

$$\begin{aligned}\Omega(G) &= -a_1 + a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta \\ &= \sum_{i=1}^{\Delta}(i-2)a_i.\end{aligned}$$

Distinctively, if G has at least one vertex whose degree is 0, then the degree sequence and omega invariant of G are defined as $D = \{0^{(a_0)}, 1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$ and $\Omega(G) = \sum_{i=0}^{\Delta}(i-2)a_i$, respectively.

Let us mention some properties of omega invariant.

Proposition 1 (Corollary 3.1, [6]) *Let* $G = (V, E)$ *be a graph with* θ *connected components. Then the cyclomatic number* $c(G)$ *of* G *satisfies the following relation:*

$$c(G) = \frac{\Omega(G)}{2} + \theta.$$

Proposition 2 (Theorem 2.1, [6]) *Let* $G = (V, E)$ *be a graph with* n *vertices and* m *edges. For the given graph* G,

$$\Omega(G) = 2(m - n).$$

Let G be a graph. K_2 deletion process that consists of the number of K_2 deletion operations is applied to get an induced subgraph without any pendant vertex or inherently any pendant edge. In every K_2 deletion operation, a pendant vertex and it is unique neighbor are deleted with their incident edges from G. In the K_2 deletion process, K_2 deletion operations are applied consecutively until reaching up to an induced subgraph without any pendant vertex and pendant edge, see [16, 17]. Stated induced subgraph is called a crucial subgraph and a crucial subgraph is defined as follows: If any pendant vertex does not exist in G, then the crucial subgraph of G is itself. If there are some pendant vertices in G, then K_2 deletion operations are applied until obtaining an induced subgraph of G without any pendant vertex and pendant edge. When the desired induced subgraph is reached, K_2 deletion process is ended and the induced subgraph is called the crucial subgraph of G. In this paper, we denote a crucial subgraph and the number of K_2 deletion operations that we need to have G' from G by G' and $\kappa(G)$, respectively. All crucial subgraphs of G are shown to be isomorphic, see for details [16, 17].

Proposition 3 (Lemma 2.7, [16]) *For a graph* G, *the following equalities exist:*

i–) $\eta(G) = \eta(G')$.
ii–) $\nu(G) = \nu(G') + \nu(G - G')$.
iii–) $\kappa(G) = \nu(G - G')$.

Let G be a pairwise cycle-vertex-disjoint graph. Let $\zeta(\mathbf{G})$ be the set of all cycles in G. Let us call the process of shrinking every cycle of G to a single vertex to get an acyclic graph T_G as a shrinking process. The vertex set of T_G is considered to be $U_G \cup W_G$ where U_G consists of all vertices of G that do not belong to any cycle of G and W_G consists of all the vertices v_{C_i} and $C_i \in \zeta(\mathbf{G})$.

There are three different types of vertex adjacency in T_G. Any two vertices in U_G are adjacent in T_G if and only if they are adjacent in G. A vertex u in U_G is adjacent to $v_{C_i} \in W_G$ in T_G if and only if u is adjacent to a vertex of corresponding cycle C_i in G. The vertices v_{C_i} and v_{C_j} are adjacent in T_G if and only if a vertex of $C_i \in \zeta(\mathbf{G})$ and a vertex of $C_j \in \zeta(\mathbf{G})$ are adjacent in G, see [17], [15].

Proposition 4 (Theorem 1.4, [15]) *Let G be a graph. G satisfies the maximal nullity condition if and only if the following conditions are all satisfied:*
i–) *All cycles of G are pairwise cycle-vertex-disjoint,*
ii–) *The length of each cycle of G is congruent to 0 modulo 4,*
iii–) $\nu(T_G) = \nu(T_G - W_G)$.

Proposition 5 (Theorem 1.4, [14]) *Let G be a graph. G satisfies the minimal nullity condition if and only if the following conditions are all satisfied:*
i–) *All cycles of G are pairwise cycle-vertex-disjoint,*
ii–) *The length of each cycle of G is odd,*
iii–) $\nu(T_G) = \nu(T_G - W_G)$.

Proposition 6 (Theorem 1.6, [17]) *Let $G = (V, E)$ be a simple graph with n vertices and m edges. G satisfies the minimal nullity condition if and only if K_2 deletion process converts G to a crucial subgraph G' that consists of disjoint $c(G)$ odd cycles and $n - 2\nu(G) - c(G)$ null graphs.*

By using Propositions 3, 4, 5 and 6, next result is obtained.

Corollary 1 *Let $G = (V, E)$ be a simple graph with n vertices and m edges. G satisfies the maximal nullity condition if and only if K_2 deletion process converts G to a crucial subgraph G' that consists of disjoint $c(G)$ $4k$-cycles (cycles with $4k$ vertices, where $k \in \mathbb{N}^+$) and $n - 2\nu(G)$ null graphs.*

Example 1 *Let G be a graph with the degree set $D(G) = \{0^2, 1^{10}, 2^9, 3^{11}, 4^1, 5^1\}$ and θ connected components as in Figure 1. By applying the K_2 deletion process we get the crucial subgraph G' as in Figure 2. Since the characteristic polynomial of G is equal to the $x^{34} - 35x^{32} + 529x^{30} - 4536x^{28} + 24370x^{26} - 85471x^{24} + 197584x^{22} - 296321x^{20} + 274868x^{18} - 141690x^{16} + 30600x^{14}$, $\eta(G) = 14$. Furthermore, since $n - 2\nu(G) + 2c(G) = 34 - 28 + 8 = 14$, G satisfies the maximal nullity condition.*

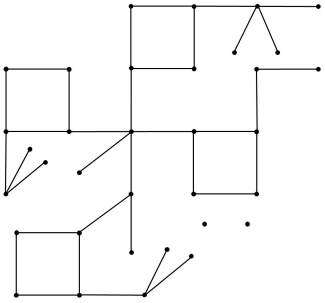


Figure 1: Graph G

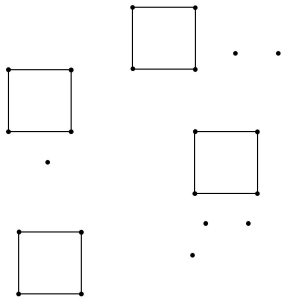


Figure 2: Graph G′

Hence, we see that $n = a_0+a_1+a_2+a_3+a_4+a_5 = 34$, $\Omega(G) = -2a_0-a_1+a_3+2a_4+3a_5 = 2$, $\theta = 3$, $c(G) = \frac{\Omega(G)}{2}+\theta = 4$ *and* $\nu(G) = 14$. *Consequently, we show an example of Corollary 1 that we have* $c(G) = 4$ 4k*-cycles and* $n-2\nu(G) = 6$ *null graphs in* G'.

Proposition 7 (Corollary 2.8.1, [13]) *Let* $G = (V, E)$ *and* G' *be a graph and a crucial subgraph of* G*, respectively. If* G *satisfies the maximal nullity or the minimal nullity condition, then* $c(G) = c(G')$.

By Proposition 7, we deduce that we can write $c(G')$ instead of $c(G)$ in Proposition 6 and in Corollary 1. From now on, when we use Proposition 6 or Corollary 1, we keep in mind this convenience.

3 Main theorems

In 2014 Wang and Wong gave a lower and an upper bound for the nullity of a graph G in [18]. In the next corollary, we restate the lower bound for the nullity $\eta(G)$ of G in terms of vertex degrees of G' also, we restate the upper bound for the nullity $\eta(G)$ of G in terms of the omega invariant of G, vertex degrees of G' and the number of connected components of G.

Corollary 2 *Let* $G=(V, E)$ *be a graph with* θ *connected components,* n *vertices,* m *edges and the degree sequence* $D(G)=\{0^{(a_0)}, 1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. *Let* $G'=(V', E')$ *be a crucial subgraph of* G *with* θ' *connected components,* n' *vertices,* m' *edges and the degree sequence* $D(G') = \{0^{(a'_0)}, 1^{(a'_1)}, 2^{(a'_2)}, \cdots, \Delta^{(a'_\Delta)}\}$. *Then*

$$a'_0 \leq \eta(G) \leq a'_0 + \Omega(G) + 2\theta.$$

Note that since $\Omega(G)$ *is equal to the* $\sum_{i=0}^{\Delta}(i-2)a_i$ *by Definition 1, the upper bound of the nullity* $\eta(G)$ *can be restated as* $a'_0 + \sum_{i=0}^{\Delta}(i-2)a_i + 2\theta$. *In the case that* G *is a connected acyclic graph, then* $\Omega(G) = -2$ *so the upper bound of* G *is equal to* a'_0. *Analogously, in the case that* G *is a connected unicyclic graph, then* $\Omega(G) = 0$ *so the upper bound of* G *is equal to* $a'_0 + 2$. *Similarly, if* G *is a connected bicyclic graph, then* $\Omega(G) = 2$ *so the upper bound of* G *is equal to* $a'_0 + 4$.

Proof. Let us denote the number of null graphs in G' by $o(G')$. By using the definition of K_2 deletion process and propositions that are given in the Section 2 we restate the upper and lower bounds of nullity of G as follows:

$$\begin{aligned}
n - 2\nu(G) - c(G) &= n' + 2\kappa(G) - 2\nu(G) - c(G') \\
&= n' - 2(\nu(G) - \kappa(G)) - c(G') \\
&= n' - 2\nu(G') - c(G') \\
&= o(G') \\
&= a'_0 \\
n - 2\nu(G) + 2c(G) &= n' - 2\nu(G') + 2c(G) \\
&= a'_0 + 2c(G) \\
&= a'_0 + 2(m - n + \theta) \\
&= a'_0 + \Omega(G) + 2\theta.
\end{aligned}$$

□

Theorem 1 *Let* $D = \left\{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *be a realizable degree sequence. If a graph* G *can be found which is a realization of* D *that satisfies the maximal nullity condition, and where all its connected components are not isomorphic to the graph* P_2, *then the* $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$.

Proof. First of all, let us show that it is sufficient to prove for only connected graphs. Assume that G is a disconnected graph with two components that are denoted by H', H''. If G satisfies the maximal nullity condition then H'

and H'' satisfy the maximal nullity condition. Also, let $a_i', a_i'', c(H'), c(H'')$ be the number of vertices whose degree is i in the connected components H', H'' and the cyclomatic numbers of H', H'', respectively. If $a_3' + a_4' + \cdots + a_\Delta' \geq 7c(H') + 1 - 2a_2'$ and $a_3'' + a_4'' + \cdots + a_\Delta'' \geq 7c(H'') + 1 - 2a_2''$ are satisfied, then by adding these inequalities we get $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$. As a result, it is clear that we can consider only connected graphs to prove. Let G be a connected realization of D that satisfies the maximal nullity condition. Then G is a pairwise cycle-vertex-disjoint graph and all lengthes of cycles (if any) of G are $4k$ by Proposition 4. Note that we focus on C_4 as $4k$-cycles by the structure of the proof. When $c(G) = 0$, then G (Every connected component of G is not P_2) is a tree. In this case, if $a_2 > 0$, then $a_3 + a_4 + \cdots + a_\Delta \geq 1 - 2a_2$ is satisfied. If $a_2 = 0$, then some of (or one of) the inequalities are satisfied $a_3 > 0, a_4 > 0, \cdots, a_\Delta > 0$ for $G \neq P_2$ so $a_3 + a_4 + \cdots + a_\Delta \geq 1 - 2a_2$ is satisfied. Hence, in the case of $c(G) = 0$, $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$ is satisfied. When $c(G) = 1$, then G is a unicyclic graph and the cycle is $4k$ by Proposition 4. Since G satisfies the maximal nullity condition, we see that every crucial subgraph G' of G that is obtained at the end of K_2 deletion process consists of one $4k$-cycle and some null graphs (if any) by using Corollary 1. As stated above, we consider C_4 as $4k$-cycle for generalizing because it can be observable that when the theorem is satisfied for C_4, then it is satisfied for every $4k$-cycle. Henceforward, we use C_4 instead of $4k$-cycles during the proof. Hence, G' consists of one C_4 and some null graphs by Corollary 1. At this point, if we rewrite the degree sequence of G' and denote by $D(G') = \left\{0^{(a_0')}, 1^{(a_1')}, 2^{(a_2')}, \cdots, \Delta^{(a_\Delta')}\right\}$ then both $7c(G')+1-2a_2'$ and $7c(G)+1-2a_2'$ are equal to zero by Proposition 7 so $a_3'+a_4'+\cdots+a_\Delta' \geq 7c(G)+1-2a_2'$ is satisfied. From this point by applying the reverse process, we add vertices and edges that are removed in the K_2 deletion process step by step and we examine cases according to degree sequence $D(G)$. For the first case, if $a_2 > 2c(G) + 2 = 4$, then since $a_3+a_4+\cdots+a_\Delta \geq 7\cdot1+1-2[2c(G)+2] = 0$ and $7\cdot1+1-2[2c(G)+2] > 7\cdot1+1-2a_2$ we have $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G)+1-2a_2$. For the second case, if $a_2 = 2c(G)+2 = 4$, then since $a_3 + a_4 + \cdots + a_\Delta \geq 7\cdot1+1-2a_2 = 0$ we have $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G)+1-2a_2$. For the third case, if $a_2 < 2c(G)+2 = 4$, then we get $2c(G) + 2 - a_2 = i$, where $i = 1, 2, 3, 4$. In this situation, graph forms are as shown below in Figure 3 and Figure 4.

Hence it is clear that for $i = 1, 2, 3, 4$, $a_3+a_4+\cdots+a_\Delta \geq 2, 4, 6, 8$, respectively. Thus, when $c(G) = 1$, $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$.

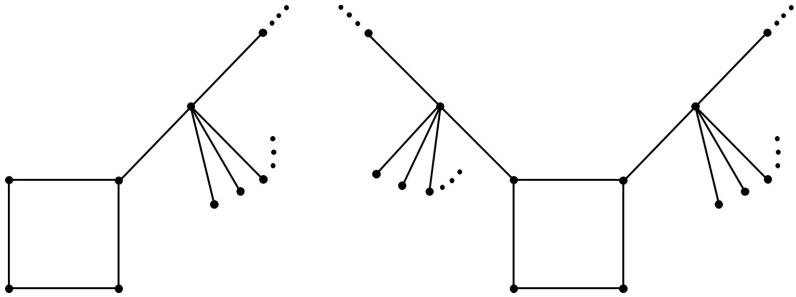


Figure 3: For $i = 1$ and $i = 2$

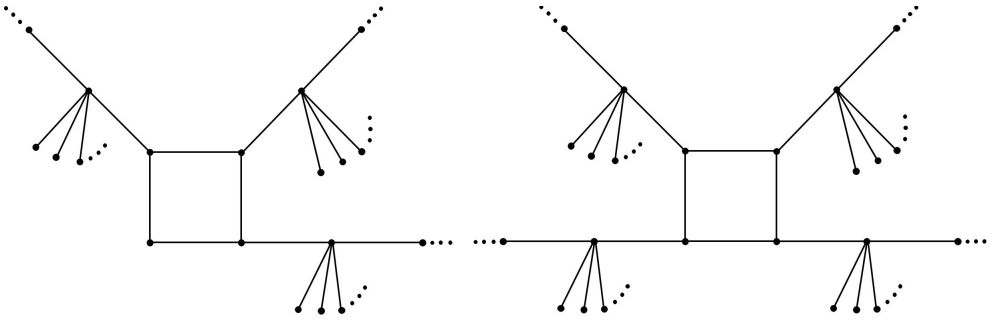


Figure 4: For $i = 3$ and $i = 4$

Now, we take care of $c(G) \geq 2$ and we prove this part by using induction on $c(G)$. Firstly, when $c(G) = 2$, since G satisfies the maximal nullity condition, then G is a pairwise cycle-vertex-disjoint graph and all lengthes of cycles are $4k$ by Proposition 4. Also, we observe that every crucial subgraph G' of G consists of two $4k$-cycles and some null graphs (If any) at the end of the K_2 deletion process by using Corollary 1. However, in this part we apply K_2 deletion operations until the last step on that C_4s are in the same connected component. After these operations, let us denote the obtained induced subgraph by G''. G'' has some null graphs (If any) and a connected component in that C_4s exist.

Let us denote the connected component of G'' by G''_c. Also, there must be at least one quasi pendant vertex on the path that exists between C_4s in the connected component G''_c. As a result, G''_c is in the form as above in Figure 5.

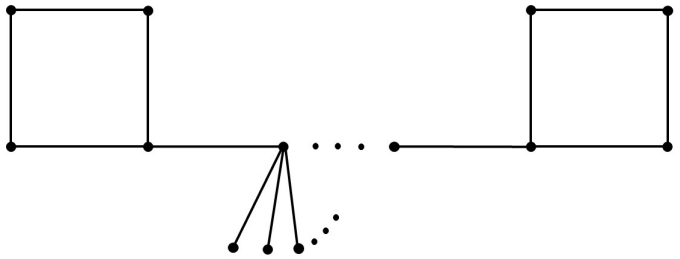


Figure 5: $c(G) = 2 = c(G_c'')$

If we assume that G_c'' is in its most basic form, then this form has two C_4 linked with a path on that exactly one quasi pendant vertex exists without any other vertex. It is clear that if the inequality is satisfied for the most basic form of G_c'', then the inequality is satisfied for every form of G_c''. Hence, let us show that the inequality is satisfied for this basic form to prove this part. Therefore, if we constitute the degree sequence of this basic form $D(G_c'') = \left\{0^{(a_0'')}, 1^{(a_1'')}, 2^{(a_2'')}, \cdots, \Delta^{(a_\Delta'')}\right\}$ then we observe that $7c(G_c'') + 1 - 2a_2''$ and $7c(G) + 1 - 2a_2''$ are equal to 3 by Proposition 7. Similar to the above reverse process, after adding vertices and edges that are removed in the K_2 operations step by step. After that, we examine cases with respect to degree sequence $D(G)$. The first case is $a_2 > 2c(G) + 2 = 6$. In this case, since for the most basic form of G_c'' the inequality $a_3 + a_4 + \cdots + a_\Delta \geq 3$ is satisfied and $a_3 + a_4 + \cdots + a_\Delta \geq 7.2 + 1 - 2[2c(G) + 2] = 3 > 7.2 + 1 - 2a_2$ then we get $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$. The second case is $a_2 = 2c(G) + 2 = 6$, then since for G_c'' the inequality $a_3 + a_4 + \cdots + a_\Delta \geq 7.2 + 1 - 2a_2 = 3$ is satisfied then we have $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$. The third case is $a_2 < 2c(G) + 2 = 6$, then we get $2c(G) + 2 - a_2 = i$, where $i = 1, 2, \cdots, 6$. In this situation, graph forms are arranged by adding vertices and edges that are adjacent and incident to the vertices of cycles of the most basic form of G_c'' according to $D(G)$ similarly to that applied in Figure 3 and Figure 4. Hence it is clear that for $i = 1, 2, \cdots, 6$, $a_3 + a_4 + \cdots + a_\Delta \geq 5, 7, 9, 11, 13, 15$ respectively. Then $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$ is satisfied for $c(G) = 2$. By the induction hypotesis, let us assume that for $c(G) = t$, the inequality $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$ is satisfied. For $c(G) = t+1$, since G satisfies the maximal nullity condition, by using Corollary 1 we deduce that we can get a disconnected induced subgraph G''' by applying number of K_2 deletion operations. Actually, the induced subgraph G''' has two connected components

excluding null graphs. Note that, we can ignore the null graphs as they do not have any effect on proof. These connected components of the induced subgraph G''' are pairwise cycle-vertex-disjoint subgraphs with the cyclomatic numbers t and 1 also cycles are C_4. Since G satisfies the maximal nullity condition, then both connected components of the induced subgraph G''' satisfy the maximal nullity condition. By the induction hypothesis, since the cyclomatic numbers of two connected components are t and 1, then for every connected component, the inequality is satisfied so the inequality $a_3''' + a_4''' + \cdots + a_\Delta''' \geq 7c(G''') + 1 - 2a_2'''$ is satisfied, where $D(G''') = \left\{0^{(a_0''')}, 1^{(a_1''')}, 2^{(a_2''')}, \cdots, \Delta^{(a_\Delta''')}\right\}$. In the same way, we attach vertices and edges that are removed during K_2 deletion operations step by step. When we add vertices and edges to disconnected subgraph G''' according to $D(G)$, since $a_2 \geq a_2''', a_3 \geq a_3''', \cdots, a_\Delta \geq a_\Delta'''$ and $c(G) = c(G''')$ by Proposition 7, the inequality $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$ is satisfied. As a result, we get the result by using induction process. □

In the Theorem 1, the inequality $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$ can be expressed as $a_3 + a_4 + \cdots + a_\Delta \geq 7\frac{\Omega(G)}{2} + 7\theta + 1 - 2a_2$ by Proposition 1. This expression is presented in the next corollary as a natural result of Theorem 1.

Corollary 3 *Let* $G = (V, E)$ *be a graph with the degree sequence* $D(G) = \left\{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *where all its connected components are not isomorphic to the graph* P_2. *If* G *satisfies the maximal nullity condition, then* $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2 = 7\frac{\Omega(G)}{2} + 7\theta + 1 - 2a_2$.

Proof. Proof is explicit by Theorem 1. □

The corollary below is a characterization of graphs that attain the given bound in Corollary 3.

Corollary 4 *Let* $D = \left\{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *be a realizable degree sequence and a graph* G *be a realization of* D *where all its connected components are not isomorphic to the graph* P_2 *and satisfies the maximal nullity condition. Then* $a_3 + a_4 + \cdots + a_\Delta = 7c(G) + 1 - 2a_2 = 7\frac{\Omega(G)}{2} + 7\theta + 1 - 2a_2$ *if and only if*

i–) G *is a connected cycle vertex disjoint graph whose all cycles are* C_4,

ii–) *For* $\forall C_4 \in G, \forall v \in C_4$, $d(v) = 2$ *or* $d(v) = 3$,

iii–) *In the case of* $d(v) = 3$, *the neighbor* u *of* v *that does not belong to corresponding* C_4 *(belongs to tree part) has either* $d(u) - 1$ *pendant vertices*

as neighbor or $d(u)-2$ *pendant vertices and* 1 *vertex that belongs to another* $C_4 \in G$ *as neighbor.*

Proof. Proof can be seen in the proof of Theorem 1. □

Furthermore, it is important to note that the converse of Theorem 1 is not true. Also, one may suspect that the converse of Corollary 3 is satisfied. The answer to this suspect is "no" because lots of realizations for a given degree sequence can be drawn. However, Corollary 3 can be considered conversely that if the inequality $a_3 + a_4 + \cdots + a_\Delta < 7c(G) + 1 - 2a_2 = 7\frac{\Omega(G)}{2} + 7\theta + 1 - 2a_2$ is satisfied for G, then G does not satisfy the maximal nullity condition.

Corollary 5 *Let* $G = (V, E)$ *be a graph with* n *vertices and the degree sequence* $D(G) = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. *If all connected components of* G *are not isomorphic to the path graph* P_2 *and* G *satisfies the maximal nullity condition, then* $n \geq 12c(G) + 4 - 3a_2 = 6\Omega(G) + 12\theta + 4 - 3a_2$.

Proof. Let G be a graph with degree sequence $D(G)$. Since G satisfies the maximal nullity condition, then inequality $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$ is satisfied by Corollary 3. By using definition of omega invariant we get $\Omega(G) = -a_1 + a_3 + 2a_4 + \cdots + (\Delta - 2)a_\Delta$. Hence we get $a_3 + a_4 + \cdots + a_\Delta = \Omega(G) + a_1 - a_4 - 2a_5 - \cdots - (\Delta - 3)a_\Delta$. If we write $\Omega(G) + a_1 - a_4 - 2a_5 - \cdots - (\Delta - 3)a_\Delta$ instead of $a_3 + a_4 + \cdots + a_\Delta$ in the inequality, we have $a_1 \geq 7c(G) + 1 - 2a_2 - \Omega(G) + a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta$. By Proposition 1, we obtain $a_1 \geq (\frac{7\Omega(G)}{2} + 7) + 1 - 2a_2 - \Omega(G) + a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta$ and by arranging we achieve $a_1 \geq 5c(G) + 3 - 2a_2 + a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta$. Since $a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta \geq 0$, then we get $a_1 \geq 5c(G) + 3 - 2a_2$. By adding the inequalites $a_1 \geq 5c(G) + 3 - 2a_2$ and $a_3 + a_4 + \cdots + a_\Delta \geq 7c(G) + 1 - 2a_2$, we have $a_1 + a_3 + a_4 + \cdots + a_\Delta \geq 12c(G) + 4 - 4a_2$ so we get $a_1 + a_2 + a_3 + a_4 + \cdots + a_\Delta \geq 12c(G) + 4 - 3a_2$ that is the desired result by considering $n = a_1 + a_2 + a_3 + a_4 + \cdots + a_\Delta$. By utilizing Proposition 1 proof is completed. □

The corollary below is a characterization of graphs that attain the given bound in Corollary 5.

Corollary 6 *Let* $G = (V, E)$ *be a graph with* n *vertices and the degree sequence* $D(G) = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. *If all connected components of* G *are not isomorphic to the path graph* P_2 *and* G *satisfies the maximal nullity condition, then* $n = 12c(G) + 4 - 3a_2 = 6\Omega(G) + 12\theta + 4 - 3a_2$ *if and only if*

i–) G *is a connected cycle vertex disjoint graph whose all cycles are* C_4,

ii–) *For* $\forall C_4 \in G, \forall v \in C_4$, $d(v) = 2$ *or* $d(v) = 3$,

iii–) *In the case of* $d(v) = 3$, *the neighbor* u *of* v *that does not belong to corresponding* C_4 *(belongs to tree part) has either* 2 *pendant vertices as neighbor or* 1 *pendant vertices and* 1 *vertex that belongs to another* $C_4 \in G$ *as neighbor.*

Proof. Proof is application of the proofs of Theorem 1, Corollary 4 and 5. □

Corollary 7 *Let* $G = (V, E)$ *be a pairwise cycle-vertex-disjoint graph with* n *vertices and all cycles (if any) are* $4k$*-cycles with the degree sequence* $D(G) = \left\{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$. *If all connected components of* G *are not isomorphic to the path graph* P_2 *and* $\nu(T_G) = \nu(T_G - W_G)$, *then* $n \geq 12c(G) + 4 - 3a_2 = 6\Omega(G) + 12\theta + 4 - 3a_2$.

Proof. Proof follows from Proposition 4 and Corollary 5. □

Conversely, if $n < 12c(G) + 4 - 3a_2 = 6\Omega(G) + 12\theta + 4 - 3a_2$, then $\nu(T_G) \neq \nu(T_G - W_G)$ and G does not satisfy the maximal nullity condition by Corollaries 5 and 7.

Theorem 2 *Let* $D = \left\{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *be a realizable degree sequence. If a graph* G *can be found which is a realization of* D *that satisfies the minimal nullity condition, and where all its connected components are not isomorphic to the path graph* P_2, *then* $a_3 + a_4 + \cdots + a_\Delta \geq 5c(G) + 1 - 2a_2$.

Proof. Let us find a graph G (all connected components of G is not P_2) as a realization of D with satisfying the minimal nullity condition. Similarly to the proof of Theorem 1, by using the induction method on $c(G)$ and by considering C_3s as cycles in graph G one can complete the proof because if the theorem is satisfied for C_3, then it is satisfied for all odd cycles. □

In the Theorem 2, the inequality $a_3 + a_4 + \cdots + a_\Delta \geq 5c(G) + 1 - 2a_2$ can be expressed as $a_3 + a_4 + \cdots + a_\Delta \geq 5\frac{\Omega(G)}{2} + 5\theta + 1 - 2a_2$ by Proposition 1.

Corollary 8 *Let* $G = (V, E)$ *be a graph with the degree sequence* $D(G) = \left\{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *where all its connected components are not isomorphic to the graph* P_2. *If* G *satisfies the minimal nullity condition, then* $a_3 + a_4 + \cdots + a_\Delta \geq 5c(G) + 1 - 2a_2 = 5\frac{\Omega(G)}{2} + 5\theta + 1 - 2a_2$.

Proof. Proof is clear by Theorem 2. □

In the next corollary, we give a characterization of graphs that attain the given bound in Corollary 8.

Corollary 9 *Let* $D = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$ *be a realizable degree sequence and a graph* G *be a realization of* D *where all its connected components are not isomorphic to the graph* P_2 *and satisfies the minimal nullity condition. Then* $a_3 + a_4 + \cdots + a_\Delta = 5c(G) + 1 - 2a_2 = 5\frac{\Omega(G)}{2} + 5\theta + 1 - 2a_2$ *if and only if*

i–) G *is a connected cycle vertex disjoint graph whose all cycles are* C_3,

ii–) *For* $\forall C_3 \in G, \forall v \in C_3$, $d(v) = 2$ *or* $d(v) = 3$,

iii–) *In the case of* $d(v) = 3$, *the neighbor* u *of* v *that does not belong to corresponding* C_3 *(belongs to tree part) has either* $d(u) - 1$ *pendant vertices as neighbor or* $d(u) - 2$ *pendant vertices and* 1 *vertex that belongs to another* $C_3 \in G$ *as neighbor.*

Proof. Proof can be derived by the proof of Theorem 2. □

Similarly to above remarks, it is trivial that the converse of Theorem 2 is not true. Moreover, the converse of Corollary 8 is not satisfied since several realizations for a given degree sequence can be formed. Corollary 8 can be considered conversely that if the inequality $a_3+a_4+\cdots+a_\Delta < 5c(G)+1-2a_2 = 5\frac{\Omega(G)}{2} + 5\theta + 1 - 2a_2$ is satisfied for G, then G does not satisfy the minimal nullity condition.

Corollary 10 *Let* $G = (V, E)$ *be a graph with* n *vertices and the degree sequence* $D(G) = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. *If all connected components of* G *are not isomorphic to the path graph* P_2 *and* G *satisfies the minimal nullity condition, then* $n \geq 8c(G) + 4 - 3a_2 = 4\Omega(G) + 8\theta + 4 - 3a_2$.

Proof. Let G (all connected components of G is not P_2) be a graph with n vertices and the degree sequence $D(G)$. Since G satisfies the minimal nullity condition, then inequality $a_3 + a_4 + \cdots + a_\Delta \geq 5c(G) + 1 - 2a_2$ is satisfied by Corollary 3. By using definition of omega invariant we get $\Omega(G) = -a_1 + a_3 + 2a_4 + \cdots + (\Delta - 2)a_\Delta$. Hence we get $a_3 + a_4 + \cdots + a_\Delta = \Omega(G) + a_1 - a_4 - 2a_5 - \cdots - (\Delta - 3)a_\Delta$. If we write $\Omega(G) + a_1 - a_4 - 2a_5 - \cdots - (\Delta - 3)a_\Delta$ instead of $a_3 + a_4 + \cdots + a_\Delta$ in the inequality, we have $a_1 \geq 5c(G) + 1 - 2a_2 - \Omega(G) + a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta$. By Proposition 1, we

obtain $a_1 \geq (\frac{5\Omega(G)}{2} + 5) + 1 - 2a_2 - \Omega(G) + a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta$ and by arranging we achieve $a_1 \geq 3c(G) + 3 - 2a_2 + a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta$. Since $a_4 + 2a_5 + \cdots + (\Delta - 3)a_\Delta \geq 0$, then we get $a_1 \geq 3c(G) + 3 - 2a_2$. By adding the inequalites $a_1 \geq 3c(G) + 3 - 2a_2$ and $a_3 + a_4 + \cdots + a_\Delta \geq 5c(G) + 1 - 2a_2$, we have $a_1 + a_3 + a_4 + \cdots + a_\Delta \geq 8c(G) + 4 - 4a_2$ so we get $a_1 + a_2 + a_3 + a_4 + \cdots + a_\Delta \geq 8c(G) + 4 - 3a_2$ that is the desired result by considering $n = a_1 + a_2 + a_3 + a_4 + \cdots + a_\Delta$. By Proposition 1 the desired result is obtained. □

The corollary below is a characterization of graphs that attain the given bound in Corollary 10.

Corollary 11 *Let* $G = (V, E)$ *be a graph with* n *vertices and the degree sequence* $D(G) = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. *If all connected components of* G *are not isomorphic to the path graph* P_2 *and* G *satisfies the minimal nullity condition, then* $n = 8c(G) + 4 - 3a_2 = 4\Omega(G) + 8\theta + 4 - 3a_2$ *if and only if*

i–) G *is a connected cycle vertex disjoint graph whose all cycles are* C_3,

ii–) *For* $\forall C_3 \in G, \forall v \in C_3$, $d(v) = 2$ *or* $d(v) = 3$,

iii–) *In the case of* $d(v) = 3$, *the neighbor* u *of* v *that does not belong to corresponding* C_3 *(belongs to tree part) has either* 2 *pendant vertices as neighbor or* 1 *pendant vertices and* 1 *vertex that belongs to another* $C_3 \in G$ *as neighbor.*

Proof. Proof can be completed by utilizing proofs of Theorem 2, Corollary 9 and 10. □

Corollary 12 *Let* $G = (V, E)$ *be a pairwise cycle-vertex-disjoint graph with* n *vertices and the length of all cycles (if any) are odd with the degree sequence* $D(G) = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. *If all connected components of* G *are not isomorphic to the path graph* P_2 *and* $\nu(T_G) = \nu(T_G - W_G)$, *then* $n \geq 8c(G) + 4 - 3a_2 = 4\Omega(G) + 8\theta + 4 - 3a_2$.

Proof. Proof follows from Proposition 5 and Corollary 10. □

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