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# Transmission-reciprocal transmission index and coindex of graphs

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**Abstract.** The transmission of a vertex $u$ in a connected graph $G$ is defined as $\sigma(u) = \sum_{v \in V(G)} d(u,v)$ and reciprocal transmission of a vertex $u$ is defined as $rs(u) = \sum_{v \in V(G)} \frac{1}{d(u,v)}$, where $d(u,v)$ is the distance between vertex $u$ and $v$ in $G$. In this paper we define new distance based topological index of a connected graph $G$ called transmission-reciprocal transmission index $TRT(G) = \sum_{uv \in E(G)} \left( \frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)} \right)$ and its coindex $\overline{TRT}(G) = \sum_{uv \notin E(G)} \left( \frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)} \right)$, where $E(G)$ is the edge set of a graph $G$ and establish the relation between $TRT(G)$ and $\overline{TRT}(G)$. Further compute this index for some standard class of graphs and obtain bounds for it.

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## 1 Introduction

Throughout this paper, we consider a finite connected graph $G$ that has no loops and multiple edges. We respectively denote by $n$ and $m$ the cardinality of its vertex set $V(G)$ and its edge set $E(G)$. As customary, we denote by $d(u)$ the degree of $u$ and by $d(u,v)$ the distance between $u$ and $v$.

In recent years a large number of topological indices have been reported and utilized for chemical documentation, isomer discrimination, study of molecular complexity, similarity/dissimilarity, drug design, and database selection, least optimization, rational combinatorial library design and for deriving multilinear regression models.

The use of topological and information theoretical indices in QSAR (Quantative structure-activity relationship) has become of growing importance seems to play an important role in situations where biological activity is determined predominantly by topological architecture of molecular structure, where simple connectivity among neighbouring atoms without considering the chemical nature of atom or nature of chemical bonding, may be the major determinant of biological activity of a molecule.

The status [11, 16] (or transmission [1]) of a vertex $u$ in $G$ is defined as the sum of the distances between $u$ and all other vertices of $G$ and is denoted by $\sigma(u)$. That is,

$$\sigma(u) = \sum_{v \in V(G)} d(u,v).$$

The reciprocal status of a vertex [15] $u \in V(G)$ is defined as the sum of reciprocal of its distance from every other vertex in $V(G)$ and is denoted by $rs(u)$. That is,

$$rs(u) = \sum_{uv \in E(G),\, u \neq v} \frac{1}{d(u,v)}.$$

The Wiener index of graph $G$ is defined as [18]

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{u \in V(G)} \sigma(u).$$

More results about Wiener index can be found in [5, 6, 10, 12, 13, 14, 17].

The first and second Zagreb indices of a graph $G$ are defined as [9]

$$Z_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)] \quad \text{and} \quad Z_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

Results on the Zagreb indices can be found in [4] and the references cited therein.

The Zagreb coindices of a graph $G$ are defined as [7]

$$\overline{Z_1}(G) = \sum_{uv \notin E(G)} [d(u) + d(v)] \quad \text{and} \quad \overline{Z_2}(G) = \sum_{uv \notin E(G)} d(u)d(v).$$

More results on Zagreb coindices can be found in [2, 3].

We introduce here new distance based index called as transmission-reciprocal transmission (T-RT) index of a graph and is defined as

$$TRT(G) = \sum_{uv \in E(G)} \left( \frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)} \right). \tag{1}$$

The transmission-reciprocal transmission coindex of a graph is defined as

$$\overline{TRT}(G) = \sum_{uv \notin E(G)} \left( \frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)} \right). \tag{2}$$

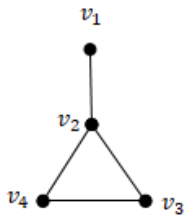



Figure 1: Graph $G$.

**Example 1** *In a graph* $G$ *of Fig. 1,* $V(G) = \{v_1, v_2, v_3, v_4\}$. *Status of vertices are* $\sigma(v_1) = 5$, $\sigma(v_2) = 3$, $\sigma(v_3) = 4$, $\sigma(v_4) = 4$ *and reciprocal status of vertices are* $rs(v_1) = 2$, $rs(v_2) = 3$, $rs(v_3) = \frac{5}{2}$, $rs(v_4) = \frac{5}{2}$. *By definition,* $TRT(G) = 11.9$ *and* $\overline{TRT}(G) = 8.2$.

## 2 Computation of transmission-reciprocal transmission index of graphs

Let $diam(G)$ denotes the diameter of a graph $G$.

**Theorem 2** *Let* $G$ *be a connected graph with* $n$ *vertices and* $m$ *edges and let* $\mathrm{diam}(G) \leq 2$*. Then*

$$TRT(G) = \sum_{uv \in E(G)} \frac{2[4n^2 - 8n + (n-1)(d(u) + d(v)) - 2d(u)d(v) + 4]}{n^2 - 2n + (n-1)(d(u) + d(v)) + d(u)d(v) + 1}.$$

**Proof.** If $\mathrm{diam}(G) \leq 2$ then $d(u)$ be the number of vertices at distance 1 from the vertex $u$ and the remaining $(n-1-d(u))$ vertices are at distance 2. Therefore for each vertex $u$ in $G$, $\sigma(u) = d(u)+2(n-1-d(u)) = 2n-2-d(u)$.

If $\mathrm{diam}(G) \leq 2$, then $d(u)$ number of vertices are at distance 1 from the vertex $u$ and the remaining $n-1-d(u)$ vertices are at distance 2. Therefore for each vertex $u$ in $G$, $rs(u) = \frac{1}{2}(n-1+d(u))$. Now by the Eq. (1) we have

$$\begin{aligned}
&TRT(G)\\
&= \sum_{uv \in E(G)} \left( \frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)} \right)\\
&= \sum_{uv \in E(G)} \left( \frac{2n-2-d(u)}{\frac{1}{2}(n-1+d(u))} + \frac{2n-2-d(v)}{\frac{1}{2}(n-1+d(v))} \right)\\
&= \sum_{uv \in E(G)} \left( \frac{2(2n-2-d(u))}{(n-1+d(u))} + \frac{2(2n-2-d(v))}{(n-1+d(v))} \right)\\
&= \sum_{uv \in E(G)} \frac{\left[ \begin{array}{l} 2(n-1+d(v))(2n-2-d(u)) + \\ 2(n-1+d(u))(2n-2-d(v)) \end{array} \right]}{(n-1+d(u))(n-1+d(v))}\\
&= \sum_{uv \in E(G)} \frac{2[4n^2 - 8n + (d(u) + d(v))(-n+1+2n-2) - 2d(u)d(v) + 4]}{n^2 - 2n + (n-1)(d(u) + d(v)) + d(u)d(v) + 1}\\
&= \sum_{uv \in E(G)} \frac{2[4n^2 - 8n + (n-1)(d(u) + d(v)) - 2d(u)d(v) + 4]}{n^2 - 2n + (n-1)(d(u) + d(v)) + d(u)d(v) + 1}.
\end{aligned}$$

□

**Corollary 3** *Let* $G$ *be a connected regular graph of degree* $r$ *on* $n$ *vertices and* $m$ *edges and let* $\mathrm{diam}(G) \leq 2$*. Then*

$$TRT(G) = \frac{2m[4n^2 - 8n + 4 + 2r(n-1) - 2r^2]}{n^2 - 2n + 2r(n-1) + r^2 + 1}.$$

**Proof.** By Theorem 2 we have

$$TRT(G) = \sum_{uv \in E(G)} \frac{2[4n^2 - 8n + (n-1)(d(u) + d(v)) - 2d(u)d(v) + 4]}{n^2 - 2n + (n-1)(d(u) + d(v)) + d(u)d(v) + 1}.$$

Since $G$ is a regular graph, $d(u) = r$, for all $u \in V(G)$. Therefore

$$\begin{aligned} TRT(G) &= \sum_{uv \in E(G)} \frac{2[4n^2 - 8n + 2r(n-1) - 2r^2 + 4]}{n^2 - 2n + 2r(n-1) + r^2 + 1} \\ &= \frac{2m[4n^2 - 8n + 2r(n-1) - 2r^2 + 4]}{n^2 - 2n + 2r(n-1) + r^2 + 1}. \end{aligned}$$

□

**Corollary 4** *For a complete graph $K_n$ on $n$ vertices,*

$$TRT(K_n) = n(n-1).$$

**Proof.** As $K_n$ is a complete graph, $d(u) = n - 1$ for all $u \in V(K_n)$. Therefore

$$\begin{aligned} TRT(K_n) &= \sum_{uv \in E(K_n)} \frac{2[4n^2 - 8n + (n-1)(d(u) + d(v)) - 2(u)d(v) + 4]}{n^2 - 2n + (n-1)(d(u) + d(v)) + d(u)d(v) + 1} \\ &= \sum_{uv \in E(K_n)} \frac{2[4n^2 - 8n + 2(n-1) - 2(n-1)^2 + 4]}{n^2 - 2n + 2(n-1)^2 + (n-1)^2 + 1} \\ &= \sum_{uv \in E(K_n)} 2 \\ &= 2\frac{n(n-1)}{2} = n(n-1). \end{aligned}$$

□

**Proposition 5** *For a complete bipartite graph $K_{p,q}$,*

$$TRT(K_{p,q}) = \frac{2pq[(p+q)(5p+5q-9) - 2pq + 4]}{(p+q)(2p+2q-3) + pq + 1}.$$

**Proof.** The graph $K_{p,q}$ has $n = p + q$ vertices and $m = pq$ edges. Also $diam(K_{p,q}) \leq 2$. The vertex set $V(K_{p,q})$ can be partitioned into two sets $V_1$ and $V_2$ such that for every edge $uv$ of $K_{p,q}$ the vertex $u \in V_1$ and $v \in V_2$, where

$|V_1| = p$ and $|V_2| = q$. Therefore $d(u) = q$ and $d(v) = p$. By Theorem 2, we have

$$\begin{aligned}
&TRT(K_{p,q}) \\
&= \sum_{uv\in E(K_{p,q})} \frac{2[4(p+q)^2 - 8(p+q) + (p+q-1)(p+q) - 2pq + 4]}{(p+q)^2 - 2(p+q) + (p+q-1)(p+q) + pq + 1} \\
&= \sum_{uv\in E(K_{p,q})} \frac{2[(p+q)(4p+4q+p+q-9) - 2pq + 4]}{(p+q)(p+q+p+q-3) + pq + 1} \\
&= \sum_{uv\in E(K_{p,q})} \frac{2[(p+q)(5p+5q-9) - 2pq + 4]}{(p+q)(2p+2q-3) + pq + 1} \\
&= \frac{2pq[(p+q)(5p+5q-9) - 2pq + 4]}{(p+q)(2p+2q-3) + pq + 1}.
\end{aligned}$$

□

**Proposition 6** *For a path $P_n$ on $n$ vertices,*

$$TRT(P_n) = \left( \frac{n(n-1)}{\sum_{j=1}^{n-1} \frac{1}{j}} \right) + \sum_{i=2}^{n-1} \left( \frac{n^2 + n + 2i(i-n-1)}{\sum_{j=1}^{i-1} \frac{1}{j} + \sum_{j=1}^{n-i} \frac{1}{j}} \right).$$

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$, where $v_i$ is adjacent to $v_{i+1}$, $i = 1, 2, \ldots, n-1$. Therefore,

$$\begin{aligned}
\sigma(v_i) &= (i-1) + (i-2) + \cdots + 1 + 1 + 2 + \cdots + (n-i) \\
&= \frac{(i-1)i}{2} + \frac{(n-i)(n-i+1)}{2} \\
&= \frac{n^2+n}{2} + i(i-n-1), \quad \text{for } i = 1, 2, 3, \ldots, n-1,
\end{aligned}$$

$$rs(v_i) = \sum_{j=1}^{i-1} \frac{1}{j} + \sum_{j=1}^{n-i} \frac{1}{j}, \qquad \text{for } i = 2, 3, \ldots, n-1$$

and

$$rs(v_1) = rs(v_n) = \sum_{j=1}^{n-1} \frac{1}{j}.$$

Therefore

$$\begin{aligned}
TRT(P_n) &= \sum_{i=1}^{n-1}\left(\frac{\sigma(v_i)}{rs(v_i)}+\frac{\sigma(v_{i+1})}{rs(v_{i+1})}\right)\\
&= \frac{\sigma(v_1)}{rs(v_1)}+2\sum_{i=2}^{n-1}\left(\frac{\sigma(v_i)}{rs(v_i)}\right)+\frac{\sigma(v_n)}{rs(v_n)}\\
&= \left(\frac{\frac{n(n-1)}{2}}{\sum_{j=1}^{n-1}\frac{1}{j}}\right)+2\sum_{i=2}^{n-1}\left(\frac{\frac{n^2+n}{2}+i(i-n-1)}{\sum_{j=1}^{i-1}\frac{1}{j}+\sum_{j=1}^{n-i}\frac{1}{j}}\right)+\left(\frac{\frac{n(n-1)}{2}}{\sum_{j=1}^{n-1}\frac{1}{j}}\right)\\
&= \left(\frac{n(n-1)}{\sum_{j=1}^{n-1}\frac{1}{j}}\right)+\sum_{i=2}^{n-1}\left(\frac{n^2+n+2i(i-n-1)}{\sum_{j=1}^{i-1}\frac{1}{j}+\sum_{j=1}^{n-i}\frac{1}{j}}\right).
\end{aligned}$$

□

**Proposition 7** *For a cycle* $C_n$ *on* $n \geq 3$ *vertices,*

$$TRT(C_n) = \begin{cases} \dfrac{n^4}{4\left(1+n\sum\limits_{i=1}^{\frac{n-2}{2}}\frac{1}{i}\right)}, & \text{if } n \text{ is even} \\ \dfrac{n(n^2-1)}{4\sum\limits_{i=1}^{\frac{n-1}{2}}\frac{1}{i}}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** If $n$ is even, then for every vertex $u$ of $C_n$

$$\sigma(u) = 2\left[1+2+3+\cdots+\frac{n-1}{2}\right]+\frac{n}{2} = \frac{n^2}{4}$$

and

$$rs(u) = \frac{2}{n}+2\sum_{i=1}^{\frac{n-2}{2}}\frac{1}{i}.$$

Therefore

$$\begin{aligned}
TRT(C_n) &= \sum_{uv\in E(C_n)} \left(\frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)}\right) \\
&= \sum_{uv\in E(C_n)} \left(\frac{\frac{n^2}{4}}{\frac{2}{n} + 2\sum\limits_{i=1}^{\frac{n-2}{2}} \frac{1}{i}} + \frac{\frac{n^2}{4}}{\frac{2}{n} + 2\sum\limits_{i=1}^{\frac{n-2}{2}} \frac{1}{i}}\right) \\
&= \sum_{uv\in E(C_n)} \frac{n^2}{2\left(\frac{2}{n} + 2\sum\limits_{i=1}^{\frac{n-2}{2}} \frac{1}{i}\right)} \\
&= \frac{n^4}{4\left(1 + n\sum\limits_{i=1}^{\frac{n-2}{2}} \frac{1}{i}\right)}.
\end{aligned}$$

If $n$ is odd, then for every vertex $u$ of $C_n$

$$\sigma(u) = 2\left[1 + 2 + \cdots + \frac{n-1}{2}\right] = \frac{n^2-1}{4} \quad \text{and} \quad rs(u) = 2\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}.$$

Therefore

$$\begin{aligned}
TRT(C_n) &= \sum_{uv\in E(C_n)} \left(\frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)}\right) \\
&= \sum_{uv\in E(C_n)} \left(\frac{\frac{n^2-1}{4}}{2\sum\limits_{i=1}^{\frac{n-1}{2}} \frac{1}{i}} + \frac{\frac{n^2-1}{4}}{2\sum\limits_{i=1}^{\frac{n-1}{2}} \frac{1}{i}}\right) \\
&= \sum_{uv\in E(C_n)} \frac{n^2-1}{4\left(\sum\limits_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\right)}
\end{aligned}$$

$$= \frac{n(n^2-1)}{4\left(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\right)}.$$

□

**Definition 8** *A wheel $W_{n+1}$, is a graph obtained from the cycle $C_n$, $n \geq 3$, by adding a new vertex and making it adjacent to all the vertices of $C_n$ (see Fig. 2).*

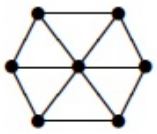


Figure 2: Wheel $W_7$.

**Definition 9** *A friendship graph (or Dutch windmill graph) $F_n$, $n \geq 2$, is a graph that can be constructed by coalescence $n$ copies of the cycle $C_3$ of length 3 with a common vertex (see Fig. 3). It has $2n+1$ vertices and $3n$ edges.*

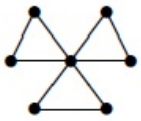


Figure 3: Friendship graph $F_3$.

**Proposition 10** *For a wheel $W_{n+1}$, $n \geq 3$,*

$$TRT(W_{n+1}) = \frac{n}{n+3}(13n-15).$$

**Proof.** The wheel $W_{n+1}$ has $n+1$ vertices, $2n$ edges and $\mathrm{diam}(w_{n+1}) = 2$. The edge set $E(W_{n+1})$ can be partitioned into two sets $E_1$ and $E_2$, such that $E_1(W_{n+1}) = \{uv \mid d(u) = n, d(v) = 3\}$ and $E_2(W_{n+1}) = \{uv \mid d(u) = 3, d(v) = 3\}$. It is easy to check that $|E_1(W_{n+1})| = n$ and $|E_2(W_{n+1})| = n$. By the Theorem 2 we have

$$\begin{aligned}
&TRT(W_{n+1}) \\
&= \sum_{uv\in E(W_{n+1})} \frac{2[4n^2-8n+(n-1)(d(u)+d(v))-2d(u)d(v)+4]}{n^2-2n+(n-1)(d(u)+d(v))+d(u)d(v)+1} \\
&= \sum_{uv\in E_1(W_{n+1})} \frac{2[4(n+1)^2-8(n+1)+n(n+3)-6n+4]}{(n+1)^2-2(n+1)+n(n+3)+3n+1} \\
&\quad + \sum_{uv\in E_2(W_{n+1})} \frac{2[4(n+1)^2-8(n+1)+n(6)-18+4]}{(n+1)^2-2(n+1)+n(6)+9+1} \\
&= \sum_{uv\in E_1(W_{n+1})} \frac{2[4n^2+4+8n-8n-8+n^2+3n-6n+4]}{n^2+1+2n-2n-2+n^2+3n+3n+1} \\
&\quad + \sum_{uv\in E_2(W_{n+1})} \frac{2[4n^2+4+8n-8n-8+6n-18+4]}{n^2+1+2n-2n-2+6n+9+1} \\
&= \frac{2n(5n^2-3n)}{2n^2+6n} + \frac{2n(4n^2+6n-18)}{n^2+6n+9} \\
&= \frac{n}{n+3}(13n-15).
\end{aligned}$$

□

**Proposition 11** *For a friendship graph* $F_n$, $n \geq 2$,

$$TRT(F_n) = \frac{6n}{n+1}(3n-1).$$

**Proof.** The edge set $E(F_n)$ can be partitioned into two sets $E_1$ and $E_2$ such that $E_1(F_n) = \{uv \mid d(u) = 2n, d(v) = 2\}$ and $E_2(F_n) = \{uv \mid d(u) = 2, d(v) = 2\}$. It is easy to check that $|E_1(F_n)| = 2n$ and $|E_2(F_n)| = n$. By the Theorem 2 we have

$$TRT(F_n) = \sum_{uv\in E(F_n)} \frac{2[4n^2-8n+(n-1)(d(u)+d(v))-2d(u)d(v)+4]}{n^2-2n+(n-1)(d(u)+d(v))+d(u)d(v)+1}$$

$$
\begin{aligned}
&= \sum_{uv\in E_1(F_n)} \frac{2[4(2n+1)^2-8(2n+1)+2n(2n+2)-8n+4]}{(2n+1)^2-2(2n+1)+2n(2n+2)+4n+1} \\
&\quad + \sum_{uv\in E_2(F_n)} \frac{2[4(2n+1)^2-8(2n+1)+8n-8+4]}{(2n+1)^2-2(2n+1)+8n+4+1} \\
&= \sum_{uv\in E_1(F_n)} \frac{2[4(4n^2+1+4n)-16n-8+4n^2+4n-8n+4]}{4n^2+1+4n-4n-2+4n^2+4n+4n+1} \\
&\quad + \sum_{uv\in E_2(F_n)} \frac{2[4(4n^2+1+4n)-16n-8+8n-8+4]}{4n^2+1+4n-4n-2+8n+4+1} \\
&= \frac{2n(5n-1)}{n+1} + \frac{4n(2n-1)}{n+1} \\
&= \frac{6n}{n+1}(3n-1).
\end{aligned}
$$

□

## 3 Transmission-reciprocal transmission index of cluster graphs

The graphs with large number of edges are referred as cluster graphs. Some of the cluster graphs obtained from complete graph are defined below.

**Definition 12** [8] *Let* $e_i$, $i = 1, 2, \ldots, k$, $1 \le k \le n-2$, *be the distinct edges of a complete graph* $K_n$, $n \ge 3$, *all being incident to a single vertex. The graph* $Ka_n(k)$ *is obtained by deleting* $e_i$, $i = 1, 2, \ldots, k$ *from* $K_n$. *In addition* $Ka_n(0) \cong K_n$.

**Definition 13** [8] *Let* $f_i$, $i = 1, 2, \ldots, k$, $1 \le k \le \lfloor \frac{n}{2} \rfloor$ *be independent edges of the complete graph* $K_n$, $n \ge 3$. *The graph* $Kb_n(k)$ *is obtained by deleting* $f_i$, $i = 1, 2, \ldots, k$ *from* $K_n$. *In addition* $Kb_n(0) \cong K_n$.

**Definition 14** [8] *Let* $V_k$ *be a* $k$*-element subset of the vertex set of the complete graph* $K_n$, $2 \le k \le n-1$, $n \ge 3$. *The graph* $Kc_n(k)$ *is obtained by deleting from* $K_n$ *all the edges connecting pairs of vertices from* $V_k$. *In addition* $Kc_n(0) \cong Kc_n(1) \cong K_n$.

**Definition 15** [8] *Let* $3 \leq k \leq n$, $n \geq 3$. *The graph* $Kd_n(k)$ *is obtained by deleting from the complete graph* $K_n$, *the edges belonging to a* $k$*-membered cycle.*

**Proposition 16** *For* $n \geq 3$ *and* $1 \leq k \leq n-2$,

$$\begin{aligned} TRT(Ka_n(k)) &= \frac{(n-k-1)(4n^2-8n+k(n-1)+4)}{2n^2-4n+k(1-n)+2} \\ &\quad + \frac{2k(k-1)(2n^2-3n)}{4n^2-12n+9} \\ &\quad + \frac{k(n-k-1)(4n^2-7n+3)}{(2n^2-5n+3)} + (n-k-1)(n-k-2). \end{aligned}$$

**Proof.** The graph $Ka_n(k)$ has $n$ vertices, $\frac{n(n-1)}{2}-k$ edges and $diam(Ka_n(k)) = 2$. The edge set $E(Ka_n(k))$ can be partitioned into four sets $E_1 = \{uv \mid d(u) = n-1-k,\ d(v) = n-1\}$, $E_2 = \{uv \mid d(u) = n-2 = d(v)\}$, $E_3 = \{uv \mid d(u) = n-2,\ d(v) = n-1\}$ and $E_4 = \{uv \mid d(u) = n-1 = d(v)\}$. It is easy to check that $|E_1| = n-k-1$, $|E_2| = \frac{k(k-1)}{2}$, $|E_3| = k(n-k-1)$ $|E_4| = \frac{(n-k-1)(n-k-2)}{2}$. Now we have

$$\begin{aligned} &TRT(Ka_n(k)) \\ &= \sum_{uv \in E(Ka_n(k))} \frac{2[4n^2-8n+(n-1)(d(u)+d(v))-2d(u)d(v)+4]}{n^2-2n+(n-1)(d(u)+d(v))+d(u)d(v)+1} \\ &= \sum_{uv \in E_1} \frac{2[4n^2-8n+(n-1)(n-1-k+n-1)-2(n-1-k)(n-1)+4]}{n^2-2n+(n-1)(n-1-k+n-1)+(n-1-k)(n-1)+1} \\ &\quad + \sum_{uv \in E_2} \frac{2[4n^2-8n+(n-1)(n-2+n-2)-2(n-2)^2+4]}{n^2-2n+(n-1)(n-2+n-2)+(n-2)^2+1} \\ &\quad + \sum_{uv \in E_3} \frac{2[4n^2-8n+(n-1)(n-2+n-1)-2(n-1)(n-2)+4]}{n^2-2n+(n-1)(n-2+n-1)+(n-2)(n-1)+1} \\ &\quad + \sum_{uv \in E_4} \frac{2[4n^2-8n+(n-1)(n-1+n-1)-2(n-1)^2+4]}{n^2-2n+(n-1)(n-1+n-1)+(n-1)^2+1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{uv\in E_1(Ka_n(k))} \frac{2(4n^2-8n+nk-k+4)}{4n^2-8n-2nk+2k+4} + \sum_{uv\in E_2(Ka_n(k))} \frac{4(2n^2-3n)}{4n^2-12n+9} \\
&\quad + \sum_{uv\in E_3(Ka_n(k))} \frac{2(4n^2-7n+3)}{4n^2-10n+6} + \sum_{uv\in E_4(Ka_n(k))} \frac{2(4n^2-8n+4)}{4n^2-8n+4} \\
&= \frac{(n-k-1)(4n^2-8n+k(n-1)+4)}{2n^2-4n+k(1-n)+2} + \frac{2k(k-1)(2n^2-3n)}{4n^2-12n+9} \\
&\quad + \frac{k(n-k-1)(4n^2-7n+3)}{(2n^2-5n+3)} + (n-k-1)(n-k-2).
\end{aligned}
$$

□

**Proposition 17** *For* $n \geq 3$ *and* $1 \leq k \leq \lfloor n/2 \rfloor$,

$$
\begin{aligned}
TRT(Kb_n(k)) &= \frac{4k(n-2k)(4n^2-7n+3)}{4n^2-9n+6} + (n-2k)(n-2k-1) \\
&\quad + \left(\left(\frac{2k(2k-1)}{2}\right) - k\right)\left(\frac{4(2n^2-3n)}{4n^2-12n+9}\right)
\end{aligned}
$$

**Proof.** The graph $Kb_n(k)$ has $n$ vertices, $\frac{n(n-1)}{2}-k$ edges and $diam(Kb_n(k)) = 2$. The edge set $E(Kb_n(k))$ can be partitioned into three sets $E_1 = \{uv \mid d(u) = n-2,\ d(v) = n-1\}$, $E_2 = \{uv \mid d(u) = n-1 = d(v)\}$ and $E_3 = \{uv \mid d(u) = n-2 = d(v)\}$. It is easy to check that $|E_1| = 2k(n-2k)$, $|E_2| = \frac{(n-2k)(n-2k-1)}{2}$, $|E_3| = \left(\frac{2k(2k-1)}{2}\right) - k$. Therefore we have

$$
\begin{aligned}
&TRT(Kb_n(k)) \\
&= \sum_{uv\in E(Kb_n(k))} \frac{2[4n^2-8n+(n-1)(d(u)+d(v))-2d(u)d(v)+4]}{n^2-2n+(n-1)(d(u)+d(v))+d(u)d(v)+1} \\
&= \sum_{uv\in E_1(Kb_n(k))} \frac{2[4n^2-8n+(n-1)(n-2+n-1)-2(n-2)(n-1)+4]}{n^2-2n+(n-1)(n-2+n-1)+(n-2)(n-1)+1} \\
&\quad + \sum_{uv\in E_2(Kb_n(k))} \frac{2[4n^2-8n+(n-1)(n-1+n-1)-2(n-1)^2+4]}{n^2-2n+(n-1)(n-1+n-1)+(n-1)^2+1} \\
&\quad + \sum_{uv\in E_3(Kb_n(k))} \frac{2[4n^2-8n+(n-1)(n-2+n-2)-2(n-2)^2+4]}{n^2-2n+(n-1)(n-2+n-2)+(n-2)^2+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{uv\in E_1(Kb_n(k))} \frac{2(4n^2-7n+3)}{4n^2-9n+6} + \sum_{uv\in E_2(Kb_n(k))} \frac{2(4n^2-8n+4)}{4n^2-8n+4} \\
&\quad + \sum_{uv\in E_3(Kb_n(k))} \frac{4(2n^2-3n)}{4n^2-12n+9} \\
&= \frac{4k(n-2k)(4n^2-7n+3)}{4n^2-9n+6} + (n-2k)(n-2k-1) \\
&\quad + \left(\left(\frac{2k(2k-1)}{2}\right) - k\right)\left(\frac{4(2n^2-3n)}{4n^2-12n+9}\right).
\end{aligned}
$$

□

**Proposition 18** *For* $n \geq 3$ *and* $2 \leq k \leq n-1$,

$$TRT(Kc_n(k)) = \frac{2k(n-k)(4n^2-9n+k(n+3)+5)}{4n^2-6n-2k(n-1)+2} + (n-k)(n-k-1).$$

**Proof.** The graph $Kc_n(k)$ has $n$ vertices and $\frac{1}{2}(n-k)(n+k-1)$ edges. Also $diam(Kc_n(k)) = 2$. The edge set $E(Kc_n(k))$ can be partitioned into two sets $E_1 = \{uv \mid d(u) = n-k,\ d(v) = n-1\}$ and $E_2 = \{uv \mid d(u) = n-1 = d(v)\}$. It is easy to check that $|E_1| = k(n-k)$, $|E_2| = \frac{(n-k)(n-k-1)}{2}$. Therefore we have

$$
\begin{aligned}
&TRT(Kc_n(k)) \\
&= \sum_{uv\in E(Kc_n(k))} \frac{2[4n^2-8n+(n-1)(d(u)+d(v))-2d(u)d(v)+4]}{n^2-2n+(n-1)(d(u)+d(v))+d(u)d(v)+1} \\
&= \sum_{uv\in E_1(Kc_n(k))} \frac{2[4n^2-8n+(n-1)(n-k+n-1)-2(n-k)(n-1)+4]}{n^2-2n+(n-1)(n-k+n-1)+(n-k)(n-1)+1} \\
&\quad + \sum_{uv\in E_2(Kc_n(k))} \frac{2[4n^2-8n+(n-1)(n-1+n-1)-2(n-1)^2+4]}{n^2-2n+(n-1)(n-1+n-1)+(n-1)^2+1} \\
&= \sum_{uv\in E_1(Kc_n(k))} \frac{(4n^2-9n+nk-k+5)}{2n^2-3n-nk+k+1} + \sum_{uv\in E_2(Kc_n(k))} \frac{2(4n^2-8n+4)}{4n^2-8n+4} \\
&= \frac{k(n-k)(4n^2-9n+nk-k+5)}{2n^2-3n-nk+k+1} + (n-k)(n-k-1).
\end{aligned}
$$

□

**Proposition 19** *For* $3 \leq k \leq n$ *and* $n \geq 5$,

$$TRT(Kd_n(k)) = \frac{k(n-k)(2n^2-3n+1)}{n^2-3n+2} + (n-k)(n-k-1) + \left(\left(\frac{k(k-1)}{2}\right)-k\right)\left(\frac{2(n^2-n-2)}{n^2-4n+4}\right).$$

**Proof.** The graph $Kd_n(k)$ has $n$ vertices and $\frac{n(n-1)}{2}-k$ edges.
Also $diam(Kd_n(k)) = 2$. The edge set $E(Kd_n(k))$ can be partitioned into three sets $E_1 = \{uv \mid d(u) = n-3 = d(v)\}$, $E_2 = \{uv \mid d(u) = n-3,\ d(v) = n-1\}$ and $E_3 = \{uv \mid d(u) = n-1 = d(v)\}$. It is easy to check that $|E_1| = \frac{k(k-1)}{2} - k$, $|E_2| = (n-k)k$, $|E_3| = \frac{(n-k-1)(n-k)}{2}$. Therefore we have

$$\begin{aligned}
&TRT(Kd_n(k)) \\
&= \sum_{uv\in E(Kd_n(k))} \frac{2[4n^2-8n+(n-1)(d(u)+d(v))-2d(u)d(v)+4]}{n^2-2n+(n-1)(d(u)+d(v))+d(u)d(v)+1} \\
&= \sum_{uv\in E_1(Kd_n(k))} \frac{2[4n^2-8n+(n-1)(n-3+n-3)-2(n-3)^2+4]}{n^2-2n+(n-1)(n-3+n-3)+(n-3)^2+1} + \\
&\quad \sum_{uv\in E_2(Kd_n(k))} \frac{2[4n^2-8n+(n-1)(n-3+n-1)-2(n-3)(n-1)+4]}{n^2-2n+(n-1)(n-3+n-1)+(n-3)(n-1)+1} \\
&\quad + \sum_{uv\in E_3(Kd_n(k))} \frac{2[4n^2-8n+(n-1)(n-1+n-1)-2(n-1)^2+4]}{n^2-2n+(n-1)(n-1+n-1)+(n-1)^2+1} \\
&= \sum_{uv\in E_1(Kd_n(k))} \frac{2(n^2-n-2)}{n^2-4n+4} + \sum_{uv\in E_2(Kd_n(k))} \frac{2n^2-3n+1}{n^2-3n+2} \\
&\quad + \sum_{uv\in E_3(Kd_n(k))} \frac{2(4n^2-8n+4)}{4n^2-8n+4} \\
&= \left(\left(\frac{k(k-1)}{2}\right)-k\right)\left(\frac{2(n^2-n-2)}{n^2-4n+4}\right) + \frac{k(n-k)(2n^2-3n+1)}{n^2-3n+2} \\
&\quad +(n-k)(n-k-1).
\end{aligned}$$

□

# 4 Bounds for transmission-reciprocal transmission index of graphs

**Theorem 20** *Let* $G$ *be a connected graph with* $n$ *vertices,* $m$ *edges and let* $diam(G) = D$. *Then*

$$\sum_{uv\in E(G)} \frac{8(n-1)^2 + 2(n-1)(d(u)+d(v)) - 4d(u)d(v)}{(n-1)^2 + (n-1)(d(u)+d(v)) + d(u)d(v)} \leq TRT(G) \leq$$
$$\sum_{uv\in E(G)} \frac{2D^2(n-1)^2 + D(D-1)^2(n-1)(d(u)+d(v)) - 2D(D-1)^2 d(u)d(v)}{(n-1)^2 + (n-1)(D-1)(d(u)+d(v)) + (D-1)^2 d(u)d(v)}.$$

**Proof.** Lower bound: For any vertex $u$ of $G$, there are $d(u)$ vertices which are at distance 1 from $u$ and remaining $n-1-d(u)$ vertices are at distance at least 2. Therefore

$$\sigma(u) \geq d(u) + 2(n-1-d(u)) = 2n-2-d(u)$$

and

$$rs(u) \leq d(u) + \frac{1}{2}(n-1-d(u)) \leq \frac{1}{2}(n-1+d(u)).$$

This implies

$$\frac{1}{rs(u)} \geq \frac{2}{(n-1+d(u))}.$$

Therefore

$$\begin{aligned}
&TRT(G)\\
&= \sum_{uv\in E(G)} \left(\frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)}\right)\\
&\geq \sum_{uv\in E(G)} \left(\frac{2(2n-2-d(u))}{n-1+d(u)} + \frac{2(2n-2-d(v))}{n-1+d(v)}\right)\\
&= \sum_{uv\in E(G)} \frac{(4n-4-2d(u))(n-1+d(v)) + (4n-4-2d(v))(n-1+d(u))}{(n-1+d(u))(n-1+d(v))}\\
&= \sum_{uv\in E(G)} \frac{8(n-1)^2 + 2(n-1)(d(u)+d(v)) - 4d(u)d(v)}{(n-1)^2 + (n-1)(d(u)+d(v)) + d(u)d(v)}.
\end{aligned}$$

Upper Bound: For any vertex $u$ of $G$, there are $d(u)$ vertices which are at distance 1 from $u$ and the remaining $(n-1-d(u))$ vertices are at distance at most $D$. Therefore

$$\sigma(u) \leq d(u) + D(n-1-d(u)) = D(n-1) - (D-1)d(u)$$

and

$$rs(u) \geq d(u) + \frac{1}{D}(n-1-d(u)) = \frac{1}{D}(n-1) + \left(1 - \frac{1}{D}\right) d(u).$$

This implies

$$\frac{1}{rs(u)} \leq \frac{D}{(n-1) + (D-1)d(u)}.$$

Therefore

$$\begin{aligned}
&TRT(G) \\
&= \sum_{uv \in E(G)} \left( \frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)} \right) \\
&\leq \sum_{uv \in E(G)} \left[ \frac{(D(n-1) - (D-1)d(u))D}{((n-1) + (D-1)d(u))} + \frac{(D(n-1) - (D-1)d(v))D}{((n-1) + (D-1)d(v))} \right] \\
&= \sum_{uv \in E(G)} \frac{\begin{bmatrix} 2D^2(n-1)^2 + D(D-1)^2(n-1)(d(u)+d(v)) \\ -2D(D-1)^2 d(u)d(v) \end{bmatrix}}{(n-1)^2 + (n-1)(D-1)(d(u)+d(v)) + (D-1)^2 d(u)d(v)}.
\end{aligned}$$

□

## 5 Transmission-reciprocal transmission coindex of graphs

**Proposition 21** *Let $G$ be a connected graph on $n$ vertices. Then*

$$\overline{TRT}(G) = (n-1) \sum_{u \in V(G)} \frac{\sigma(u)}{rs(u)} - TRT(G).$$

**Proof.**

$$\begin{aligned}\overline{TRT}(G) &= \sum_{uv\notin E(G)}\left(\frac{\sigma(u)}{rs(u)}+\frac{\sigma(v)}{rs(v)}\right)\\ &= \sum_{\{u,v\}\subseteq V(G)}\left(\frac{\sigma(u)}{rs(u)}+\frac{\sigma(v)}{rs(v)}\right)-\sum_{uv\in E(G)}\left(\frac{\sigma(u)}{rs(u)}+\frac{\sigma(v)}{rs(v)}\right)\\ &= (n-1)\sum_{u\in V(G)}\frac{\sigma(u)}{rs(u)}-TRT(G).\end{aligned}$$

□

**Corollary 22** *Let* $G$ *be a connected graph on* $n$ *vertices and* $diam(G)\leq 2$. *Then*

$$\begin{aligned}\overline{TRT}(G) &= (n-1)\sum_{u\in V(G)}\frac{4n-4-2d(u)}{n-1+d(u)}\\ &\quad -\sum_{uv\in E(G)}\frac{2[4n^2-8n+(n-1)(d(u)+d(v))-2d(u)d(v)+4]}{n^2-2n+(n-1)(d(u)+d(v))+d(u)d(v)+1}.\end{aligned}$$

**Proof.** For a graph with $diam(G)\leq 2$, we know that $\sigma(u)=2n-2-d(u)$ and $rs(u)=\frac{1}{2}(n-1+d(u))$. By Proposition 21 result follows. □

**Proposition 23** *For any connected graph with* $n$ *vertices and* $diam(G)\leq 2$,

$$\overline{TRT}(G)=\sum_{uv\notin E(G)}\frac{8(n-1)^2+2(n-1)(d(u)+d(v))-4d(u)d(v)}{(n-1)^2+(n-1)(d(u)+d(v))+d(u)d(v)}.$$

**Proof.** For any graph with $diam(G)\leq 2$, $\sigma(u)=2n-2-d(u)$ and $rs(u)=\frac{1}{2}(n-1+d(u))$. Therefore

$$\begin{aligned}&\overline{TRT}(G)\\ &= \sum_{uv\notin E(G)}\left(\frac{2(2n-2-d(u))}{n-1+d(u)}+\frac{2(2n-2-d(v))}{n-1+d(v)}\right)\\ &= \sum_{uv\notin E(G)}\frac{(4n-4-2d(u))(n-1+d(v))+(4n-4-2d(v))(n-1+d(u))}{(n-1+d(u))(n-1+d(v))}\\ &= \sum_{uv\notin E(G)}\frac{8(n-1)^2+2(n-1)(d(u)+d(v))-4d(u)d(v)}{(n-1)^2+(n-1)(d(u)+d(v))+d(u)d(v)}.\end{aligned}$$

□

**Proposition 24** *Let* $G$ *be a graph with* $n$ *vertices. Let* $\overline{G}$ *be the complement of* $G$ *and it is connected. Then*

$$TRT(\overline{G}) \geq \sum_{uv \in E(\overline{G})} \frac{8(n-1)^2 - 6(n-1)(d(u)+d(v)) + 4d(u)d(v)}{4(n-1)^2 - 2(n-1)(d(u)+d(v)) + d(u)d(v)}.$$

**Proof.** For any vertex $u$ in $\overline{G}$, there are $n-1-d_G(u)$ vertices which are at distance 1 and the remaining $d_G(u)$ vertices are at distance at least 2. Therefore

$$\sigma_{\overline{G}}(u) \geq [n-1-d_G(u)] + 2d_G(u) = n-1-d_G(u)$$

and

$$rs_{\overline{G}}(u) \leq (n-1-d_G(u)) + \frac{1}{2}d_G(u) = n-1-\frac{1}{2}d_G(u).$$

Therefore

$$\begin{aligned}
&TRT(\overline{G}) \\
&= \sum_{uv \in E(\overline{G})} \left( \frac{\sigma(u)}{rs(u)} + \frac{\sigma(v)}{rs(v)} \right) \\
&\geq \sum_{uv \in E(\overline{G})} \left( \frac{2n-2-2d(u)}{2n-2-d(u)} + \frac{2n-2-2d(v)}{2n-2-d(v)} \right) \\
&= \sum_{uv \in E(\overline{G})} \frac{8n^2 - 16n + 8 - (6n-6)(d(u)+d(v)) + 4d(u)d(v)}{4n^2 - 8n + 4 - (2n-2)(d(u)+d(v)) + d(u)d(v)} \\
&= \sum_{uv \in E(\overline{G})} \frac{8(n-1)^2 - 6(n-1)(d(u)+d(v)) + 4d(u)d(v)}{4(n-1)^2 - 2(n-1)(d(u)+d(v)) + d(u)d(v)}.
\end{aligned}$$

□

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