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On Seidel Laplacian matrix and energy of graphs

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Abstract. In this work, the Seidel Laplacian spectrum of graphs are determined. Then new bounds are presented for the Seidel Laplacian energy of regular graphs and graphs by using their Seidel Laplacian spectrum and other techniques. Further, the Seidel Laplacian energy of specific graphs are computed.

1 Introduction

Let G be a simple graph with vertex set and edge set of cardinality n and e, respectively. The adjacent vertices are denoted by $v_i \sim v_j$, otherwise $v_i \nsim v_j$. Degree of a vertex v_i is denoted by d_i. The adjacency matrix $A(G)$ is defined with (i,j)-entries as 1 and 0, if $v_i \sim v_j$ and otherwise, respectively. The energy of a graph is introduced in [8] as $En(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i are the eigenvalues of $A(G)$, namely $En(G) = En(A(G))$. Graph energy and its types have long been an interesting topic for chemists and mathematicians (see [3],[11],[15],[18]).

Let $D(G) = diag(d_1, d_2, ..., d_n)$ be the diagonal matrix of vertex degrees of G. The Laplacian matrix $L(G)$ is defined as $L(G) = D(G) - A(G)$ with eigenvalues $n \geq \vartheta_1 \geq \vartheta_2 \geq ... \geq \vartheta_n = 0$ which are called as Laplacian eigenvalues of G. The Laplacian energy of G is introduced in [6] as $En(L(G)) = \sum_{i=1}^{n} \left|\vartheta_i - \frac{2e}{n}\right|$.

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The Seidel matrix of G is defined by $S(G) = A(G^c) - A(G)$, where G^c is the complement of the graph G. The Seidel energy $En(S(G))$ of G is proposed in [9] as $En(S(G)) = \sum_{i=1}^{n} |\xi_i|$, where ξ_i are the Seidel eigenvalues of G. In [15], the relations between the Seidel energy and energy of graphs are given.

The Seidel Laplacian matrix $S_{\mathcal{L}}(G)$ of G is introduced in [17] as

$$S_{\mathcal{L}}(G) = D_s(G) - S(G), \tag{1}$$

where $D_s(G)$ be the diagonal matrix with main diagonal entries $n - 1 - 2d_i$. Also, $D_s(G) = D(G^c) - D(G)$ and $S_{\mathcal{L}}(G) = L(G^c) - L(G)$. The eigenvalues $\xi_1^L, \xi_2^L, ..., \xi_n^L$ of $S_{\mathcal{L}}(G)$ are called as the Seidel Laplacian eigenvalues of G. Further, the Seidel Laplacian energy of G is proposed in [17] as

$$En(S_{\mathcal{L}}(G)) = \sum_{i=1}^{n} \left| \xi_i^L - \frac{n(n-1) - 4e}{n} \right|$$

and some bounds for the Seidel Laplacian energy are presented by using mathematical inequalities. The eigenvalues ξ_i^L with multiplicities form the Seidel Laplacian spectrum of G and ξ_1^L denotes the spectral radius of $S_{\mathcal{L}}(G)$. Further studies on Seidel Laplacian energy see [1],[5]. Throughout this work, the order of all matrices is n.

Let $t_i := \xi_i^L - X$, where $X = \frac{n(n-1)-4e}{n}$. Hence,

$$En(S_{\mathcal{L}}(G)) = \sum_{i=1}^{n} |t_i|. \tag{2}$$

In this work, we initially determine the Seidel Laplacian spectrum of graphs. Then we propose new bounds for the Seidel Laplacian energy of graphs and regular graphs. Besides, we calculate the Seidel Laplacian energy of specific graphs.

In [17], it is shown that

$$\sum_{i=1}^{n} t_i = 0, \tag{3}$$

$$\sum_{i=1}^{n} t_i^2 = n(n-1) + 4Zg - \frac{16e^2}{n}, \tag{4}$$

where $Zg = Zg(G) = \sum_{u \sim v} d_u + d_v$ is the first Zagreb index of G. The inequality (5) is given in [1].

$$|t_1| \geq |X| \tag{5}$$

Lemma 1 ([17]) *If ξ_i^L are the Seidel Laplacian eigenvalues of G, then $-\xi_i^L$ are the Seidel Laplacian eigenvalues of G^c, for $i = 1, 2, ..., n$.*

Lemma 2 ([17]) *If $\xi_1, \xi_2, ..., \xi_n$ are the Seidel eigenvalues of a r-regular graph G, then $n-1-2r-\xi_i$ $(i = 1, 2, ..., n)$ are the Seidel Laplacian eigenvalues of G.*

2 Seidel Laplacian eigenvalues

Let $\vartheta_1 \geq \vartheta_2 \geq ... \geq \vartheta_n$ be the Laplacian eigenvalues of G and associated Laplacian eigenvectors be $\vec{c_1}, \vec{c_2}, ..., \vec{c_n}$ are column-vectors of dimension n. Let I be the unit matrix and J be the all-ones matrix.

Lemma 3 ([7]) *The vector $\vec{j}$ with all components are 1 is a Laplacian eigenvector of any order n graph corresponding to the eigenvalue of $\vartheta_n = 0$.*

According to the previous lemma $\vec{c_n} = \vec{j}$ can be choosen. Assume that for $i = 1, 2, ..., n-1$ the Laplacian eigenvectors $\vec{c_i}$ are orthogonal to $\vec{c_n}$. For any vector $\vec{c}$, the scalar product is $\vec{j}^t \cdot \vec{c} = \rho\left(\vec{c}\right)$.

Lemma 4 ([7]) *If G is a graph, then $\rho\left(\vec{c_i}\right) = 0$ for any $1 \leq i \leq n-1$.*

Now we give the relationship between the Seidel Laplacian and Laplacian eigenvalues as follows.

Theorem 5 *If $\vec{c_i}$ is a Laplacian eigenvector of the graph G, then $\vec{c_i}$ is a Seidel Laplacian eigenvector of the graph G and $\xi_i^L = n - 2\vartheta_i$, for $1 \leq i \leq n-1$. If $i = n$, then $\xi_i^L = \vartheta_i = 0$.*

Proof. Clearly $S_{\mathcal{L}}(G) = L(G^c) - L(G) = nI - J - 2L(G)$. Thus from Lemma 4, we have

$$\begin{aligned} S_{\mathcal{L}}(G)\vec{c_i} &= [nI - J - 2L(G)]\,\vec{c_i} \\ &= nI\vec{c_i} - J\vec{c_i} - 2L(G)\vec{c_i} \\ &= n\vec{c_i} - \rho\left(\vec{c_i}\right)\vec{j} - 2\vartheta_i\vec{c_i} \\ &= (n - 2\vartheta_i)\,\vec{c_i}. \end{aligned} \tag{6}$$

From (6) obviously we have $\xi_i^L = n - 2\vartheta_i$ for $1 \leq i \leq n-1$. If $i = n$, then $\xi_i^L = \vartheta_i = 0$ see [1]. □

Corollary 6 *The following assertions hold for the Seidel Laplacian eigenvalues of* G *and* G^c :

(i) $\overline{\xi_i^L} = -n + 2\vartheta_{n-i}$, *for* $1 \leq i \leq n-1$ *and* $\overline{\xi_i^L} = \overline{\vartheta_i} = \vartheta_i = 0$ *for* $i = n$, *where* $\overline{\xi_i^L}$ *and* $\overline{\vartheta_i}$ *are the Seidel Laplacian and Laplacian eigenvalues of* G^c, *respectively.*

(ii) The Seidel Laplacian eigenvalues of G *and* G^c *can be given respectively as*

$$n - 2\vartheta_1, n - 2\vartheta_2, ..., n - 2\vartheta_{n-1}, 0 \tag{7}$$

$$-n + 2\vartheta_{n-1}, -n + 2\vartheta_{n-2}, ..., -n + 2\vartheta_1, 0. \tag{8}$$

Proof.

(i) Obviously for $i = n$, $\overline{\vartheta_i} = \vartheta_i = 0$ and for $1 \leq i \leq n-1$, $\overline{\vartheta_i} = n - \vartheta_{n-i}$. From Theorem 5, $\xi_i^L = n - 2\vartheta_i$ for $1 \leq i \leq n-1$, we also have $\overline{\xi_i^L} = n - 2\overline{\vartheta_i} = n - 2(n - \vartheta_{n-i}) = -n + 2\vartheta_{n-i}$. For $i = n$, $\overline{\xi_i^L} = \overline{\vartheta_i} = \vartheta_i = 0$.

(ii) The proof of (7) and (8) can be seen as a result of Theorem 5 and (i), respectively.

□

Remark 7 ([2]) *Let* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ *be the eigenvalues of a* r*-regular graph, then* $\lambda_1 = r$ *and its Laplacian eigenvalues are*

$$0, r - \lambda_2, ..., r - \lambda_n,$$

and the Seidel eigenvalues are

$$n - 1 - 2r, -1 - 2\lambda_2, ..., -1 - 2\lambda_n. \tag{9}$$

Corollary 8 *If* G *is a* r*-regular graph, then the Seidel Laplacian eigenvalues of* G *are*

$$0, n - 2r + 2\lambda_2, ..., n - 2r + 2\lambda_n. \tag{10}$$

Proof. From Lemma 2, $n - 1 - 2r - \xi_i$ are the Seidel Laplacian eigenvalues of G and using (9) yields the result. □

We will determine the Seidel Laplacian spectrum of certain graphs by using Theorem 5 and their Laplacian spectrum.

Proposition 9 *If S_n is a star, then the Seidel Laplacian spectrum of S_n is* $\left\{-n, 0, \underbrace{n-2, ..., n-2}_{n-2}\right\}$.

Proof. Laplacian spectrum of S_n is $\left\{n, 0, \underbrace{1, ..., 1}_{n-2}\right\}$. Then by using $\xi_i^L = n - 2\vartheta_i$, for $1 \leq i \leq n-1$ and $\xi_n^L = \vartheta_n = 0$, the proof can be seen. □

Proposition 10 *If C_n is a cycle graph, then the Seidel Laplacian spectrum of C_n is $n - 4\left(1 - \cos\frac{2\pi i}{n}\right)$, $i = 0, 1, ..., n-1$.*

Proof. Laplacian spectrum of C_n is $2\left(1 - \cos\frac{2\pi i}{n}\right)$ for $i = 0, 1, ..., n-1$ (see [19]). Using Theorem 5 completes the proof. □

Proposition 11 *If P_n is a path graph, then the Seidel Laplacian spectrum of P_n is $n - 4\left(1 - \cos\frac{\pi i}{n}\right)$, $i = 0, 1, ..., n-1$.*

Proof. Laplacian spectrum of P_n is $2\left(1 - \cos\frac{\pi i}{n}\right)$ for $i = 0, 1, ..., n-1$ (see [19]), using Theorem 5 yields the result. □

Proposition 12 *If $K_{m,n}$ be a complete bipartite graph, then Seidel Laplacian eigenvalues of $K_{m,n}$ are $-(m+n), 0$; $n-m$ of multiplicity $n-1$; $m-n$ of multiplicity $m-1$.*

Proof. Laplacian eigenvalues of $K_{m,n}$ are $m+n, 0$; m of multiplicity $n-1$; n of multiplicity $m-1$. As $K_{m,n}$ has $m+n$ vertices, then from Theorem 5 $\xi_i^L = m + n - 2\vartheta_i$ for $1 \leq i \leq m+n-1$ and $\xi_{m+n}^L = \vartheta_{m+n} = 0$. □

The friendship graph is formed by s triangles with a common vertex, which has $2s+1$ vertices and $3s$ edges.

Proposition 13 *If F_s $(s \geq 1)$ is a friendship graph, then the Seidel Laplacian spectrum of F_s is $-(2s+1), 0$; $2s-5$ of multiplicity s; $2s-1$ of multiplicity $s-1$.*

Proof. The friendship graph has Laplacian eigenvalues $2s+1, 0$; 3 of multiplicity s; 1 of multiplicity $s-1$ (see [19]). The proof is clear from Theorem 5. □

Lemma 14 ([10]) *If* B *is an irreducible non-negative matrix with spectral radius* $\theta(B)$ *and* $s_i(B)$ *be the* i^{th} *row sum of* B*, then*

$$\theta(B) \leq \max_{1\leq i\leq n} s_i(B).$$

Now we present an upper bound for ξ_1^L.

Theorem 15 *Let* G *be an order* n *connected graph with maximum degree* Δ*. Then*

$$\xi_1^L \leq 2\Delta.$$

Proof. Using the previous lemma we get

$$\begin{aligned}\xi_1^L \leq & \theta(|S_{\mathcal{L}}|)\\ \leq & \max_{1\leq i\leq n} s_i(|S_{\mathcal{L}}|)\\ = & \max_{1\leq i\leq n} \{|n-1-2d_i|+n-1\}\end{aligned}$$

If $n-1 \geq 2d_i$, then $s_i(|S_{\mathcal{L}}|) = 2(n-1-d_i)$ and if $n-1 < 2d_i$, then $s_i(|S_{\mathcal{L}}|) = 2d_i$. Therefore we have

$$\begin{aligned}\xi_1^L \leq & 2\max_{1\leq i\leq n} \{d_i, n-1-d_i\}\\ \leq & 2\max_{1\leq i\leq n} \{\Delta, n-1-\delta\} = 2\Delta,\end{aligned}$$

where δ is the minimum degree. □

3 Bounds for Seidel Laplacian energy

Several bounds will be obtained for the Seidel Laplacian energy of graphs via their Seidel Laplacian spectrum, mathematical inequalities and Ky Fan theorem in this section. Now, we can begin with the following theorem which states that the Seidel Laplacian energy and Seidel energy coincides for regular graphs.

Theorem 16 *If* G *is a* r*-regular graph of order* n*, then*

$$En(S_{\mathcal{L}}(G)) = En(S(G)).$$

Proof. We have $X = n-1-2r$ and by Lemma 2, $\xi_i^L = n-1-2r-\xi_i$ which yields

$$\begin{aligned} En\left(S_{\mathcal{L}}\left(G\right)\right) =& \sum_{i=1}^{n} \left|n-1-2r-\xi_i-(n-1-2r)\right| \\ =& En\left(S\left(G\right)\right). \end{aligned}$$

□

Theorem 17 ([12]) *If* G *is a graph of order* n *with* e *edges, then*

$$En\left(G\right) \leq \sqrt{2en}. \tag{11}$$

Here, we can give two bounds for the Seidel Laplacian energy of regular graphs.

Theorem 18 *If* G *is a* r*-regular graph of order* n*, then*

$$En\left(S_{\mathcal{L}}\left(G\right)\right) \quad \leq \quad \left|n-1-2r\right| + n-1-2r+2En\left(G\right). \tag{12}$$

Proof. From (10) and as $X = n-1-2r$, we get

$$\begin{aligned} En\left(S_{\mathcal{L}}\left(G\right)\right) =& \sum_{i=1}^{n} \left|\xi_i^L-(n-1-2r)\right| \\ =& \left|n-1-2r\right| + \sum_{i=2}^{n} \left|n-2r+2\lambda_i-(n-1-2r)\right| \\ \leq& \left|n-1-2r\right| + \sum_{i=2}^{n} \left(2\left|\lambda_i\right|+1\right) \\ =& \left|n-1-2r\right| + n-1+2\left(En(G)-\left|\lambda_1\right|\right). \end{aligned}$$

The proof is clear by using $\lambda_1 = r$. □

Corollary 19 *If* G *is a* r*-regular graph of order* n*, then*

$$En\left(S_{\mathcal{L}}\left(G\right)\right) \leq \left|n-1-2r\right| + n-1-2r+2n\sqrt{r}.$$

Proof. From (11), $En\left(G\right) \leq n\sqrt{r}$. By using this fact in (12) yields the result. □

Theorem 20 *Let* G *be a graph of order* n *with* e *edges. Then*

$$En\left(S_{\mathcal{L}}\left(G\right)\right) \geq \sqrt{n(n-1)+4Zg-\frac{16e^2}{n}}.$$

Proof. If we apply the Radon inequality [16], then

$$\sum_{i=1}^{n} |t_i| = \sum_{i=1}^{n} \frac{|t_i|^2}{|t_i|} \geq \frac{\sum_{i=1}^{n} |t_i|^2}{\sum_{i=1}^{n} |t_i|}.$$

Hence, from (2) and (4) the proof is clear. □

Theorem 21 *Let* G *be a graph. Then*

$$En(S_{\mathcal{L}}(G)) \geq 2|X|.$$

Proof. Clearly

$$\begin{aligned} En(S_{\mathcal{L}}(G)) &= |t_1| + \sum_{i=2}^{n} |t_i| \\ &\geq |t_1| + \left| \sum_{i=2}^{n} t_i \right|. \end{aligned}$$

From (3), $|t_1| = \left| -\sum_{i=2}^{n} t_i \right|$. Thus by (5)

$$En(S_{\mathcal{L}}(G)) \geq 2|t_1| \geq 2\left| n - 1 - \frac{4e}{n} \right|,$$

completes the proof. □

Theorem 22 *If* G *is a graph with* n *vertices and* e *edges, then*

$$En(S_{\mathcal{L}}(G)) \geq \frac{2\left(n(n-1) + 4Zg - \frac{16e^2}{n}\right)}{\xi_1^L}.$$

Proof. Let $(a_i), (b_i)$, $i = 1, 2, ..., n$ be real numbers with $\sum_{i=1}^{n} |a_i| = 1$ and $\sum_{i=1}^{n} a_i = 0$. Then

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \frac{1}{2} \left(\max_{1 \leq i \leq n} b_i - \min_{1 \leq i \leq n} b_i \right) \tag{13}$$

holds ([13],p.346).

Setting $b_i := \xi_i^L - X$ and $a_i = \frac{\xi_i^L - X}{\sum_{i=1}^{n} |\xi_i^L - X|}$ satisfies the conditions for inequality (13). Thus

$$\left| \frac{\sum_{i=1}^{n} \left(\xi_i^L - X\right)^2}{\sum_{i=1}^{n} \left|\xi_i^L - X\right|} \right| \leq \frac{1}{2} \left(\max_{1 \leq i \leq n} \left(\xi_i^L - X\right) - \min_{1 \leq i \leq n} \left(\xi_i^L - X\right) \right),$$

then, by (4)

$$\frac{n(n-1) + 4Zg - \frac{16e^2}{n}}{En(S_{\mathcal{L}}(G))} \leq \frac{1}{2} \left(\xi_1^L - X - (0 - X) \right),$$

yields

$$En(S_{\mathcal{L}}(G)) \geq \frac{2\left(n(n-1) + 4Zg - \frac{16e^2}{n}\right)}{\xi_1^L}.$$

□

Combining Theorem 15 with Theorem 22 results as follows.

Corollary 23 *Let* G *be a connected graph. Then*

$$En(S_{\mathcal{L}}(G)) \geq \frac{n(n-1) + 4Zg - \frac{16e^2}{n}}{\Delta}.$$

Let H be a symmetric matrix, $y_i(H)$ be its eigenvalues and $\sigma_i(H)$ be its singular values, $i = 1, 2, ..., n$. Then $\sigma_i(H) = |y_i(H)|$ for $i = 1, 2, ..., n$. In [14], the energy of a graph G is reconsidered as sum of the singular values of its adjacency matrix $A(G)$. This point of view has brought a new approach to the graph energy theory and made the following theorem proved by Fan [4] to be included in the graph energy theory.

Theorem 24 ([4]) *If* A, B *and* C *are square matrices of order* n*, then*

$$\sum_{i=1}^{n} \sigma_i(A) \leq \sum_{i=1}^{n} \sigma_i(B) + \sum_{i=1}^{n} \sigma_i(C),$$

where $A = B + C$.

Theorem 25 *Let* G *be a graph, then*

$$En\left(S_{\mathcal{L}}(G)\right) \leq En\left(S(G)\right) + 2\sum_{i=1}^{n}\left|\overline{d} - d_i\right|, \tag{14}$$

where $\overline{d}$ *is the average degree of* G.

Proof. From (1), we can write

$$\left(S_{\mathcal{L}}(G) - \frac{n(n-1)-4e}{n}I\right) = (-S(G)) + \left(D_s(G) - \frac{n(n-1)-4e}{n}I\right).$$

Also, $\left(D_s(G) - \frac{n(n-1)-4e}{n}I\right)$ is a diagonal matrix with eigenvalues $\frac{4e}{n} - 2d_i$ that is, $2\left(\overline{d} - d_i\right)$ for $i = 1, 2, ..., n$ where $\overline{d}=\frac{2e}{n}$. Then by (2) and applying Theorem 24 satisfies the result. □

Corollary 26 *If* G *is a graph of order* n, *then*

$$En\left(S_{\mathcal{L}}(G)\right) \leq \sqrt{n\left(n^2-n\right)} + 2\sum_{i=1}^{n}\left|\overline{d} - d_i\right|.$$

Proof. In [11], it is shown that $En\left(S(G)\right) \leq \sqrt{n\left(n^2-n\right)}$. If we use this bound in (14), then we get the result. □

Different bounds for $En\left(S_{\mathcal{L}}(G)\right)$ can also be derived by using the bounds for Seidel energy.

4 Seidel Laplacian energy of specific graphs

The Seidel Laplacian energy of certain graphs will be calculated here by using their Seidel Laplacian spectrum.

Proposition 27 *If* S_n *is a star with* $n\,(\geq 4)$ *vertices, then* $En\left(S_{\mathcal{L}}(S_n)\right) = 6n + \frac{16}{n} - 20$.

Proof. *Star graph has* $n-1$ *edges, thus* $X = \frac{n^2-5n+4}{n}$. *From Proposition 9*

$$\begin{aligned} En\left(S_{\mathcal{L}}(S_n)\right) &= \left|-n - \frac{n^2-5n+4}{n}\right| + \left|\frac{n^2-5n+4}{n}\right| \\ &\quad + (n-2)\left|n-2-\frac{n^2-5n+4}{n}\right| \\ &= \left|5 - \left(2n + \frac{4}{n}\right)\right| + \left|n + \frac{4}{n} - 5\right| + (n-2)\left|3 - \frac{4}{n}\right|. \end{aligned}$$

Since $5-\left(2n+\frac{4}{n}\right)<0$, $3-\frac{4}{n}>0$ *and* $n+\frac{4}{n}-5\geq 0$ *for* $n\geq 4$, *we obtain*

$$\begin{aligned}En\left(S_{\mathcal{L}}\left(S_n\right)\right)=&-5+2n+\frac{4}{n}+n+\frac{4}{n}-5+(n-2)\left(3-\frac{4}{n}\right)\\=&6n+\frac{16}{n}-20.\end{aligned}$$

□

Proposition 28 *If* C_n *is the cycle graph with* n *vertices, then*

$$En\left(S_{\mathcal{L}}\left(C_n\right)\right)=\sum_{i=0}^{n-1}\left|1+4\cos\frac{2\pi i}{n}\right|.$$

Proof. From Proposition 10 and $X=n-5$, we have

$$En\left(S_{\mathcal{L}}\left(C_n\right)\right)=\sum_{i=0}^{n-1}\left|n-4\left(1-\cos\frac{2\pi i}{n}\right)-(n-5)\right|=\sum_{i=0}^{n-1}\left|1+4\cos\frac{2\pi i}{n}\right|.$$

□

Proposition 29 *If* P_n *is the path graph with* n *vertices, then*

$$En\left(S_{\mathcal{L}}\left(P_n\right)\right)=\sum_{i=0}^{n-1}\left|1+4\left(\cos\frac{\pi i}{n}\right)-\frac{4}{n}\right|.$$

Proof. From Proposition 11 and as $X=\frac{n^2-5n+4}{n}$, we obtain

$$\begin{aligned}En\left(S_{\mathcal{L}}\left(P_n\right)\right)=&\sum_{i=0}^{n-1}\left|n-4\left(1-\cos\frac{\pi i}{n}\right)-\frac{n^2-5n+4}{n}\right|\\=&\sum_{i=0}^{n-1}\left|1+4\left(\cos\frac{\pi i}{n}\right)-\frac{4}{n}\right|.\end{aligned}$$

□

Proposition 30 *If* $K_{m,n}$ $(1<n<m)$ *is a complete bipartite graph, then*

$$En\left(S_{\mathcal{L}}\left(K_{m,n}\right)\right)=\begin{cases}\frac{4\left(m^2n-n^2m+n^2\right)+2\left(m^2-m-n-mn\right)}{m+n}, & \text{if } X>0\\ \frac{4\left(m^2n-n^2m\right)+2\left(n^2+mn\right)}{m+n}, & \text{if } X<0\end{cases},$$

where $X = \frac{(m-n)^2-(m+n)}{m+n}$.

Proof. $K_{m,n}$ has $m+n$ vertices, mn edges, then $X = \frac{(m-n)^2-(m+n)}{m+n}$. Thus from Proposition 12

$$\begin{aligned} En\left(S_{\mathcal{L}}\left(K_{m,n}\right)\right) = &\left|-(m+n) - \frac{(m-n)^2-(m+n)}{m+n}\right| \\ &+ (n-1)\left|n-m-\frac{(m-n)^2-(m+n)}{m+n}\right| \\ &+ (m-1)\left|m-n-\frac{(m-n)^2-(m+n)}{m+n}\right| \\ &+ \left|\frac{(m-n)^2-(m+n)}{m+n}\right|. \end{aligned} \tag{15}$$

Denote the terms in RHS of (15) as A_1, A_2, A_3, A_4, respectively. Then

$$A_1 = (m+n) + \frac{(m-n)^2-(m+n)}{m+n} = \frac{2m^2+2n^2-m-n}{m+n},$$

as $-(m+n) - \frac{(m-n)^2-(m+n)}{m+n} = \frac{(m+n)-\left[(m-n)^2+(m+n)^2\right]}{m+n} < 0$.

$$\begin{aligned} A_2 &= (n-1)\left[\frac{-(m+n)+\left(m^2-n^2\right)+(m-n)^2}{m+n}\right] \\ &= (n-1)\left(\frac{2m^2-2mn-m-n}{m+n}\right) \\ &= \frac{2m^2n-2n^2m-2m^2-n^2+mn+m+n}{m+n}, \end{aligned}$$

as $n-m-\frac{(m-n)^2-(m+n)}{m+n} = \frac{m+n-\left[\left(m^2-n^2\right)+(m-n)^2\right]}{m+n} < 0$.

$$\begin{aligned} A_3 &= (m-1)\left(m-n-\frac{(m-n)^2-(m+n)}{m+n}\right) \\ &= (m-1)\left(\frac{-2n^2+2mn+m+n}{m+n}\right) \\ &= \frac{-2n^2m+2m^2n+m^2+2n^2-mn-m-n}{m+n}, \end{aligned}$$

as $m-n-\frac{(m-n)^2-(m+n)}{m+n}=\frac{m^2-n^2-(m-n)^2+m+n}{m+n}=\frac{2n(m-n)+m+n}{m+n}>0$. Now consider $A_4=\left|\frac{(m-n)^2-(m+n)}{m+n}\right|=|X|$. Obviously if $X>0$, then $A_4=\frac{m^2-2mn+n^2-m-n}{m+n}$ and if $X<0$, then $A_4=-\frac{m^2-2mn+n^2-m-n}{m+n}$. If we sum A_1,A_2,A_3,A_4 in case $X>0$, then

$$En\left(S_{\mathcal{L}}\left(K_{m,n}\right)\right)=\frac{4m^2n-4n^2m+2m^2+4n^2-2m-2n-2mn}{m+n}.$$

Repeating above operations in case $X<0$, we have

$$En\left(S_{\mathcal{L}}\left(K_{m,n}\right)\right)=\frac{4m^2n-4n^2m+2n^2+2mn}{m+n}.$$

□

Example 31 *Let $K_{4,3}$ be a complete bipartite graph. Then the Seidel Laplacian eigenvalues of $K_{4,3}$ are $-7,0$; -1 of multiplicity 2; 1 of multiplicity 3. Since $X=\frac{1^2-7}{7}=-\frac{6}{7}<0$, from Proposition 30 we have $En\left(S_{\mathcal{L}}\left(K_{m,n}\right)\right)=\frac{4\left(m^2n-n^2m\right)+2\left(n^2+mn\right)}{m+n}$, thus $En\left(S_{\mathcal{L}}\left(K_{4,3}\right)\right)=\frac{90}{7}$. Consider $K_{5,2}$, then the Seidel Laplacian eigenvalues of $K_{5,2}$ are $-7,0$; -3 of multiplicity 1; 3 of multiplicity 4. Since $X=\frac{3^2-7}{7}=\frac{2}{7}>0$, from Proposition 30 we have $En\left(S_{\mathcal{L}}\left(K_{m,n}\right)\right)=\frac{4\left(m^2n-n^2m+n^2\right)+2\left(m^2-m-n-mn\right)}{m+n}$, then $En\left(S_{\mathcal{L}}\left(K_{5,2}\right)\right)=\frac{152}{7}$.*

Proposition 32 *If F_s $(s\geq 3)$ is the friendship graph, then $En\left(S_{\mathcal{L}}\left(F_s\right)\right)=\frac{24s^2-32s+21}{2s+1}$.*

Proof. By Proposition 13 and as $X=\frac{(2s+1)2s-12s}{2s+1}=\frac{4s^2-10s}{2s+1}$, we have

$$\begin{aligned}
En\left(S_{\mathcal{L}}\left(F_s\right)\right)=&\sum_{i=0}^{2s+1}\left|\xi_i^L-X\right|\\
=&\left|-(2s+1)-\frac{4s^2-10s}{2s+1}\right|+s\left|2s-5-\frac{4s^2-10s}{2s+1}\right|\\
&+(s-1)\left|2s-1-\frac{4s^2-10s}{2s+1}\right|+\left|\frac{4s^2-10s}{2s+1}\right|\\
=&\frac{1}{2s+1}\left(\left|8s^2-6s+1\right|+s\left|2s-5\right|+(s-1)\left|10r-1\right|+\left|4s^2-10s\right|\right).
\end{aligned}\tag{16}$$

In RHS of (16), all of the expressions in absolute value are positive for $s \geq 3$, thus

$$En\left(S_{\mathcal{L}}\left(F_s\right)\right) = \frac{1}{2s+1}\left(24s^2 - 32s + 2\right).$$

□

Corollary 33 *If* G *is the friendship graph with* n *vertices, then*

$$En\left(S_{\mathcal{L}}\left(G\right)\right) = \frac{6n^2 - 28n + 24}{n}.$$

Proof. Setting $s = \frac{n-1}{2}$ in Proposition 32 yields the result. □

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